Higher-Order Functional Reactive Programming without Spacetime Leaks

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$$
\begin{array}{llll}
\text{Types} & A & ::= & b \mid A \times B \mid A + B \mid A \to B \mid \bullet A \mid \hat{\mu}\alpha.\, A \mid \Box A \mid \mathsf{S}\,A \mid \mathsf{alloc} \\
\text{Terms} & e & ::= & \mathsf{fst}\, e \mid \mathsf{snd}\, e \mid (e, e') \\
 & & \mid & \mathsf{inl}\, e \mid \mathsf{inr}\, e \mid \mathsf{case}(e, \mathsf{inl}\, x \to e', \mathsf{inr}\, y \to e'') \\
 & & \mid & \lambda x.\, e \mid e\; e' \\
 & & \mid & \delta_{e'}(e) \mid \mathsf{let}\ \delta(x) = e\ \mathsf{in}\ e' \\
 & & \mid & \mathsf{into}\, e \mid \mathsf{out}\, e \\
 & & \mid & \mathsf{stable}(e) \mid \mathsf{let}\ \mathsf{stable}(x) = e\ \mathsf{in}\ e' \\
 & & \mid & \mathsf{cons}(e, e') \mid \mathsf{let}\ \mathsf{cons}(x, xs) = e\ \mathsf{in}\ e' \\
 & & \mid & \mathsf{fix}\, x.\, e \mid x \mid \mathsf{promote}(e) \\
 & & \mid & l \mid\ !l \mid \diamond \\
\text{Values} & v & ::= & (v, v') \mid \mathsf{inl}\, v \mid \mathsf{inr}\, v' \mid \lambda x.\, e \mid l \mid \mathsf{into}\, v \mid \mathsf{stable}(v) \mid \mathsf{cons}(v, v') \mid \diamond \\
\text{Stores} & \sigma & ::= & \cdot \mid \sigma, l : v\ \mathsf{now} \mid \sigma, l : e\ \mathsf{later} \mid \sigma, l : \mathsf{null}
\end{array}
$$

Figure 1: Syntax

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2.1 Operational Properties

Lemma 1 (Extension). *If* $\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle$, *then there exists* σ'' *such that* $\sigma' = \sigma \cdot \sigma''$.

Lemma 2 (Uniformity). *If* $\langle \cdot; e \rangle \Downarrow \langle \cdot; v \rangle$, *then* $\langle \sigma; e \rangle \Downarrow \langle \sigma; v \rangle$.

Lemma 3 (Permutability). *We have that:*

1. *If* $\pi \in \text{Perm}$ *and* $\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle$ *then* $\langle \pi(\sigma); \pi(e) \rangle \Downarrow \langle \pi(\sigma'); \pi(v) \rangle$.
2. *If* $\pi \in \text{Perm}$ *and* $\sigma \Longrightarrow \sigma'$ *then* $\pi(\sigma) \Longrightarrow \pi(\sigma')$.

Lemma 4 (Supportedness). *We have that:*

$$\boxed{\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle}$$

$$\frac{}{\langle \sigma; v \rangle \Downarrow \langle \sigma; v \rangle}$$

$$\frac{\langle \sigma; e_1 \rangle \Downarrow \langle \sigma'; v_1 \rangle \qquad \langle \sigma'; e_2 \rangle \Downarrow \langle \sigma''; v_2 \rangle}{\langle \sigma; (e_1, e_2) \rangle \Downarrow \langle \sigma''; (v_1, v_2) \rangle} \qquad \frac{\langle \sigma; e \rangle \Downarrow \langle \sigma'; (v_1, v_2) \rangle}{\langle \sigma; \mathsf{fst}\, e \rangle \Downarrow \langle \sigma'; v_1 \rangle} \qquad \frac{\langle \sigma; e \rangle \Downarrow \langle \sigma'; (v_1, v_2) \rangle}{\langle \sigma; \mathsf{snd}\, e \rangle \Downarrow \langle \sigma'; v_2 \rangle}$$

$$\frac{\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle}{\langle \sigma; \mathsf{inl}\, e \rangle \Downarrow \langle \sigma'; \mathsf{inl}\, v \rangle} \qquad \frac{\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle}{\langle \sigma; \mathsf{inr}\, e \rangle \Downarrow \langle \sigma'; \mathsf{inr}\, v \rangle}$$

$$\frac{\langle \sigma; e \rangle \Downarrow \langle \sigma'; \mathsf{inl}\, v \rangle \qquad \langle \sigma'; [v/x]e' \rangle \Downarrow \langle \sigma''; v'' \rangle}{\langle \sigma; \mathsf{case}(e, \mathsf{inl}\, x \to e', \mathsf{inr}\, y \to e'') \rangle \Downarrow \langle \sigma''; v'' \rangle} \qquad \frac{\langle \sigma; e \rangle \Downarrow \langle \sigma'; \mathsf{inr}\, v \rangle \qquad \langle \sigma'; [v/y]e'' \rangle \Downarrow \langle \sigma''; v'' \rangle}{\langle \sigma; \mathsf{case}(e, \mathsf{inl}\, x \to e', \mathsf{inr}\, y \to e'') \rangle \Downarrow \langle \sigma''; v'' \rangle}$$

$$\frac{\langle \sigma; e_1 \rangle \Downarrow \langle \sigma'; \lambda x.\, e_1' \rangle \qquad \langle \sigma'; e_2 \rangle \Downarrow \langle \sigma''; v_2 \rangle \qquad \langle \sigma''; [v_2/x]e_1' \rangle \Downarrow \langle \sigma'''; v \rangle}{\langle \sigma; e_1\ e_2 \rangle \Downarrow \langle \sigma'''; v \rangle}$$

$$\frac{\langle \sigma; e' \rangle \Downarrow \langle \sigma'; \diamond \rangle \qquad l \notin \mathrm{dom}(\sigma')}{\langle \sigma; \delta_{e'}(e) \rangle \Downarrow \langle (\sigma', l : e\ \mathsf{later}); l \rangle} \qquad \frac{\langle \sigma; e \rangle \Downarrow \langle \sigma'; l \rangle \qquad \langle \sigma'; [!l/x]e' \rangle \Downarrow \langle \sigma''; v \rangle}{\langle \sigma; \mathsf{let}\ \delta(x) = e\ \mathsf{in}\ e' \rangle \Downarrow \langle \sigma''; v \rangle} \qquad \frac{l : v\ \mathsf{now} \in \sigma}{\langle \sigma; !l \rangle \Downarrow \langle \sigma; v \rangle}$$

$$\frac{\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle}{\langle \sigma; \mathsf{into}\, e \rangle \Downarrow \langle \sigma'; \mathsf{into}\, v \rangle} \qquad \frac{\langle \sigma; e \rangle \Downarrow \langle \sigma'; \mathsf{into}\, v \rangle}{\langle \sigma; \mathsf{out}\, e \rangle \Downarrow \langle \sigma'; v \rangle}$$

$$\frac{\langle \sigma; e \rangle \Downarrow \langle \sigma; v \rangle}{\langle \sigma; \mathsf{stable}(e) \rangle \Downarrow \langle \sigma; \mathsf{stable}(v) \rangle} \qquad \frac{\langle \sigma; e \rangle \Downarrow \langle \sigma'; \mathsf{stable}(v) \rangle \qquad \langle \sigma'; [v/x]e' \rangle \Downarrow \langle \sigma''; v'' \rangle}{\langle \sigma; \mathsf{let\ stable}(x) = e\ \mathsf{in}\ e' \rangle \Downarrow \langle \sigma''; v'' \rangle}$$

$$\frac{\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle \qquad \langle \sigma'; e' \rangle \Downarrow \langle \sigma''; v' \rangle}{\langle \sigma; \mathsf{cons}(e, e') \rangle \Downarrow \langle \sigma''; \mathsf{cons}(v, v') \rangle} \qquad \frac{\langle \sigma; e \rangle \Downarrow \langle \sigma'; \mathsf{cons}(v, l) \rangle \qquad \langle \sigma'; [v/x, l/xs]e' \rangle \Downarrow \langle \sigma''; v'' \rangle}{\langle \sigma; \mathsf{let\ cons}(x, xs) = e\ \mathsf{in}\ e' \rangle \Downarrow \langle \sigma''; v'' \rangle}$$

$$\frac{\langle \sigma; [\mathsf{fix}\, x.\ e/x]e \rangle \Downarrow \langle \sigma'; v \rangle}{\langle \sigma; \mathsf{fix}\, x.\ e \rangle \Downarrow \langle \sigma'; v \rangle} \qquad \frac{\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle}{\langle \sigma; \mathsf{promote}(e) \rangle \Downarrow \langle \sigma'; \mathsf{stable}(v) \rangle}$$

Figure 2: Expression Semantics

$$\boxed{\sigma \Longrightarrow \sigma'}$$

$$\frac{}{\cdot \Longrightarrow \cdot} \qquad \frac{\sigma \Longrightarrow \sigma' \qquad l \notin \mathrm{dom}(\sigma')}{\sigma, l : v\ \mathsf{now} \Longrightarrow \sigma', l : \mathsf{null}} \qquad \frac{\sigma \Longrightarrow \sigma' \qquad l \notin \mathrm{dom}(\sigma')}{\sigma, l : \mathsf{null} \Longrightarrow \sigma', l : \mathsf{null}}$$

$$\frac{\sigma \Longrightarrow \sigma' \qquad \langle \sigma'; e \rangle \Downarrow \langle \sigma''; v \rangle \qquad l \notin \mathrm{dom}(\sigma'')}{\sigma, l : e\ \mathsf{later} \Longrightarrow \sigma'', l : v\ \mathsf{now}}$$

Figure 3: Tick Semantics

$$
\begin{array}{llll}
\text{Qualifiers} & q & ::= & \mathsf{now} \mid \mathsf{stable} \mid \mathsf{later} \\
\text{Contexts} & \Gamma & ::= & \cdot \mid \Gamma, x : A\ q
\end{array}
$$

$$
\begin{array}{lcl}
(\cdot)^{\bullet} & = & \cdot \\
(\Gamma, x : A\ \mathsf{later})^{\bullet} & = & \Gamma^{\bullet}, x : A\ \mathsf{now} \\
(\Gamma, x : A\ \mathsf{stable})^{\bullet} & = & \Gamma^{\bullet}, x : A\ \mathsf{stable} \\
(\Gamma, x : A\ \mathsf{now})^{\bullet} & = & \Gamma^{\bullet} \\
\\
(\cdot)^{\square} & = & \cdot \\
(\Gamma, x : A\ \mathsf{stable})^{\square} & = & \Gamma^{\square}, x : A\ \mathsf{stable} \\
(\Gamma, x : A\ \mathsf{later})^{\square} & = & \Gamma^{\square} \\
(\Gamma, x : A\ \mathsf{now})^{\square} & = & \Gamma^{\square}
\end{array}
$$

Figure 4: Hypotheses, Contexts and Operations on Them

1. *If* $e \sqsubseteq \sigma$ *and* $\sigma' \leq \sigma$ *then* $e \sqsubseteq \sigma'$.

2. *If* $e \sqsubseteq \sigma$ *and* $e' \sqsubseteq \sigma$ *then* $[e/x]e' \sqsubseteq \sigma$.

3. *If* σ supported *and* $e \sqsubseteq \sigma$ *and* $\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle$ *then* $v \sqsubseteq \sigma'$ *and* σ' supported.

4. *If* σ supported *and* $\sigma \Longrightarrow \sigma'$ *then* σ' supported.

Lemma 5 (Quasi-determinacy)**.** *We have that:*

1. *If* $\langle \sigma; e \rangle \Downarrow \langle \sigma'; v' \rangle$ *and* $\langle \sigma; e \rangle \Downarrow \langle \sigma''; v'' \rangle$ *and* σ supported *and* $e \sqsubseteq \sigma$, *then there is a* $\pi \in \mathrm{Perm}$ *such that* $\pi'(\sigma') = \sigma''$ *and* $\pi(\sigma) = \sigma$.

2. *If* $\sigma \Longrightarrow \sigma'$ *and* $\sigma \Longrightarrow \sigma''$ *and* σ supported, *then there is a* $\pi \in \mathrm{Perm}$ *such that* $\pi(\sigma') = \sigma''$ *and* $\pi(\sigma) = \sigma$.

2.2 Semantic Properties

Lemma 6 (Order Permutation)**.** *If* $\sigma' \leq \sigma$ *and* $\pi \in \mathrm{Perm}$ *then* $\pi(\sigma') \leq \pi(\sigma)$.

Lemma 7 (Heap Renaming)**.** *For all* $\pi \in \mathrm{Perm}$ *and* $\sigma \in \mathrm{Heap}_n$, $\pi(\sigma) \in \mathrm{Heap}_n$.

Lemma 8 (Kripke Monotoncity)**.** *If* ρ *is a monotone environment and* $w' \leq w$, *then* $\mathcal{V}\,[\![A]\!]\,\rho\,w' \supseteq \mathcal{V}\,[\![A]\!]\,\rho\,w$.

Lemma 9 (Renaming)**.** *We have that:*

1. *If* ρ *is a permutable environment and* $\pi \in \mathrm{Perm}$ *and* $v \in \mathcal{V}\,[\![A]\!]\,\rho\,w$ *then* $\pi(v) \in \mathcal{V}\,[\![A]\!]\,\rho\,\pi(w)$.

2. *If* ρ *is a permutable environment and* $\pi \in \mathrm{Perm}$ *and* $e \in \mathcal{E}\,[\![A]\!]\,\rho\,w$ *then* $\pi(e) \in \mathcal{E}\,[\![A]\!]\,\rho\,\pi(w)$.

3. *If* ρ *is a permutable environment and* $\pi \in \mathrm{Perm}$ *and* $e \in \mathcal{L}\,[\![A]\!]\,\rho\,w$ *then* $\pi(e) \in \mathcal{L}\,[\![A]\!]\,\rho\,\pi(w)$.

Lemma 10 (Supportedness of the Logical Relation)**.** *If* ρ *is a supported environment and* $v \in \mathcal{V}\,[\![A]\!]\,\rho\,w$ *then* $v \sqsubseteq w.\sigma$.

Lemma 11 (Weakening)**.** *Assuming* ρ *is a type environment, we have that:*

1. *If* $\mathrm{FV}(A) \subseteq \mathrm{dom}(\rho)$ *then* $\mathcal{V}\,[\![A]\!]\,\rho\,w = \mathcal{V}\,[\![A]\!]\,(\rho, \rho')\,w$.

2. *If* $\mathrm{FV}(A) \subseteq \mathrm{dom}(\rho)$ *then* $\mathcal{E}\,[\![A]\!]\,\rho\,w = \mathcal{E}\,[\![A]\!]\,(\rho, \rho')\,w$.

$$\boxed{\Gamma \vdash e : A\ q}$$

$$\frac{\Gamma \vdash e : A\ \mathsf{now} \qquad \Gamma \vdash e' : B\ \mathsf{now}}{\Gamma \vdash (e, e') : A \times B\ \mathsf{now}}\ \times\mathrm{I} \qquad \frac{\Gamma \vdash e : A \times B\ \mathsf{now}}{\Gamma \vdash \mathsf{fst}\, e : A\ \mathsf{now}}\ \times\mathrm{LE} \qquad \frac{\Gamma \vdash e : A \times B\ \mathsf{now}}{\Gamma \vdash \mathsf{snd}\, e : B\ \mathsf{now}}\ \times\mathrm{RE}$$

$$\frac{\Gamma \vdash e : A\ \mathsf{now}}{\Gamma \vdash \mathsf{inl}\, e : A + B\ \mathsf{now}}\ +\mathrm{LI} \qquad \frac{\Gamma \vdash e : B\ \mathsf{now}}{\Gamma \vdash \mathsf{inr}\, e : A + B\ \mathsf{now}}\ +\mathrm{RI}$$

$$\frac{\Gamma \vdash e : A + B\ \mathsf{now} \qquad \Gamma, x : A\ \mathsf{now} \vdash e' : C\ \mathsf{now} \qquad \Gamma, y : B\ \mathsf{now} \vdash e'' : C\ \mathsf{now}}{\Gamma \vdash \mathsf{case}(e, \mathsf{inl}\, x \to e', \mathsf{inr}\, y \to e'') : C\ \mathsf{now}}\ +\mathrm{E}$$

$$\frac{\Gamma, x : A\ \mathsf{now} \vdash e : B\ \mathsf{now}}{\Gamma \vdash \lambda x.\ e : A \to B\ \mathsf{now}}\ \to\mathrm{I} \qquad \frac{\Gamma \vdash e : A \to B\ \mathsf{now} \qquad \Gamma \vdash e' : A\ \mathsf{now}}{\Gamma \vdash e\ e' : B\ \mathsf{now}}\ \to\mathrm{E}$$

$$\frac{\Gamma \vdash e : A\ \mathsf{later} \qquad \Gamma \vdash e' : \mathsf{alloc}\ \mathsf{now}}{\Gamma \vdash \delta_{e'}(e) : \bullet A\ \mathsf{now}}\ \bullet\mathrm{I} \qquad \frac{\Gamma \vdash e : \bullet A\ \mathsf{now} \qquad \Gamma, x : A\ \mathsf{later} \vdash e' : C\ \mathsf{now}}{\Gamma \vdash \mathsf{let}\ \delta(x) = e\ \mathsf{in}\ e' : C\ \mathsf{now}}\ \bullet\mathrm{E}$$

$$\frac{\Gamma \vdash e : [\bullet(\hat{\mu}\alpha.\ A)/\alpha]A\ \mathsf{now}}{\Gamma \vdash \mathsf{into}\, e : \hat{\mu}\alpha.\ A\ \mathsf{now}}\ \mu\mathrm{I} \qquad \frac{\Gamma \vdash e : \hat{\mu}\alpha.\ A\ \mathsf{now}}{\Gamma \vdash \mathsf{out}\, e : [\bullet(\hat{\mu}\alpha.\ A)/\alpha]A\ \mathsf{now}}\ \mu\mathrm{E}$$

$$\frac{\Gamma \vdash e : A\ \mathsf{stable}}{\Gamma \vdash \mathsf{stable}(e) : \square A\ \mathsf{now}}\ \square\mathrm{I} \qquad \frac{\Gamma \vdash e : \square A\ \mathsf{now} \qquad \Gamma, x : A\ \mathsf{stable} \vdash e' : C\ \mathsf{now}}{\Gamma \vdash \mathsf{let}\ \mathsf{stable}(x) = e\ \mathsf{in}\ e' : C\ \mathsf{now}}\ \square\mathrm{E}$$

$$\frac{\Gamma \vdash e : A\ \mathsf{now} \qquad \Gamma \vdash e' : \bullet(\mathsf{S}\,A)\ \mathsf{now}}{\Gamma \vdash \mathsf{cons}(e, e') : \mathsf{S}\,A\ \mathsf{now}}\ \mathrm{SI}$$

$$\frac{\Gamma \vdash e : \mathsf{S}\,A\ \mathsf{now} \qquad \Gamma, x : A\ \mathsf{now}, xs : \bullet(\mathsf{S}\,A)\ \mathsf{now} \vdash e' : C\ \mathsf{now}}{\Gamma \vdash \mathsf{let}\ \mathsf{cons}(x, xs) = e\ \mathsf{in}\ e' : C\ \mathsf{now}}\ \mathrm{SE}$$

$$\frac{x : A\ q \qquad q \in \{\mathsf{now}, \mathsf{stable}\}}{\Gamma \vdash x : A\ \mathsf{now}}\ \textsc{Hyp}$$

$$\frac{\Gamma^{\square}, x : A\ \mathsf{later} \vdash e : A\ \mathsf{now}}{\Gamma \vdash \mathsf{fix}\, x.\ e : A\ \mathsf{now}}\ \textsc{Fix} \qquad \frac{\Gamma \vdash e : A\ \mathsf{now} \qquad A\ \mathsf{stable}}{\Gamma \vdash \mathsf{promote}(e) : \square A\ \mathsf{now}}\ \textsc{Promote}$$

$$\frac{\Gamma^{\square} \vdash e : A\ \mathsf{now}}{\Gamma \vdash e : A\ \mathsf{stable}}\ \textsc{TStable} \qquad \frac{\Gamma^{\bullet} \vdash e : A\ \mathsf{now}}{\Gamma \vdash e : A\ \mathsf{later}}\ \textsc{TLater}$$

$$\boxed{A\ \mathsf{stable}}$$

$$\frac{A\ \mathsf{stable} \qquad B\ \mathsf{stable}}{A \times B\ \mathsf{stable}} \qquad \frac{A\ \mathsf{stable} \qquad B\ \mathsf{stable}}{A + B\ \mathsf{stable}} \qquad \frac{}{\square A\ \mathsf{stable}}$$

Figure 5: Typing

$$\boxed{e \sqsubseteq \sigma}$$

$$\frac{}{x \sqsubseteq \sigma} \qquad \frac{e_1 \sqsubseteq \sigma \qquad e_2 \sqsubseteq \sigma}{e_1\ e_2 \sqsubseteq \sigma} \qquad \frac{e \sqsubseteq \sigma}{\lambda x.\ e \sqsubseteq \sigma} \qquad \frac{e_1 \sqsubseteq \sigma \qquad e_2 \sqsubseteq \sigma}{(e_1, e_2) \sqsubseteq \sigma} \qquad \frac{e \sqsubseteq \sigma}{\mathsf{fst}\ e \sqsubseteq \sigma} \qquad \frac{e \sqsubseteq \sigma}{\mathsf{snd}\ e \sqsubseteq \sigma} \qquad \frac{e \sqsubseteq \sigma}{\mathsf{inl}\ e \sqsubseteq \sigma}$$

$$\frac{e \sqsubseteq \sigma}{\mathsf{inr}\ e \sqsubseteq \sigma} \qquad \frac{e \sqsubseteq \sigma \qquad e_1 \sqsubseteq \sigma \qquad e_2 \sqsubseteq \sigma}{\mathsf{case}(e, \mathsf{inl}\ x \to e_1, \mathsf{inr}\ y \to e_2) \sqsubseteq \sigma} \qquad \frac{e \sqsubseteq \sigma}{\mathsf{into}\ e \sqsubseteq \sigma} \qquad \frac{e \sqsubseteq \sigma}{\mathsf{out}\ e \sqsubseteq \sigma} \qquad \frac{e \sqsubseteq \sigma \qquad e' \sqsubseteq \sigma}{\mathsf{cons}(e, e') \sqsubseteq \sigma}$$

$$\frac{e \sqsubseteq \sigma \qquad e' \sqsubseteq \sigma}{\delta_{e'}(e) \sqsubseteq \sigma} \qquad \frac{e \sqsubseteq \sigma}{\mathsf{stable}(e) \sqsubseteq \sigma} \qquad \frac{e \sqsubseteq \sigma \qquad e' \sqsubseteq \sigma}{\mathsf{let}\ \delta(x) = e\ \mathsf{in}\ e' \sqsubseteq \sigma} \qquad \frac{e \sqsubseteq \sigma \qquad e' \sqsubseteq \sigma}{\mathsf{let}\ \mathsf{stable}(x) = e\ \mathsf{in}\ e' \sqsubseteq \sigma}$$

$$\frac{e \sqsubseteq \sigma \qquad e' \sqsubseteq \sigma}{\mathsf{let}\ \mathsf{cons}(x, xs) = e\ \mathsf{in}\ e' \sqsubseteq \sigma} \qquad \frac{e \sqsubseteq \sigma}{\mathsf{promote}(e) \sqsubseteq \sigma} \qquad \frac{e \sqsubseteq \sigma}{\mathsf{fix}\ x.\ e \sqsubseteq \sigma} \qquad \frac{l \in \mathrm{dom}(\sigma)}{l \sqsubseteq \sigma} \qquad \frac{l \in \mathrm{dom}(\sigma)}{!l \sqsubseteq \sigma}$$

$$\boxed{\sigma\ \mathsf{supported}}$$

$$\frac{}{\cdot\ \mathsf{supported}} \qquad \frac{\sigma\ \mathsf{supported} \qquad v \sqsubseteq \sigma}{\sigma, l : v\ \mathsf{now}\ \mathsf{supported}} \qquad \frac{\sigma\ \mathsf{supported} \qquad e \sqsubseteq \sigma}{\sigma, l : e\ \mathsf{later}\ \mathsf{supported}} \qquad \frac{\sigma\ \mathsf{supported}}{\sigma, l : \mathsf{null}\ \mathsf{supported}}$$

Figure 6: Definition of Store-supportedness

$$
\begin{array}{lcl}
\mathrm{Heap}_0 & = & \{\sigma \in \mathrm{Store} \mid \sigma\ \mathsf{supported}\} \\
\mathrm{Heap}_{n+1} & = & \left\{\sigma \in \mathrm{Store} \;\middle|\; \sigma\ \mathsf{supported} \wedge \exists \sigma'.\ \sigma \Longrightarrow \sigma' \wedge \sigma' \in \mathrm{Heap}_n\right\} \\
\sigma' \leq \sigma & \iff & \exists \sigma_0.\ \sigma \cdot \sigma_0 = \sigma' \\
\mathrm{Cap} & = & \{\top, \bot\} \\
a' \leq a & \iff & a = a' \vee (a' = \bot \wedge a = \top) \\
\mathrm{World} & = & \left\{(n, \sigma, a) \;\middle|\; n \in \mathbb{N} \wedge \sigma \in \mathrm{Heap}_n \wedge a \in \mathrm{Cap}\right\} \\
(n', \sigma', a') \leq (n, \sigma, a) & \iff & n' \leq n \wedge \sigma' \leq \sigma \wedge a' \leq a \\
\mathrm{Type} & = & \left\{ X \in \mathrm{World} \to \mathcal{P}(\mathrm{Value}) \;\middle|\; \begin{array}{l} \forall w, w'.\ w' \leq w \implies X\ w' \supseteq X\ w \wedge \\ \forall \pi \in \mathrm{Perm}, w.\ \pi(X\ w) = X\ (\pi(w)) \wedge \\ \forall w, v \in X\ w.\ v \sqsubseteq w.\sigma \end{array} \right\}
\end{array}
$$

Figure 7: Definition of Worlds

$$
\begin{array}{lcl}
\mathcal{V}\,[\![\alpha]\!]\,\rho\,w & = & \rho\;\alpha\;w \\
\mathcal{V}\,[\![\hat{\mu}\alpha.\,A]\!]\,\rho\,w & = & \{\mathsf{into}\,v \mid v \in \mathcal{V}\,[\![A]\!]\,(\rho, \mathcal{V}\,[\![\bullet(\hat{\mu}\alpha.\,A)]\!]\,\rho\,w/\alpha)\;w\} \\
\mathcal{V}\,[\![A+B]\!]\,\rho\,w & = & \{\mathsf{inl}\,v \mid v \in \mathcal{V}\,[\![A]\!]\,\rho\,w\} \cup \{\mathsf{inr}\,v \mid v \in \mathcal{V}\,[\![B]\!]\,\rho\,w\} \\
\mathcal{V}\,[\![A\times B]\!]\,\rho\,w & = & \{(v_1, v_2) \mid v_1 \in \mathcal{V}\,[\![A]\!]\,\rho\,w \wedge v_2 \in \mathcal{V}\,[\![B]\!]\,\rho\,w\} \\
\mathcal{V}\,[\![A\to B]\!]\,\rho\,w & = & \left\{ \lambda x.\,e \;\middle|\; \begin{array}{l} \lambda x.\,e \sqsubseteq w.\sigma \wedge \\ \forall \pi \in \mathrm{Perm}, w' \leq w, e' \in \mathcal{E}\,[\![A]\!]\,\rho\;\pi(w').\;[e'/x]\pi(e) \in \mathcal{E}\,[\![B]\!]\,\rho\;\pi(w') \end{array} \right\} \\
\mathcal{V}\,[\![\bullet A]\!]\,\rho\,w & = & \{l \mid w.\sigma = (\sigma_0, l : e\;\mathsf{later}, \sigma_1) \wedge \forall \pi \in \mathrm{Perm}, w' \leq (w.n, \sigma_0, w.a).\;\pi(e) \in \mathcal{L}\,[\![A]\!]\,\rho\;\pi(w')\} \\
\mathcal{V}\,[\![\mathsf{S}\,A]\!]\,\rho\,w & = & \{\mathsf{cons}(v, v') \mid v \in \mathcal{V}\,[\![A]\!]\,\rho\,w \wedge v' \in \mathcal{V}\,[\![\bullet \mathsf{S}\,A]\!]\,\rho\,w\} \\
\mathcal{V}\,[\![\Box A]\!]\,\rho\,w & = & \{\mathsf{stable}(v) \mid v \in \mathcal{V}\,[\![A]\!]\,\rho\;(w.n, \cdot, \top)\} \\
\mathcal{V}\,[\![\mathsf{alloc}]\!]\,\rho\,w & = & \{\diamond \mid w.a = \bot\} \\
 & & \\
\mathcal{E}\,[\![A]\!]\,\rho\;(n, \sigma, a) & = & \left\{ e \;\middle|\; \begin{array}{l} e \sqsubseteq \sigma \wedge \\ \forall \sigma' \leq \sigma.\; \exists \sigma'' \leq \sigma', v \in \mathcal{V}\,[\![A]\!]\,\rho\;(n, \sigma'', a). \\ \quad \langle \sigma'; e\rangle \Downarrow \langle \sigma''; v\rangle \wedge (a = \top \implies \sigma'' = \sigma') \end{array} \right\} \\
\mathcal{L}\,[\![A]\!]\,\rho\;(0, \sigma, a) & = & \{e \in \mathrm{Expr} \mid e \sqsubseteq \sigma\} \\
\mathcal{L}\,[\![A]\!]\,\rho\;(n+1, \sigma, a) & = & \{e \in \mathrm{Expr} \mid e \sqsubseteq \sigma \wedge \sigma \Longrightarrow \sigma' \wedge \forall w' \leq (n, \sigma', a).\; e \in \mathcal{E}\,[\![A]\!]\,\rho\;w'\} \\
 & & \\
\mathrm{Env}(\cdot)\;w & = & \{()\} \\
\mathrm{Env}(\Gamma, x : A\;\mathsf{now})\;w & = & \{(\gamma, e/x) \mid \gamma \in \mathrm{Env}(\Gamma)\;w \wedge \forall w' \leq w.\; e \in \mathcal{E}\,[\![A]\!]\cdot(w')\} \\
\mathrm{Env}(\Gamma, x : A\;\mathsf{stable})\;w & = & \{(\gamma, e/x) \mid \gamma \in \mathrm{Env}(\Gamma)\;w \wedge \forall w' \leq w.\; e \in \mathcal{E}\,[\![A]\!]\cdot(w'.n, \cdot, \top)\} \\
\mathrm{Env}(\Gamma, x : A\;\mathsf{later})\;w & = & \{(\gamma, e/x) \mid \gamma \in \mathrm{Env}(\Gamma)\;w \wedge \forall w' \leq w.\; e \in \mathcal{L}\,[\![A]\!]\cdot(w')\}
\end{array}
$$

Figure 8: The Logical Relation

$$
\begin{array}{lcl}
()^{\bullet}_{\cdot} & = & () \\
(\gamma, e/x)^{\bullet}_{\Gamma, x:A\;\mathsf{stable}} & = & \gamma^{\bullet}_{\Gamma}, e/x \\
(\gamma, e/x)^{\bullet}_{\Gamma, x:A\;\mathsf{now}} & = & \gamma^{\bullet}_{\Gamma} \\
(\gamma, e/x)^{\bullet}_{\Gamma, x:A\;\mathsf{later}} & = & \gamma^{\bullet}_{\Gamma}, e/x \\
 & & \\
()^{\Box}_{\cdot} & = & () \\
(\gamma, e/x)^{\Box}_{\Gamma, x:A\;\mathsf{stable}} & = & \gamma^{\Box}_{\Gamma}, e/x \\
(\gamma, e/x)^{\Box}_{\Gamma, x:A\;\mathsf{now}} & = & \gamma^{\Box}_{\Gamma} \\
(\gamma, e/x)^{\Box}_{\Gamma, x:A\;\mathsf{later}} & = & \gamma^{\Box}_{\Gamma}
\end{array}
$$

Figure 9: Operations on Environments

3. *If* $\mathrm{FV}(A) \subseteq \mathrm{dom}(\rho)$ *then* $\mathcal{L}[\![A]\!]\,\rho\,w = \mathcal{L}[\![A]\!]\,(\rho,\rho')\,w$.

Lemma 12 (Type Substitution). *We have that:*

1. *For all type environments* ρ *and* w, $\mathcal{V}[\![B]\!]\,(\rho, \mathcal{V}[\![A]\!]\,\rho\,/\alpha)\,w = \mathcal{V}[\![[A/\alpha]B]\!]\,\rho\,w$.
2. *For all type environments* ρ *and* w, $\mathcal{E}[\![B]\!]\,(\rho, \mathcal{V}[\![A]\!]\,\rho\,/\alpha)\,w = \mathcal{E}[\![[A/\alpha]B]\!]\,\rho\,w$.
3. *For all type environments* ρ *and* w, $\mathcal{L}[\![B]\!]\,(\rho, \mathcal{V}[\![A]\!]\,\rho\,/\alpha)\,w = \mathcal{L}[\![[A/\alpha]B]\!]\,\rho\,w$.

Lemma 13 (Value Inclusion). *If* $v \in \mathcal{V}[\![A]\!]\,\rho\,w$ *then* $v \in \mathcal{E}[\![A]\!]\,\rho\,w$.

Lemma 14 (Kripke Monotoncity for Environments). *If* $w' \leq w$, *then* $\mathrm{Env}(\Gamma)\,w' \supseteq \mathrm{Env}(\Gamma)\,w$.

Lemma 15 (Renaming for Environments). *If* $\pi \in \mathrm{Perm}$ *and* $\gamma \in \mathrm{Env}(A)\,w$ *then* $\pi(\gamma) \in \mathrm{Env}(A)\,\pi(w)$.

Lemma 16 (Environment Shift). *Suppose* $\gamma \in \mathrm{Env}(\Gamma)\,w$. *Then:*

1. $\gamma^{\square}_{\Gamma} \in \mathrm{Env}(\Gamma^{\square})\,(w.n, \cdot, \top)$.
2. *If* $w = (n+1, \sigma, a)$ *and* $\sigma \Longrightarrow \sigma'$, *then* $\gamma^{\bullet}_{\Gamma} \in \mathrm{Env}(\Gamma^{\bullet})\,(n, \sigma', a)$.

Lemma 17 (Stability). *If* A stable *and* $v \in \mathcal{V}[\![A]\!]\,\rho\,w$, *then* $v \in \mathcal{V}[\![A]\!]\,\rho\,(w.n, \cdot, \top)$.

Theorem 1 (Fundamental Property). *The following properties hold:*

1. *If* $\Gamma \vdash e : A$ later *and* $\gamma \in \mathrm{Env}(\Gamma)\,w$, *then* $\gamma(e) \in \mathcal{L}[\![A]\!] \cdot w$.
2. *If* $\Gamma \vdash e : A$ stable *and* $\gamma \in \mathrm{Env}(\Gamma)\,w$, *then* $\gamma(e) \in \mathcal{E}[\![A]\!] \cdot (w.n, \cdot, \top)$.
3. *If* $\Gamma \vdash e : A$ now *and* $\gamma \in \mathrm{Env}(\Gamma)\,w$, *then* $\gamma(e) \in \mathcal{E}[\![A]\!] \cdot w$.

3 Proofs

3.1 Operational Properties

Lemma 1 (Extension). *If* $\langle \sigma; e\rangle \Downarrow \langle \sigma'; v\rangle$, *then there exists* σ'' *such that* $\sigma' = \sigma \cdot \sigma''$.

Proof. This follows by induction on the evaluation derivation.

- Case $\langle \sigma; v\rangle \Downarrow \langle \sigma; v\rangle$:
  Take the existential witness to be $\cdot$.

- Case $\langle \sigma; e_1\ e_2\rangle \Downarrow \langle \sigma'''; v\rangle$:
  By inversion, $\langle \sigma; e_1\rangle \Downarrow \langle \sigma'; \lambda x.\ e_1'\rangle$ and $\langle \sigma'; e_2\rangle \Downarrow \langle \sigma''; v_2\rangle$ and $\langle \sigma''; [v_2/x]e'\rangle \Downarrow \langle \sigma'''; v\rangle$.
  By induction, $\sigma' = \sigma \cdot \sigma_0$.
  By induction, $\sigma'' = \sigma' \cdot \sigma_1$.
  By induction, $\sigma''' = \sigma'' \cdot \sigma_2$.
  Hence $\sigma''' = \sigma \cdot \sigma_0 \cdot \sigma_1 \cdot \sigma_2$.
  Take the existential witness to be $\sigma_0 \cdot \sigma_1 \cdot \sigma_2$.

- Case $\langle \sigma_0; \mathsf{cons}(e_1, e_2)\rangle \Downarrow \langle \sigma_2; \mathsf{cons}(v_1, v_2)\rangle$:
  By inversion, $\langle \sigma_0; e_1\rangle \Downarrow \langle \sigma_1; v_1\rangle$ and $\langle \sigma_1; e_2\rangle \Downarrow \langle \sigma_2; v_2\rangle$.
  By induction, $\sigma_1 = \sigma_0 \cdot \sigma_0'$.
  By induction, $\sigma_2 = \sigma_1 \cdot \sigma_1'$.
  Hence $\sigma_2 = \sigma_0 \cdot \sigma_0' \cdot \sigma_1'$.
  Take the existential witness to be $\sigma_0' \cdot \sigma_1'$.

- Case $\langle \sigma_0; \mathsf{let\ cons}(x, xs) = e \mathsf{\ in\ } e' \rangle \Downarrow \langle \sigma_2; v' \rangle$:
  By inversion, $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1; \mathsf{cons}(v, l) \rangle$ and $\langle \sigma_1; [v/x, l/xs]e' \rangle \Downarrow \langle \sigma_2; v' \rangle$.
  By induction, $\sigma_1 = \sigma_0 \cdot \sigma_0'$.
  By induction, $\sigma_2 = \sigma_1 \cdot \sigma_1'$.
  Take the existential witness to be $\sigma_0' \cdot \sigma_1'$.

- Case $\langle \sigma; \delta_{e'}(e) \rangle \Downarrow \langle \sigma', l : e \mathsf{\ later}; l \rangle$:
  By inversion, $\langle \sigma; e' \rangle \Downarrow \langle \sigma'; \diamond \rangle$.
  By induction, $\sigma' = \sigma \cdot \sigma''$.
  Take the existential witness to be $\sigma'', l : e \mathsf{\ later}$.

- Case $\langle \sigma_0; \mathsf{let\ } \delta(x) = e \mathsf{\ in\ } e' \rangle \Downarrow \langle \sigma_2; v \rangle$:
  By inversion, $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1; l \rangle$ and $\langle \sigma_1; [!l/x]e' \rangle \Downarrow \langle \sigma_2; v \rangle$.
  By induction, $\sigma_1 = \sigma_0 \cdot \sigma_0'$.
  By induction, $\sigma_2 = \sigma_1 \cdot \sigma_1'$.
  Take the existential witness to be $\sigma_0' \cdot \sigma_1'$.

- Case $\langle \sigma; !l \rangle \Downarrow \langle \sigma; v \rangle$:
  Take the existential witness to be $\cdot$.

- Case $\langle \sigma; \mathsf{stable}(e) \rangle \Downarrow \langle \sigma; \mathsf{stable}(v) \rangle$:
  Take the existential witness to be $\cdot$.

- Case $\langle \sigma_0; \mathsf{let\ stable}(x) = e \mathsf{\ in\ } e' \rangle \Downarrow \langle \sigma_2; v' \rangle$:
  By inversion, $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1; \mathsf{stable}(v) \rangle$ and $\langle \sigma_1; [v/x]e' \rangle \Downarrow \langle \sigma_2; v' \rangle$.
  By induction, $\sigma_1 = \sigma_0 \cdot \sigma_0'$.
  By induction, $\sigma_2 = \sigma_1 \cdot \sigma_1'$.
  Take the existential witness to be $\sigma_0' \cdot \sigma_1'$.

- Case $\langle \sigma; \mathsf{fix}\, x.\ e \rangle \Downarrow \langle \sigma'; v \rangle$:
  By inversion, $\langle \sigma; [\mathsf{fix}\, x.\ e/x]e \rangle \Downarrow \langle \sigma'; v \rangle$.
  By induction, $\sigma' = \sigma \cdot \sigma''$.
  Take the existential witness to be σ''.

- Case $\langle \sigma; \mathsf{promote}(e) \rangle \Downarrow \langle \sigma'; \mathsf{stable}(v) \rangle$:
  By inversion, $\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle$.
  By induction, $\sigma' = \sigma \cdot \sigma''$.
  Take the existential witness to be σ''.

- Case $\langle \sigma; \mathsf{inl}\, e \rangle \Downarrow \langle \sigma'; \mathsf{inl}\, v \rangle$:
  By inversion, $\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle$. By induction, $\sigma' = \sigma \cdot \sigma''$.
  Take the existential witness to be σ''.

- Case $\langle \sigma; \mathsf{inr}\, e \rangle \Downarrow \langle \sigma'; \mathsf{inr}\, v \rangle$:
  By inversion, $\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle$. By induction, $\sigma' = \sigma \cdot \sigma''$.
  Take the existential witness to be σ''.

- Case $\langle \sigma; \mathsf{into}\, e \rangle \Downarrow \langle \sigma'; \mathsf{into}\, v \rangle$:
  By inversion, $\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle$. By induction, $\sigma' = \sigma \cdot \sigma''$.
  Take the existential witness to be σ''.

- Case $\langle \sigma; \mathsf{out}\, e \rangle \Downarrow \langle \sigma'; v \rangle$:
  By inversion, $\langle \sigma; e \rangle \Downarrow \langle \sigma'; \mathsf{into}\, v \rangle$. By induction, $\sigma' = \sigma \cdot \sigma''$.
  Take the existential witness to be σ''.

- Case $\langle \sigma; \mathsf{case}(e, \mathsf{inl}\, x \to e', \mathsf{inr}\, y \to e'') \rangle \Downarrow \langle \sigma''; v' \rangle$:
  By inversion,
  either $\langle \sigma; e \rangle \Downarrow \langle \sigma'; \mathsf{inl}\, v \rangle$ and $\langle \sigma'; [v/x]e' \rangle \Downarrow \langle \sigma''; v' \rangle$ or $\langle \sigma; e \rangle \Downarrow \langle \sigma'; \mathsf{inl}\, v \rangle$ and $\langle \sigma'; [v/y]e'' \rangle \Downarrow \langle \sigma''; v' \rangle$.
  Suppose $\langle \sigma; e \rangle \Downarrow \langle \sigma'; \mathsf{inl}\, v \rangle$ and $\langle \sigma'; [v/x]e' \rangle \Downarrow \langle \sigma''; v' \rangle$.
  Then by induction, $\sigma' = \sigma \cdot \sigma_0$.
  Then by induction, $\sigma'' = \sigma' \cdot \sigma_1$.
  Take the existential witness to be $\sigma_0 \cdot \sigma_1$. Suppose $\langle \sigma; e \rangle \Downarrow \langle \sigma'; \mathsf{inr}\, v \rangle$ and $\langle \sigma'; [v/y]e'' \rangle \Downarrow \langle \sigma''; v' \rangle$.
  Then by induction, $\sigma' = \sigma \cdot \sigma_0$.
  Then by induction, $\sigma'' = \sigma' \cdot \sigma_1$.
  Take the existential witness to be $\sigma_0 \cdot \sigma_1$, so that $\sigma'' = \sigma \cdot (\sigma_0 \cdot \sigma_1)$.

- Case $\langle \sigma; (e_1, e_2) \rangle \Downarrow \langle \sigma''; (v_1, v_2) \rangle$:
  By inversion, $\langle \sigma; e_1 \rangle \Downarrow \langle \sigma'; v_1 \rangle$ and $\langle \sigma'; e_2 \rangle \Downarrow \langle \sigma''; v_2 \rangle$.
  By induction, $\sigma' = \sigma \cdot \sigma_0$.
  Then by induction, $\sigma'' = \sigma' \cdot \sigma_1$.
  Then by induction, $\sigma'' = \sigma' \cdot \sigma_1$.
  Take the existential witness to be $\sigma_0 \cdot \sigma_1$, so that $\sigma'' = \sigma \cdot (\sigma_0 \cdot \sigma_1)$.

- Case $\langle \sigma; \mathsf{fst}\, e \rangle \Downarrow \langle \sigma'; v \rangle$:
  By inversion, $\langle \sigma; e \rangle \Downarrow \langle \sigma'; (v, v') \rangle$.
  By induction, $\sigma' = \sigma \cdot \sigma''$.
  Take the existential witness to be σ''.

- Case $\langle \sigma; \mathsf{snd}\, e \rangle \Downarrow \langle \sigma'; v' \rangle$:
  By inversion, $\langle \sigma; e \rangle \Downarrow \langle \sigma'; (v, v') \rangle$.
  By induction, $\sigma' = \sigma \cdot \sigma''$.
  Take the existential witness to be σ''.

□

Lemma 2 (Uniformity). *If* $\langle \cdot; e \rangle \Downarrow \langle \cdot; v \rangle$, *then* $\langle \sigma; e \rangle \Downarrow \langle \sigma; v \rangle$.

Proof. This follows by induction on the evaluation derivation.

- Case $\langle \cdot; v \rangle \Downarrow \langle \cdot; v \rangle$:
  By rule, $\langle \sigma; v \rangle \Downarrow \langle \sigma; v \rangle$.

- Case $\langle \cdot; e_1\ e_2 \rangle \Downarrow \langle \cdot; v \rangle$:
  By inversion, $\langle \cdot; e_1 \rangle \Downarrow \langle \sigma'; \lambda x.\ e' \rangle$ and $\langle \sigma'; e_2 \rangle \Downarrow \langle \sigma''; v_2 \rangle$ and $\langle \sigma''; [v_2/x]e' \rangle \Downarrow \langle \cdot; v \rangle$.
  By uniformity, we know that $\cdot$ extends σ'', hence $\sigma'' = \cdot$.
  By uniformity, we know that σ'' extends σ', hence $\sigma' = \cdot$.
  By induction, $\langle \sigma; e_1 \rangle \Downarrow \langle \sigma; \lambda x.\ e' \rangle$.
  By induction, $\langle \sigma; e_2 \rangle \Downarrow \langle \sigma; v_2 \rangle$.
  By induction, $\langle \sigma; [v_2/x]e' \rangle \Downarrow \langle \sigma; v \rangle$.
  By rule $\langle \sigma; e_1\ e_2 \rangle \Downarrow \langle \sigma; v \rangle$.

- Case $\langle \cdot; \mathsf{cons}(e_1, e_2) \rangle \Downarrow \langle \cdot; \mathsf{cons}(v_1, v_2) \rangle$:
  By inversion $\langle \cdot; e_1 \rangle \Downarrow \langle \sigma'; \lambda x.\ e' \rangle$ and $\langle \sigma'; e_2 \rangle \Downarrow \langle \cdot; v_2 \rangle$.
  By uniformity, we know $\cdot$ extends σ', and so $\sigma' = \cdot$.
  By induction, $\langle \sigma; e_1 \rangle \Downarrow \langle \sigma; v_1 \rangle$.
  By induction, $\langle \sigma; e_2 \rangle \Downarrow \langle \sigma; v_2 \rangle$.
  By rule, $\langle \sigma; \mathsf{cons}(e_1, e_2) \rangle \Downarrow \langle \sigma; \mathsf{cons}(v_1, v_2) \rangle$.

- Case $\langle \cdot; \mathsf{let}\ \mathsf{cons}(x, xs) = e\ \mathsf{in}\ e' \rangle \Downarrow \langle \cdot; v' \rangle$:
  By inversion, $\langle \cdot; e \rangle \Downarrow \langle \sigma'; \mathsf{cons}(v, l) \rangle$ and $\langle \sigma'; [v/x, l/xs]e' \rangle \Downarrow \langle \cdot; v' \rangle$.

By uniformity, we know $\cdot$ extends σ', and so $\sigma' = \cdot$.
By induction, $\langle \sigma; e \rangle \Downarrow \langle \sigma; \mathsf{cons}(v, l) \rangle$.
By induction, $\langle \sigma; [v/x, l/xs]e' \rangle \Downarrow \langle \sigma; v' \rangle$.
By rule, $\langle \sigma; \mathsf{let}\ \mathsf{cons}(x, xs) = e\ \mathsf{in}\ e' \rangle \Downarrow \langle \sigma; v' \rangle$.

- Case $\langle \cdot; \delta_{e'}(e) \rangle \Downarrow \langle \cdot; l \rangle$:
  This case is impossible, since the returned store cannot be empty.

- Case $\langle \cdot; \mathsf{let}\ \delta(x) = e\ \mathsf{in}\ e' \rangle \Downarrow \langle \cdot; v \rangle$:
  By inversion, $\langle \cdot; e \rangle \Downarrow \langle \sigma'; l \rangle$ and $\langle \sigma'; [!l/x]e' \rangle \Downarrow \langle \cdot; v' \rangle$.
  By uniformity, we know $\cdot$ extends σ', and so $\sigma' = \cdot$.
  By induction, $\langle \sigma; e \rangle \Downarrow \langle \sigma; \mathsf{cons}(v, l) \rangle$.
  By induction, $\langle \sigma; [!l/x]e' \rangle \Downarrow \langle \sigma; v' \rangle$.
  By rule, $\langle \sigma; \mathsf{let}\ \delta(x) = e\ \mathsf{in}\ e' \rangle \Downarrow \langle \sigma; v \rangle$.

- Case $\langle \cdot; !l \rangle \Downarrow \langle \cdot; v \rangle$:
  This case is impossible, since the input store cannot be empty.

- Case $\langle \cdot; \mathsf{stable}(e) \rangle \Downarrow \langle \cdot; \mathsf{stable}(v) \rangle$:
  By inversion, $\langle \cdot; e \rangle \Downarrow \langle \cdot; v \rangle$.
  By induction, $\langle \sigma; e \rangle \Downarrow \langle \sigma; v \rangle$.
  By rule, $\langle \sigma; \mathsf{stable}(e) \rangle \Downarrow \langle \sigma; \mathsf{stable}(v) \rangle$.

- Case $\langle \cdot; \mathsf{let}\ \mathsf{stable}(x) = e\ \mathsf{in}\ e' \rangle \Downarrow \langle \cdot; v' \rangle$:
  By inversion, $\langle \cdot; e \rangle \Downarrow \langle \sigma'; \mathsf{stable}(v) \rangle$ and $\langle \sigma'; [v/x]e' \rangle \Downarrow \langle \cdot; v' \rangle$.
  By uniformity, we know $\cdot$ extends σ', and so $\sigma' = \cdot$.
  By induction, $\langle \sigma; e \rangle \Downarrow \langle \sigma; \mathsf{stable}(v) \rangle$.
  By induction, $\langle \sigma; [v/x]e' \rangle \Downarrow \langle \sigma; v' \rangle$.
  By rule, $\langle \sigma; \mathsf{let}\ \mathsf{stable}(x) = e\ \mathsf{in}\ e' \rangle \Downarrow \langle \sigma; v \rangle$.

- Case $\langle \cdot; \mathsf{fix}\, x.\ e \rangle \Downarrow \langle \cdot; v \rangle$:
  By inversion, $\langle \cdot; [\mathsf{fix}\, x.\ e/x]e \rangle \Downarrow \langle \cdot; v \rangle$.
  By induction, $\langle \sigma; [\mathsf{fix}\, x.\ e/x]e \rangle \Downarrow \langle \sigma; v \rangle$.
  By rule, $\langle \sigma; \mathsf{fix}\, x.\ e \rangle \Downarrow \langle \sigma; v \rangle$.

- Case $\langle \cdot; \mathsf{promote}(e) \rangle \Downarrow \langle \cdot; \mathsf{stable}(v) \rangle$:
  By inversion, $\langle \cdot; e \rangle \Downarrow \langle \cdot; v \rangle$.
  By induction, $\langle \sigma; e \rangle \Downarrow \langle \sigma; v \rangle$.
  By rule, $\langle \sigma; \mathsf{promote}(e) \rangle \Downarrow \langle \sigma; \mathsf{stable}(v) \rangle$.

- Case $\langle \cdot; \mathsf{inl}\, e \rangle \Downarrow \langle \cdot; \mathsf{inl}\, v \rangle$:
  By inversion, $\langle \cdot; e \rangle \Downarrow \langle \cdot; v \rangle$.
  By induction,$\langle \sigma; e \rangle \Downarrow \langle \sigma; v \rangle$.
  By rule $\langle \sigma; \mathsf{inl}\, e \rangle \Downarrow \langle \sigma; \mathsf{inl}\, v \rangle$.

- Case $\langle \cdot; \mathsf{inr}\, e \rangle \Downarrow \langle \cdot; \mathsf{inr}\, v \rangle$:
  By inversion, $\langle \cdot; e \rangle \Downarrow \langle \cdot; v \rangle$.
  By induction,$\langle \sigma; e \rangle \Downarrow \langle \sigma; v \rangle$.
  By rule $\langle \sigma; \mathsf{inr}\, e \rangle \Downarrow \langle \sigma; \mathsf{inr}\, v \rangle$.

- Case $\langle \cdot; \mathsf{case}(e, \mathsf{inl}\, x \to e', \mathsf{inr}\, y \to e'') \rangle \Downarrow \langle \cdot; v' \rangle$:
  By inversion, either $\langle \cdot; e \rangle \Downarrow \langle \cdot; \mathsf{inl}\, v \rangle$ and $\langle \cdot; [v/x]e' \rangle \Downarrow \langle \cdot; v' \rangle$ or $\langle \cdot; e \rangle \Downarrow \langle \cdot; \mathsf{inl}\, v \rangle$ and $\langle \cdot; [v/y]e'' \rangle \Downarrow \langle \cdot; v' \rangle$.
  Suppose $\langle \cdot; e \rangle \Downarrow \langle \cdot; \mathsf{inl}\, v \rangle$ and $\langle \cdot; [v/x]e' \rangle \Downarrow \langle \cdot; v' \rangle$.
  By induction, $\langle \sigma; e \rangle \Downarrow \langle \sigma; \mathsf{inl}\, v \rangle$.

By induction, $\langle \sigma; [v/x]e' \rangle \Downarrow \langle \sigma; v' \rangle$.
By rule $\langle \sigma; \mathsf{case}(e, \mathsf{inl}\, x \to e', \mathsf{inr}\, y \to e'') \rangle \Downarrow \langle \sigma; v' \rangle$.

Suppose $\langle \cdot; e \rangle \Downarrow \langle \cdot; \mathsf{inr}\, v \rangle$ and $\langle \cdot; [v/y]e'' \rangle \Downarrow \langle \cdot; v' \rangle$.
By induction, $\langle \sigma; e \rangle \Downarrow \langle \sigma; \mathsf{inl}\, v \rangle$.
By induction, $\langle \sigma; [v/y]e'' \rangle \Downarrow \langle \sigma; v' \rangle$.
By rule $\langle \sigma; \mathsf{case}(e, \mathsf{inl}\, x \to e', \mathsf{inr}\, y \to e'') \rangle \Downarrow \langle \sigma; v' \rangle$.

- Case $\langle \cdot; (e_1, e_2) \rangle \Downarrow \langle \cdot; (v_1, v_2) \rangle$:
  By inversion, $\langle \cdot; e_1 \rangle \Downarrow \langle \cdot; v_1 \rangle$ and $\langle \cdot; e_2 \rangle \Downarrow \langle \cdot; v_2 \rangle$.
  By induction $\langle \sigma; e_1 \rangle \Downarrow \langle \sigma; v_1 \rangle$.
  By induction $\langle \sigma; e_2 \rangle \Downarrow \langle \sigma; v_2 \rangle$.
  By rule, $\langle \sigma; (e_1, e_2) \rangle \Downarrow \langle \sigma; (v_1, v_2) \rangle$.

- Case $\langle \cdot; \mathsf{fst}\, e \rangle \Downarrow \langle \cdot; v \rangle$:
  By inversion, $\langle \cdot; e \rangle \Downarrow \langle \cdot; (v, v') \rangle$.
  By induction, $\langle \sigma; e \rangle \Downarrow \langle \sigma; (v, v') \rangle$.
  By rule, $\langle \sigma; \mathsf{fst}\, e \rangle \Downarrow \langle \sigma; v \rangle$.

- Case $\langle \cdot; \mathsf{snd}\, e \rangle \Downarrow \langle \cdot; v' \rangle$:
  By inversion, $\langle \cdot; e \rangle \Downarrow \langle \cdot; (v, v') \rangle$.
  By induction, $\langle \sigma; e \rangle \Downarrow \langle \sigma; (v, v') \rangle$.
  By rule, $\langle \sigma; \mathsf{snd}\, e \rangle \Downarrow \langle \sigma; v' \rangle$.

- Case $\langle \cdot; \mathsf{into}\, e \rangle \Downarrow \langle \cdot; \mathsf{into}\, v \rangle$:
  By inversion, $\langle \cdot; e \rangle \Downarrow \langle \cdot; v \rangle$.
  By induction,$\langle \sigma; e \rangle \Downarrow \langle \sigma; v \rangle$.
  By rule $\langle \sigma; \mathsf{inl}\, e \rangle \Downarrow \langle \sigma; \mathsf{inl}\, v \rangle$.

- Case $\langle \cdot; \mathsf{out}\, e \rangle \Downarrow \langle \cdot; v \rangle$:
  By inversion, $\langle \cdot; e \rangle \Downarrow \langle \cdot; \mathsf{into}\, v \rangle$.
  By induction,$\langle \sigma; e \rangle \Downarrow \langle \sigma; \mathsf{into}\, v \rangle$.
  By rule $\langle \sigma; \mathsf{inl}\, e \rangle \Downarrow \langle \sigma; v \rangle$.

□

Lemma 3 (Permutability)**.** *We have that:*

1. *If* $\pi \in \text{Perm}$ *and* $\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle$ *then* $\langle \pi(\sigma); \pi(e) \rangle \Downarrow \langle \pi(\sigma'); \pi(v) \rangle$.
2. *If* $\pi \in \text{Perm}$ *and* $\sigma \Longrightarrow \sigma'$ *then* $\pi(\sigma) \Longrightarrow \pi(\sigma')$.

Proof. 1. We proceed by induction on the evaluation relation.

- Case $\langle \sigma; v \rangle \Downarrow \langle \sigma; v \rangle$:
  Assume we have $\pi \in \text{Perm}$.
  Then by rule $\langle \pi(\sigma); \pi(v) \rangle \Downarrow \langle \pi(\sigma); \pi(v) \rangle$.
- Case $\langle \sigma; e_1\ e_2 \rangle \Downarrow \langle \sigma'''; v \rangle$:
  By inversion, $\langle \sigma; e_1 \rangle \Downarrow \langle \sigma'; \lambda x.\ e_1' \rangle$ and $\langle \sigma'; e_2 \rangle \Downarrow \langle \sigma''; v_2 \rangle$ and $\langle \sigma''; [v_2/x]e' \rangle \Downarrow \langle \sigma'''; v \rangle$.
  Assume we have $\pi \in \text{Perm}$.
  By induction, $\langle \pi(\sigma); \pi(e_1) \rangle \Downarrow \langle \pi(\sigma'); \pi(\lambda x.\ e_1') \rangle$
  and $\langle \pi(\sigma'); \pi(e_2) \rangle \Downarrow \langle \pi(\sigma''); \pi(v_2) \rangle$
  and $\langle \pi(\sigma''); \pi([v_2/x]e') \rangle \Downarrow \langle \pi(\sigma'''); \pi(v) \rangle$.
  Note that $\pi(e_1\ e_2) = \pi(e_1)\ \pi(e_2)$.
  Note that $\pi(\lambda x.\ e') = \lambda x.\ \pi(e')$.
  Note that $\pi([v_2/x]e') = [\pi(v_2)/x]\pi(e')$.
  Hence by rule, $\langle \pi(\sigma); \pi(e_1\ e_2) \rangle \Downarrow \langle \pi(\sigma'''); \pi(v) \rangle$.

- Case $\langle \sigma_0; \mathsf{cons}(e_1, e_2)\rangle \Downarrow \langle \sigma_2; \mathsf{cons}(v_1, v_2)\rangle$:
By inversion, $\langle \sigma_0; e_1\rangle \Downarrow \langle \sigma_1; v_1\rangle$ and $\langle \sigma_1; e_2\rangle \Downarrow \langle \sigma_2; v_2\rangle$.
Assume we have $\pi \in \text{Perm}$.
By induction, $\langle \pi(\sigma_0); \pi(e_1)\rangle \Downarrow \langle \pi(\sigma_1); \pi(v_1)\rangle$.
By induction, $\langle \pi(\sigma_1); \pi(e_2)\rangle \Downarrow \langle \pi(\sigma_2); \pi(v_2)\rangle$.
Note that $\pi(\mathsf{cons}(e_1, e_2)) = \mathsf{cons}(\pi(e_1), \pi(e_2))$.
Note that $\pi(\mathsf{cons}(v_1, v_2)) = \mathsf{cons}(\pi(v_1), \pi(v_2))$.
Hence by rule $\langle \pi(\sigma_0); \pi(\mathsf{cons}(e_1, e_2))\rangle \Downarrow \langle \pi(\sigma_2); \pi(\mathsf{cons}(v_1, v_2))\rangle$.
- Case $\langle \sigma_0; \mathsf{let}\ \mathsf{cons}(x, xs) = e\ \mathsf{in}\ e'\rangle \Downarrow \langle \sigma_2; v'\rangle$:
By inversion, $\langle \sigma_0; e\rangle \Downarrow \langle \sigma_1; \mathsf{cons}(v, l)\rangle$ and $\langle \sigma_1; [v/x, l/xs]e'\rangle \Downarrow \langle \sigma_2; v'\rangle$.
Assume we have $\pi \in \text{Perm}$.
By induction, $\left\langle \pi(\sigma_)); \pi(e)\right\rangle \Downarrow \langle \pi(\sigma_1); \pi(\mathsf{cons}(v, l))\rangle$.
By induction, $\left\langle \pi(\sigma_)); \pi([v/x, l/xs]e')\right\rangle \Downarrow \langle \pi(\sigma_2); \pi(v')\rangle$.
Note that $\pi(\mathsf{cons}(v, l)) = \mathsf{cons}(\pi(v), \pi(l))$.
Note that $\pi([v/x, l/xs]e') = [\pi(v)/x, \pi(l)/xs]\pi(e')$.
Note that $\pi(\mathsf{let}\ \mathsf{cons}(x, xs) = e\ \mathsf{in}\ e') = \mathsf{let}\ \mathsf{cons}(x, xs) = \pi(e)\ \mathsf{in}\ \pi(e')$.
Hence by rule $\langle \pi(\sigma_0); \pi(\mathsf{let}\ \mathsf{cons}(x, xs) = e\ \mathsf{in}\ e')\rangle \Downarrow \langle \pi(\sigma_2); \pi(v')\rangle$.
- Case $\langle \sigma; \delta_{e'}(e)\rangle \Downarrow \langle \sigma', l : e\ \mathsf{later}; l\rangle$:
By inversion, $\langle \sigma; e'\rangle \Downarrow \langle \sigma'; \diamond\rangle$ and $l \notin \text{dom}(\sigma')$.
Assume we have $\pi \in \text{Perm}$.
By induction, $\langle \pi(\sigma); \pi(e')\rangle \Downarrow \langle \pi(\sigma'; \diamond\rangle$.
By properties of permutations, if $l \notin \text{dom}(\sigma')$, then $\pi(l) \notin \text{dom}(\pi(\sigma'))$.
Note that $\pi(\delta_{e'}(e)) = \delta_{\pi(e')}(\pi(e))$.
Note that $\pi(\sigma', l : e\ \mathsf{later}) = \pi(\sigma'), \pi(l) : \pi(e)\ \mathsf{later}$.
Hence by rule, $\langle \pi(\sigma); \pi(\delta_{e'}(e))\rangle \Downarrow \langle \pi(\sigma', l : e\ \mathsf{later}); \pi(l)\rangle$.
- Case $\langle \sigma_0; \mathsf{let}\ \delta(x) = e\ \mathsf{in}\ e'\rangle \Downarrow \langle \sigma_2; v\rangle$:
By inversion, $\langle \sigma_0; e\rangle \Downarrow \langle \sigma_1; l\rangle$ and $\langle \sigma_1; [!l/x]e'\rangle \Downarrow \langle \sigma_2; v\rangle$.
Assume we have $\pi \in \text{Perm}$.
By induction, $\langle \pi(\sigma_0); \pi(e)\rangle \Downarrow \langle \pi(\sigma_1); \pi(l)\rangle$.
By induction, $\langle \pi(\sigma_1); \pi([!l/x]e')\rangle \Downarrow \langle \pi(\sigma_2); \pi(v)\rangle$.
Note that $\pi([!l/x]e') = [!\pi(l)/x]\pi(e')$.
Note that $\pi(\mathsf{let}\ \delta(x) = e\ \mathsf{in}\ e')$ equals $\mathsf{let}\ \delta(x) = \pi(e)\ \mathsf{in}\ \pi(e')$.
Hence by rule, $\langle \pi(\sigma_0); \pi(\mathsf{let}\ \delta(x) = e\ \mathsf{in}\ e')\rangle \Downarrow \langle \pi(\sigma_2); \pi(v)\rangle$.
- Case $\langle \sigma; !l\rangle \Downarrow \langle \sigma; v\rangle$:
By inversion, we know that $l : v\ \mathsf{now} \in \sigma$.
Assume we have $\pi \in \text{Perm}$.
Note that $\pi(l) : \pi(v)\ \mathsf{now} \in \pi(\sigma)$.
Note that $\pi(!l) = !\pi(l)$.
Hence by rule, $\langle \pi(\sigma); \pi(!l)\rangle \Downarrow \langle \pi(\sigma); \pi(v)\rangle$.
- Case $\langle \sigma; \mathsf{stable}(e)\rangle \Downarrow \langle \sigma; \mathsf{stable}(v)\rangle$:
By inversion, we know that $\langle \sigma; e\rangle \Downarrow \langle \sigma; v\rangle$.
Assume we have $\pi \in \text{Perm}$.
By induction, $\langle \pi(\sigma); \pi(e)\rangle \Downarrow \langle \pi(\sigma); \pi(v)\rangle$.
Note that $\pi(\mathsf{stable}(e)) = \mathsf{stable}(\pi(e))$ and $\pi(\mathsf{stable}(v)) = \mathsf{stable}(\pi(v))$.
Hence by rule, $\langle \pi(\sigma); \pi(\mathsf{stable}(e))\rangle \Downarrow \langle \pi(\sigma); \pi(\mathsf{stable}(v))\rangle$.
- Case $\langle \sigma_0; \mathsf{let}\ \mathsf{stable}(x) = e\ \mathsf{in}\ e'\rangle \Downarrow \langle \sigma_2; v'\rangle$:
By inversion, $\langle \sigma_0; e\rangle \Downarrow \langle \sigma_1; \mathsf{stable}(v)\rangle$ and $\langle \sigma_1; [v/x]e'\rangle \Downarrow \langle \sigma_2; v'\rangle$.
Assume we have $\pi \in \text{Perm}$.
By induction, $\langle \pi(\sigma_0); \pi(e)\rangle \Downarrow \langle \pi(\sigma_1); \pi(\mathsf{stable}(v))\rangle$.
By induction, $\langle \pi(\sigma_1); \pi([v/x]e')\rangle \Downarrow \langle \pi(\sigma_2); \pi(v')\rangle$.

Note that $\pi(\mathsf{stable}(v)) = \mathsf{stable}(\pi(v))$.
Note that $\pi([v/x]e') = [\pi(v)/x]\pi(e')$.
Note that $\pi(\mathsf{let\ stable}(x) = e \mathsf{\ in\ } e')$ equals $\mathsf{let\ stable}(x) = \pi(e) \mathsf{\ in\ } \pi(e')$.
Hence by rule, $\langle \pi(\sigma_0); \pi(\mathsf{let\ stable}(x) = e \mathsf{\ in\ } e')\rangle \Downarrow \langle \pi(\sigma_2); \pi(v')\rangle$.

- Case $\langle \sigma; \mathsf{fix}\, x.\ e\rangle \Downarrow \langle \sigma'; v\rangle$:
  By inversion, $\langle \sigma; [\mathsf{fix}\, x.\ e/x]e\rangle \Downarrow \langle \sigma'; v\rangle$.
  Assume we have $\pi \in$ Perm.
  By induction, $\langle \pi(\sigma); \pi([\mathsf{fix}\, x.\ e/x]e)\rangle \Downarrow \langle \pi(\sigma'); \pi(v)\rangle$.
  Note that $\pi(\mathsf{fix}\, x.\ e) = \mathsf{fix}\, x.\ \pi(e)$.
  Note that $\pi([\mathsf{fix}\, x.\ e/x]e) = [\mathsf{fix}\, x.\ \pi(e)/x]\pi(e)$.
  Hence by rule, $\langle \pi(\sigma); \pi(\mathsf{fix}\, x.\ e)\rangle \Downarrow \langle \pi(\sigma'); \pi(v)\rangle$.

- Case $\langle \sigma; \mathsf{promote}(e)\rangle \Downarrow \langle \sigma'; \mathsf{stable}(v)\rangle$:
  By inversion, $\langle \sigma; e\rangle \Downarrow \langle \sigma'; v\rangle$.
  Assume we have $\pi \in$ Perm.
  By induction, $\langle \pi(\sigma); \pi(e)\rangle \Downarrow \langle \pi(\sigma'); \pi(v)\rangle$.
  Note that $\pi(\mathsf{promote}(e)) = \mathsf{promote}(\pi(e))$ and $\pi(\mathsf{promote}(v)) = \mathsf{promote}(v)$.
  Hence by rule, $\langle \pi(\sigma); \pi(\mathsf{promote}(e))\rangle \Downarrow \langle \pi(\sigma'); \pi(\mathsf{stable}(v))\rangle$.

- Case $\langle \sigma; \mathsf{inl}\, e\rangle \Downarrow \langle \sigma'; \mathsf{inl}\, v\rangle$:
  By inversion, $\langle \sigma; e\rangle \Downarrow \langle \sigma'; v\rangle$.
  Assume we have $\pi \in$ Perm.
  By induction, $\langle \pi(\sigma); \pi(e)\rangle \Downarrow \langle \pi(\sigma'); \pi(v)\rangle$.
  By rule, we have $\langle \pi(\sigma); \mathsf{inr}\, \pi(e)\rangle \Downarrow \langle \pi(\sigma'); \mathsf{inr}\, \pi(v)\rangle$.
  By definition, $\langle \pi(\sigma); \pi(\mathsf{inr}\, e)\rangle \Downarrow \langle \pi(\sigma'); \pi(\mathsf{inr}\, v)\rangle$.

- Case $\langle \sigma; \mathsf{inr}\, e\rangle \Downarrow \langle \sigma'; \mathsf{inr}\, v\rangle$:
  By inversion, $\langle \sigma; e\rangle \Downarrow \langle \sigma'; v\rangle$.
  Assume we have $\pi \in$ Perm.
  By induction, $\langle \pi(\sigma); \pi(e)\rangle \Downarrow \langle \pi(\sigma'); \pi(v)\rangle$.
  By rule, we have $\langle \pi(\sigma); \mathsf{inl}\, \pi(e)\rangle \Downarrow \langle \pi(\sigma'); \mathsf{inl}\, \pi(v)\rangle$.
  By definition, $\langle \pi(\sigma); \pi(\mathsf{inl}\, e)\rangle \Downarrow \langle \pi(\sigma'); \pi(\mathsf{inl}\, v)\rangle$.

- Case $\langle \sigma; \mathsf{case}(e, \mathsf{inl}\, x \to e', \mathsf{inr}\, y \to e'')\rangle \Downarrow \langle \sigma''; v'\rangle$:
  By inversion, either $\langle \sigma; e\rangle \Downarrow \langle \sigma'; \mathsf{inl}\, v\rangle$ and $\langle \sigma'; [v/x]e'\rangle \Downarrow \langle \sigma''; v'\rangle$ or $\langle \sigma; e\rangle \Downarrow \langle \sigma'; \mathsf{inl}\, v\rangle$ and $\langle \sigma'; [v/y]e''\rangle \Downarrow \langle \sigma''; v'\rangle$.

  Suppose $\langle \sigma; e\rangle \Downarrow \langle \sigma'; \mathsf{inl}\, v\rangle$ and $\langle \sigma'; [v/x]e'\rangle \Downarrow \langle \sigma''; v'\rangle$.
  By induction, $\langle \pi(\sigma); \pi(e)\rangle \Downarrow \langle \pi(\sigma'); \pi(\mathsf{inl}\, v)\rangle$.
  By definition, $\langle \pi(\sigma); \pi(e)\rangle \Downarrow \langle \pi(\sigma'); \mathsf{inl}\, \pi(v)\rangle$.
  By induction, $\langle \pi(\sigma'); \pi([v/x]e')\rangle \Downarrow \langle \pi(\sigma''); \pi(v')\rangle$.
  By definition, $\langle \pi(\sigma'); [\pi(v)/x]\pi(e'))\rangle \Downarrow \langle \pi(\sigma''); \pi(v')\rangle$.
  By rule, $\langle \pi(\sigma); \mathsf{case}(\pi(e), \mathsf{inl}\, x \to \pi(e'), \mathsf{inr}\, y \to \pi(e''))\rangle \Downarrow \langle \sigma''; \pi(v')\rangle$.
  By definition, $\langle \pi(\sigma); \pi(\mathsf{case}(e, \mathsf{inl}\, x \to e', \mathsf{inr}\, y \to e''))\rangle \Downarrow \langle \sigma''; \pi(v')\rangle$.

  Suppose $\langle \sigma; e\rangle \Downarrow \langle \sigma'; \mathsf{inr}\, v\rangle$ and $\langle \sigma'; [v/y]e''\rangle \Downarrow \langle \sigma''; v'\rangle$.
  Suppose $\langle \sigma; e\rangle \Downarrow \langle \sigma'; \mathsf{inr}\, v\rangle$ and $\langle \sigma'; [v/x]e'\rangle \Downarrow \langle \sigma''; v'\rangle$.
  By induction, $\langle \pi(\sigma); \pi(e)\rangle \Downarrow \langle \pi(\sigma'); \pi(\mathsf{inr}\, v)\rangle$.
  By definition, $\langle \pi(\sigma); \pi(e)\rangle \Downarrow \langle \pi(\sigma'); \mathsf{inr}\, \pi(v)\rangle$.
  By induction, $\langle \pi(\sigma'); \pi([v/y]e'')\rangle \Downarrow \langle \pi(\sigma''); \pi(v')\rangle$.
  By definition, $\langle \pi(\sigma'); [\pi(v)/y]\pi(e''))\rangle \Downarrow \langle \pi(\sigma''); \pi(v')\rangle$.
  By rule, $\langle \pi(\sigma); \mathsf{case}(\pi(e), \mathsf{inl}\, x \to \pi(e'), \mathsf{inr}\, y \to \pi(e''))\rangle \Downarrow \langle \sigma''; \pi(v')\rangle$.
  By definition, $\langle \pi(\sigma); \pi(\mathsf{case}(e, \mathsf{inl}\, x \to e', \mathsf{inr}\, y \to e''))\rangle \Downarrow \langle \sigma''; \pi(v')\rangle$.

- Case $\langle \sigma; (e_1, e_2)\rangle \Downarrow \langle \sigma'; (v_1, v_2)\rangle$:
  By inversion, $\langle \sigma; e_1\rangle \Downarrow \langle \sigma'; v_1\rangle$ and $\langle \sigma'; e_2\rangle \Downarrow \langle \sigma''; v_2\rangle$.
  By induction, $\langle \pi(\sigma); \pi(e_1)\rangle \Downarrow \langle \pi(\sigma'); \pi(v_1)\rangle$.
  By induction, $\langle \pi(\sigma'); \pi(e_2)\rangle \Downarrow \langle \pi(\sigma''); \pi(v_2)\rangle$.
  By rule, $\langle \pi(\sigma); (\pi(e_1), \pi(e_2))\rangle \Downarrow \langle \pi(\sigma''); (\pi(v_1), \pi(v_2))\rangle$.
  By definition, $\langle \pi(\sigma); \pi(e_1, e_2)\rangle \Downarrow \langle \pi(\sigma''); \pi(v_1, v_2)\rangle$.
- Case $\langle \sigma; \mathsf{fst}\, e\rangle \Downarrow \langle \sigma'; v\rangle$:
  By inversion, $\langle \sigma; e\rangle \Downarrow \langle \sigma'; (v, v')\rangle$.
  By induction, $\langle \pi(\sigma); \pi(e)\rangle \Downarrow \langle \pi(\sigma'); \pi(v, v')\rangle$.
  By definition, $\langle \pi(\sigma); \pi(e)\rangle \Downarrow \langle \pi(\sigma'); (\pi(v), \pi(v'))\rangle$.
  By rule, $\langle \pi(\sigma); \mathsf{fst}\, \pi(e)\rangle \Downarrow \langle \pi(\sigma'); \pi(v)\rangle$.
  By definition, $\langle \pi(\sigma); \pi(\mathsf{fst}\, e)\rangle \Downarrow \langle \pi(\sigma'); \pi(v)\rangle$.
- Case $\langle \sigma; \mathsf{snd}\, e\rangle \Downarrow \langle \sigma'; v'\rangle$:
  By inversion, $\langle \sigma; e\rangle \Downarrow \langle \sigma'; (v, v')\rangle$.
  By induction, $\langle \pi(\sigma); \pi(e)\rangle \Downarrow \langle \pi(\sigma'); \pi(v, v')\rangle$.
  By definition, $\langle \pi(\sigma); \pi(e)\rangle \Downarrow \langle \pi(\sigma'); (\pi(v), \pi(v'))\rangle$.
  By rule, $\langle \pi(\sigma); \mathsf{snd}\, \pi(e)\rangle \Downarrow \langle \pi(\sigma'); \pi(v')\rangle$.
  By definition, $\langle \pi(\sigma); \pi(\mathsf{fst}\, e)\rangle \Downarrow \langle \pi(\sigma'); \pi(v')\rangle$.
- Case $\langle \sigma; \mathsf{into}\, e\rangle \Downarrow \langle \sigma'; \mathsf{into}\, v\rangle$:
  By inversion, $\langle \sigma; e\rangle \Downarrow \langle \sigma'; v\rangle$.
  Assume we have $\pi \in$ Perm.
  By induction, $\langle \pi(\sigma); \pi(e)\rangle \Downarrow \langle \pi(\sigma'); \pi(v)\rangle$.
  By rule, we have $\langle \pi(\sigma); \mathsf{into}\, \pi(e)\rangle \Downarrow \langle \pi(\sigma'); \mathsf{into}\, \pi(v)\rangle$.
  By definition, $\langle \pi(\sigma); \pi(\mathsf{into}\, e)\rangle \Downarrow \langle \pi(\sigma'); \pi(\mathsf{into}\, v)\rangle$.
- Case $\langle \sigma; \mathsf{out}\, e\rangle \Downarrow \langle \sigma'; v\rangle$:
  By inversion, $\langle \sigma; e\rangle \Downarrow \langle \sigma'; \mathsf{into}\, v\rangle$.
  Assume we have $\pi \in$ Perm.
  By induction, $\langle \pi(\sigma); \pi(e)\rangle \Downarrow \langle \pi(\sigma'); \pi(\mathsf{into}\, v)\rangle$.
  By definition, $\langle \pi(\sigma); \pi(e)\rangle \Downarrow \langle \pi(\sigma'); \mathsf{into}\, \pi(v)\rangle$.
  By rule, we have $\langle \pi(\sigma); \mathsf{out}\, \pi(e)\rangle \Downarrow \langle \pi(\sigma'); \pi(v)\rangle$.
  By definition, $\langle \pi(\sigma); \pi(\mathsf{out}\, e)\rangle \Downarrow \langle \pi(\sigma'); \pi(v)\rangle$.

2. Assume we have $\pi \in$ Perm. Now we proceed by induction on the derivation of $\sigma \Longrightarrow \sigma'$.

   - Case $\cdot \Longrightarrow \cdot$:
     Immediate, since $\pi(\cdot) = \cdot$.
   - Case $\sigma, l : v\ \mathsf{now} \Longrightarrow \sigma', l : \mathsf{null}$:
     By inversion, $\sigma \Longrightarrow \sigma'$.
     By induction, $\pi(\sigma) \Longrightarrow \pi(\sigma')$.
     Note that $\pi(\sigma, l : v\ \mathsf{now}) = \pi(\sigma), \pi(l) : \pi(v)\ \mathsf{now}$ and $\pi(\sigma, l : \mathsf{null}) = \pi(\sigma), \pi(l) : \mathsf{null}$.
     Hence by rule, $\pi(\sigma, l : v\ \mathsf{now}) \Longrightarrow \pi(\sigma')$.
   - Case $\sigma, l : \mathsf{null} \Longrightarrow \sigma', l : \mathsf{null}$:
     By inversion, $\sigma \Longrightarrow \sigma'$.
     By induction, $\pi(\sigma) \Longrightarrow \pi(\sigma')$.
     Note that $\pi(\sigma, l : \mathsf{null}) = \pi(\sigma), \pi(l) : \mathsf{null}$.
     Hence by rule, $\pi(\sigma, l : v\ \mathsf{now}) \Longrightarrow \pi(\sigma')$.
   - Case $\sigma, l : e\ \mathsf{later} \Longrightarrow \sigma''$:
     By inversion, $\sigma \Longrightarrow \sigma'$ and $\langle \sigma'; e\rangle \Downarrow \langle \sigma''; v\rangle$ and $l \notin \mathrm{dom}(\sigma'')$.
     By induction, $\pi(\sigma) \Longrightarrow \pi(\sigma')$.
     By expression permutation lemma, $\langle \pi(\sigma'); \pi(e)\rangle \Downarrow \langle \pi(\sigma''); \pi(v)\rangle$.

By properties of permutations, $\pi(l) \notin \mathrm{dom}(\pi(\sigma''))$.
Hence by rule, $\pi(\sigma, l : e \text{ later}) \Longrightarrow \pi(\sigma'')$.

□

Lemma 4 (Supportedness)**.** *We have that:*

1. *If* $e \sqsubseteq \sigma$ *and* $\sigma' \leq \sigma$ *then* $e \sqsubseteq \sigma'$.
2. *If* $e \sqsubseteq \sigma$ *and* $e' \sqsubseteq \sigma$ *then* $[e/x]e' \sqsubseteq \sigma$.
3. *If* σ supported *and* $e \sqsubseteq \sigma$ *and* $\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle$ *then* $v \sqsubseteq \sigma'$ *and* σ' supported.
4. *If* σ supported *and* $\sigma \Longrightarrow \sigma'$ *then* σ' supported.

Proof. We proceed as follows:

1. This follows from an induction on the syntax of e.
2. This follows from an induction on the syntax of e'.
3. This can be proven by induction on the derivation of $\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle$. The two interesting cases are:
   - Case $e = \delta_{e_2}(e_1)$:
     By hypothesis, we know that $\langle \sigma; \delta_{e_2}(e_1) \rangle \Downarrow \langle \sigma', l : e_1 \text{ later}; l \rangle$.
     By inversion, we know that $\langle \sigma; e_2 \rangle \Downarrow \langle \sigma'; \diamond \rangle$ and $l \notin \mathrm{dom}(\sigma')$.
     By inversion on $\delta_{e_2}(e_1) \sqsubseteq \sigma$, we know that $e_1 \sqsubseteq \sigma$ and $e_2 \sqsubseteq \sigma$.
     By induction on $\langle \sigma; e_2 \rangle \Downarrow \langle \sigma'; \diamond \rangle$, we know σ' supported.
     Since $\sigma' \leq \sigma$ and $e_1 \sqsubseteq \sigma$, we know that $e_1 \sqsubseteq \sigma'$.
     Hence we know that $(\sigma', l : e_1 \text{ later})$ supported.
     Note that $l \in \mathrm{dom}(\sigma', l : e \text{ later})$.
   - Case $e = !l$:
     By hypothesis, we know that $\sigma = \sigma_0, l : v \text{ now}, \sigma_1$.
     Since σ supported, we know that $v \sqsubseteq \sigma_0$.
     Since $\sigma \leq \sigma_0$, we know that $v \sqsubseteq \sigma$.
     By assumption σ supported.
4. This follows by induction on the derivation of $\sigma \Longrightarrow \sigma'$.

□

Lemma 5 (Quasi-determinacy)**.** *We have that:*

1. *If* $\langle \sigma; e \rangle \Downarrow \langle \sigma'; v' \rangle$ *and* $\langle \sigma; e \rangle \Downarrow \langle \sigma''; v'' \rangle$ *and* σ supported *and* $e \sqsubseteq \sigma$, *then there is a* $\pi \in \mathrm{Perm}$ *such that* $\pi'(\sigma') = \sigma''$ *and* $\pi(\sigma) = \sigma$.
2. *If* $\sigma \Longrightarrow \sigma'$ *and* $\sigma \Longrightarrow \sigma''$ *and* σ supported, *then there is a* $\pi \in \mathrm{Perm}$ *such that* $\pi(\sigma') = \sigma''$ *and* $\pi(\sigma) = \sigma$.

Proof. 1. We proceed by induction on the evaluation relation of $\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle$.

   - Case $\langle \sigma; v \rangle \Downarrow \langle \sigma; v \rangle$:
     In this case, we have also have $\langle \pi(\sigma); \pi(v) \rangle \Downarrow \langle \pi(\sigma); \pi(v) \rangle$.
     Hence the permutation π' is π.

- Case $\langle \sigma_0; e_1\ e_2 \rangle \Downarrow \langle \sigma_3; v_3' \rangle$:
  By inversion, $\langle \sigma_0; e_1 \rangle \Downarrow \langle \sigma_1; \lambda x.\ \hat{e}_1 \rangle$ and $\langle \sigma_1; e_2 \rangle \Downarrow \langle \sigma_2; v_2 \rangle$ and $\langle \sigma_2; [v_2/x]\hat{e}_1 \rangle \Downarrow \langle \sigma_3; v_3 \rangle$.
  By inversion, $\langle \sigma_0; e_1 \rangle \Downarrow \langle \sigma_1'; \lambda x.\ \hat{e}_1' \rangle$ and $\langle \sigma_1'; e_2 \rangle \Downarrow \langle \sigma_2'; v_2' \rangle$ and $\langle \sigma_2'; [v_2'/x]\hat{e}_1' \rangle \Downarrow \langle \sigma_3; v_3' \rangle$.
  By induction, we have a π_1 such that $\pi_1(\sigma_1) = \sigma_1'$ and $\pi_1(\sigma_0) = \sigma_0$.
  Note that $\pi_1(e_2) = e_2$, since $\pi_1(e_1\ e_2) = e_1\ e_2$.
  By permutability, $\langle \pi_1(\sigma_1); e_2 \rangle \Downarrow \langle \pi_1(\sigma_2); \pi_1(v_2) \rangle$.
  By induction, we get π_2 such that $\pi_2(\pi_1(\sigma_2)) = \sigma_2'$ and $\pi_2(\pi_1(v_2)) = v_2'$ and $\pi_2(\pi_1(\sigma_1)) = \pi_1(\sigma_1)$.
  We need to show that $\pi_2(\pi_1([v_2/x]\hat{e}_1)) = [v_2'/x]\hat{e}_1'$.
  Note that $\pi_2(\pi_1(v_2)) = v_2'$.
  Note that $\pi_1(\hat{e}_1) = \hat{e}_1'$, and that $\hat{e}_1'$ is supported by σ_1'.
  Hence $\pi_2(\pi_1(\hat{e}_1)) = \pi_1(\hat{e}_1)$.
  Hence $\pi_2(\pi_1([v_2/x]\hat{e}_1)) = [v_2'/x]\hat{e}_1'$.
  Hence by permutability, $\langle \pi_2(\pi_1(\sigma_2)); \pi_2(\pi_1([v_2/x]\hat{e}_1)) \rangle \Downarrow \langle \pi_2(\pi_1(\sigma_3)); \pi_2(\pi_1(v_3)) \rangle$.
  Hence by induction, there is a π_3 such that $\sigma_3' = \pi_3(\pi_2(\pi_1(\sigma_2)))$ and $v_3' = \pi_3(\pi_2(\pi_1(v_2)))$.
  We take $\pi_3 \circ \pi_2 \circ \pi_1$ as our permutation witness.
  Since π_3 is safe with respect to σ_2, and π_2 is safe with respect to σ_1,
  both are safe with respect to σ_0.
  Hence $\pi_3 \circ \pi_2 \circ \pi_1(\sigma_0) = \sigma_0$.
- Case $\langle \sigma_0; \mathsf{cons}(e_1, e_2) \rangle \Downarrow \langle \sigma_2; \mathsf{cons}(v_1, v_2) \rangle$:
  By inversion, $\langle \sigma_0; e_1 \rangle \Downarrow \langle \sigma_1; v_1 \rangle$ and $\langle \sigma_1; e_2 \rangle \Downarrow \langle \sigma_2; l_2 \rangle$.
  By inversion, $\langle \sigma_0; e_1 \rangle \Downarrow \langle \sigma_1'; v_1' \rangle$ and $\langle \sigma_1'; e_2 \rangle \Downarrow \langle \sigma_2'; l_2' \rangle$.
  By induction, we have a π_1 such that $\pi_1(\sigma_1) = \sigma_1'$ and $\pi_1(\sigma_0) = \sigma_0$ and $\pi_1(v_1) = v_1'$.
  Note that $\pi_1(e_2) = e_2$, since $\pi_1(\mathsf{cons}(e_1, e_2)) = \mathsf{cons}(e_1, e_2)$.
  By permutability, $\langle \pi_1(\sigma_1); e_2 \rangle \Downarrow \langle \pi_1(\sigma_2); \pi_1(v_2) \rangle$.
  Hence by induction, we have a π_2 such that $\pi_2(\pi_1(\sigma_2)) = \sigma_2'$ and $\pi_2(\pi_1(v_2)) = v_2'$.
  We take $\pi_2 \circ \pi_1$ as our permutation witness.

- Case $\langle \sigma_0; \mathsf{let\ cons}(x, xs) = e\ \mathsf{in}\ e' \rangle \Downarrow \langle \sigma_2; v_2 \rangle$:
  By inversion, $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1; \mathsf{cons}(v, l) \rangle$ and $\langle \sigma_1; [v/x, l/xs]e' \rangle \Downarrow \langle \sigma_2; v_2 \rangle$.
  By inversion, $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1'; \mathsf{cons}(v', l') \rangle$ and $\langle \sigma_1'; [v'/x, l'/xs]e' \rangle \Downarrow \langle \sigma_2'; v_2' \rangle$.
  By induction, we have π_1 such that $\pi_1(\sigma_1) = \sigma_1'$ and $\pi_1(v) = v'$ and $\pi_1(l) = l'$ and $\pi_1(\sigma_0) = \sigma_0$.
  Note that $\pi(e') = e'$, since it is supported by σ_0.
  Hence $\pi_1([v/x, l/x]e') = [v'/x, l'/x]\pi_1(e') = [v'/x, l'/x]e'$.
  Hence by permutability, $\langle \pi_1(\sigma_1); \pi_1([v/x, l/xs]e') \rangle \Downarrow \langle \pi_1(\sigma_2); \pi_1(v_2) \rangle$.
  By induction, we get π_2 such that $\pi_2(\pi_1(\sigma_2)) = \sigma''$ and $\pi_2(\pi_1(v_2)) = v_2'$ and $\pi_2(\sigma_1') = \sigma_1'$.
  We take $\pi_2 \circ \pi_1$ as the permutation witness.

- Case $\langle \sigma_0; \delta_{e'}(e) \rangle \Downarrow \langle \sigma_1, l : e\ \mathsf{later}; l \rangle$:
  By inversion, $\langle \sigma_0; e' \rangle \Downarrow \langle \sigma_1; \diamond \rangle$ and $l \notin \mathrm{dom}(\sigma_1)$.
  By inversion, $\langle \sigma_0; e' \rangle \Downarrow \langle \sigma_1'; \diamond \rangle$ and $l' \notin \mathrm{dom}(\sigma_1')$.
  By induction, we get π_1 such that $\pi_1(\sigma_1) = \sigma_1'$ and $\pi_1(\sigma) = \sigma$.
  Note $l \notin \mathrm{dom}(\sigma_1)$ and $l' \notin \mathrm{dom}(\sigma_1')$.
  We can take our permutation witness to be $\pi' \triangleq \pi \circ (l\ l')$.
  Since e' is supported by σ_0, it follows that $\pi'(\sigma_1, l : e'\ \mathsf{later}) = \sigma_1', l' : e'\ \mathsf{later}$.
  By definition $\pi'(l) = l'$.
  Hence there is a π' such that $\langle \sigma_0; \delta_{e'}(e) \rangle \Downarrow \langle \sigma_1, l : e\ \mathsf{later}; l \rangle$ and $\langle \sigma_0; \delta_{e'}(e) \rangle \Downarrow \langle \pi'(\sigma_1, l : e\ \mathsf{later}); \pi'(l) \rangle$:
  Obviously $\pi'(\sigma_0) = \sigma_0$, since $l \notin \mathrm{dom}(\sigma_0)$.

- Case $\langle \sigma_0; \mathsf{let}\ \delta(x) = e\ \mathsf{in}\ e' \rangle \Downarrow \langle \sigma_2; v \rangle$:
  By inversion, $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1; l \rangle$ and $\langle \sigma_1; [!l/x]e' \rangle \Downarrow \langle \sigma_2; v \rangle$.
  By inversion, $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1'; l' \rangle$ and $\langle \sigma_1'; [!l'/x]e' \rangle \Downarrow \langle \sigma_2'; v' \rangle$.

By induction, there is a π_1 such that $\pi_1(\sigma_1) = \pi_1'$ and $\pi_1(l) = l'$ and $\pi_1(\sigma_0) = \sigma_0$.
Note that $\pi_1(e') = e'$, since e' is supported by σ_0.
Note that $\pi_1([!l/x]e') = [!l'/x]\pi_1(e') = [!l'/x]e'$.
Hence $\langle \pi_1(\sigma_1); [!l'/x]e' \rangle \Downarrow \langle \pi_1(\sigma_2); \pi_1(v) \rangle$.
By induction, we have π_2 such that $\pi_2(\pi_1(\sigma_1)) = \sigma_2'$ and $\pi_2(\pi_1(v)) = v'$ and $\pi_2(\pi_1(\sigma_2)) = \pi_1(\sigma_1)$.
Hence $\langle \sigma_0; \mathsf{let}\ \delta(x) = e\ \mathsf{in}\ e' \rangle \Downarrow \langle \sigma_2; v \rangle$ and $\langle \sigma_0; \mathsf{let}\ \delta(x) = e\ \mathsf{in}\ e' \rangle \Downarrow \langle \pi_2(\pi_1(\sigma_2)); \pi_2(\pi_1(v)) \rangle$. We take $\pi_2 \circ \pi_1$ as our permutation witness.

- Case $\langle \sigma; !l \rangle \Downarrow \langle \sigma; v \rangle$:
  By inversion, we know that $l : v\ \mathsf{now} \in \sigma$.
  By inversion, we know that $l : v\ \mathsf{now} \in \sigma$.
  We take the identity permutation as our permutation witness.
- Case $\langle \sigma_0; \mathsf{stable}(e) \rangle \Downarrow \langle \sigma_1; \mathsf{stable}(v) \rangle$:
  By inversion, we know that $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1; v \rangle$.
  By inversion, we know that $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1'; v' \rangle$.
  By induction, we know that there is a π_1 such that $\pi_1(\sigma_1) = \sigma_1'$ and $\pi_1(v) = v'$.
  Note that $\pi_1(\mathsf{stable}(v)) = \mathsf{stable}(v')$.
  Hence we can take π_1 as our permutation witness.
- Case $\langle \sigma_0; \mathsf{let}\ \mathsf{stable}(x) = e\ \mathsf{in}\ e' \rangle \Downarrow \langle \sigma_2; v_2 \rangle$:
  By inversion, $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1; \mathsf{stable}(v_1) \rangle$ and $\langle \sigma_1; [v_1/x]e' \rangle \Downarrow \langle \sigma_2; v_2 \rangle$.
  By inversion, $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1'; \mathsf{stable}(v_1') \rangle$ and $\langle \sigma_1'; [v_1'/x]e' \rangle \Downarrow \langle \sigma_2'; v_2' \rangle$.
  By induction, we get π_1 such that $\pi_1(\sigma_1) = \sigma_1'$ and $\pi_1(v_1) = v_1'$ and $\pi_1(\sigma_0) = \sigma_0$.
  Note that $\pi_1(e') = e'$ since e' is supported by σ_0.
  Hence $\pi_1([v_1/x]e') = [v_1'/x]e'$. By permutability, $\langle \sigma_1'; [v_1'/x]e' \rangle \Downarrow \langle \pi_1(\sigma_2); \pi_1(v_2) \rangle$.
  By induction, we get π_2 such that $\pi_2(\pi_1(\sigma_2)) = \sigma_2'$ and $\pi_2(\pi_1(v_2)) = v_2'$.
  Hence $\langle \sigma_0; \mathsf{let}\ \mathsf{stable}(x) = e\ \mathsf{in}\ e' \rangle \Downarrow \langle \sigma_2; v' \rangle$ and $\langle \sigma_0; \mathsf{let}\ \mathsf{stable}(x) = e\ \mathsf{in}\ e' \rangle \Downarrow \langle \pi_2(\pi_1(\sigma_2)); \pi_2(\pi_1(v_2)) \rangle$.
  We can take $\pi_2 \circ \pi_1$ as our permutation witness.
- Case $\langle \sigma_0; \mathsf{fix}\, x.\ e \rangle \Downarrow \langle \sigma_1; v \rangle$:
  By inversion, $\langle \sigma_0; [\mathsf{fix}\, x.\ e/x]e \rangle \Downarrow \langle \sigma_1; v \rangle$.
  By inversion, $\langle \sigma_0; [\mathsf{fix}\, x.\ e/x]e \rangle \Downarrow \langle \sigma_1'; v' \rangle$.
  By induction, we get π_1 such that $\pi_1(\sigma_1) = \sigma_1$ and $\pi_1(v) = v'$ and $\pi_1(\sigma_0) = \sigma_0$.
  We can take π_1 to be our permutation witness.
- Case $\langle \sigma_0; \mathsf{promote}(e) \rangle \Downarrow \langle \sigma_1; \mathsf{stable}(v) \rangle$:
  By inversion, we know that $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1; v \rangle$.
  By inversion, we know that $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1'; v' \rangle$.
  By induction, we know that there is a π_1 such that $\pi_1(\sigma_1) = \sigma_1'$ and $\pi_1(v) = v'$.
  Note that $\pi_1(\mathsf{stable}(v)) = \mathsf{stable}(v')$.
  Hence we can take π_1 as our permutation witness.
- Case $\langle \sigma_0; \mathsf{inl}\, e \rangle \Downarrow \langle \sigma_1; \mathsf{inl}\, v \rangle$:
  The other derivation must be of the form $\langle \sigma_0; \mathsf{inl}\, e \rangle \Downarrow \langle \sigma_1'; \mathsf{inl}\, v' \rangle$.
  By inversion, $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1; v \rangle$.
  By inversion, $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1'; v' \rangle$.
  By induction, we have a π_1 such that $\pi_1(\sigma_1) = \sigma_1'$ and $\pi(\sigma_0) = \sigma_0$ and $\pi_1(v) = v'$.
  Note that $\pi_1(\mathsf{inl}\, v) = \mathsf{inl}\, \pi_1(v) = \mathsf{inl}\, v'$.
  Hence we can take π_1 as our permutation witness.
- Case $\langle \sigma_0; \mathsf{inr}\, e \rangle \Downarrow \langle \sigma_1; \mathsf{inr}\, v \rangle$:
  The other derivation must be of the form $\langle \sigma_0; \mathsf{inr}\, e \rangle \Downarrow \langle \sigma_1'; \mathsf{inr}\, v' \rangle$.
  By inversion, $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1; v \rangle$.
  By inversion, $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1'; v' \rangle$.
  By induction, we have a π_1 such that $\pi_1(\sigma_1) = \sigma_1'$ and $\pi(\sigma_0) = \sigma_0$ and $\pi_1(v) = v'$.

Note that $\pi_1(\mathsf{inr}\, v) = \mathsf{inr}\, \pi_1(v) = \mathsf{inr}\, v'$.
Hence we can take π_1 as our permutation witness.

- Case $\langle \sigma_0; \mathsf{case}(e, \mathsf{inl}\, x \to e_1, \mathsf{inr}\, y \to e_2) \rangle \Downarrow \langle \sigma_2; v \rangle$:
  By inversion, either $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1; \mathsf{inl}\, v_1 \rangle$ and $\langle \sigma_1; [v_1/x]e' \rangle \Downarrow \langle \sigma_2; v \rangle$ or $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1; \mathsf{inr}\, v_2 \rangle$ and $\langle \sigma_1; [v_2/y]e_2 \rangle \Downarrow \langle \sigma_2; v \rangle$.
  By inversion, either $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1'; \mathsf{inl}\, v_1' \rangle$ and $\langle \sigma_1'; [v_1'/x]e_1 \rangle \Downarrow \langle \sigma_2'; v' \rangle$ or $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1; \mathsf{inr}\, v_2 \rangle$ and $\langle \sigma_1; [v_2/y]e_2 \rangle \Downarrow \langle \sigma_2; v' \rangle$.

  Suppose $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1; \mathsf{inl}\, v_1 \rangle$ and $\langle \sigma_1; [v_1/x]e_1 \rangle \Downarrow \langle \sigma_2; v \rangle$.
  Suppose $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1'; \mathsf{inr}\, v_2' \rangle$ and $\langle \sigma_1'; [v_2'/y]e_2 \rangle \Downarrow \langle \sigma_2'; v' \rangle$.
  Then by induction there is a π such that $\pi(\mathsf{inl}\, v_1) = \mathsf{inr}\, v_2'$, which is impossible.

  Suppose $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1; \mathsf{inl}\, v_2 \rangle$ and $\langle \sigma_1; [v_2/x]e_2 \rangle \Downarrow \langle \sigma_2; v \rangle$.
  Suppose $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1'; \mathsf{inr}\, v_1' \rangle$ and $\langle \sigma_1'; [v_1'/y]e_1 \rangle \Downarrow \langle \sigma_2'; v' \rangle$.
  Then by induction there is a π such that $\pi(\mathsf{inr}\, v_2) = \mathsf{inl}\, v_1'$, which is impossible.

  Suppose $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1; \mathsf{inl}\, v_1 \rangle$ and $\langle \sigma_1; [v_1/x]e_1 \rangle \Downarrow \langle \sigma_2; v \rangle$.
  Suppose $\langle \sigma_0'; e \rangle \Downarrow \langle \sigma_1'; \mathsf{inl}\, v_1' \rangle$ and $\langle \sigma_1'; [v_1'/x]e_1 \rangle \Downarrow \langle \sigma_2'; v' \rangle$.
  By induction, we have π_1 such that $\pi_1(\sigma_1) = \sigma_1'$ and $\pi_1(v_1) = v_1'$ and $\pi_1(\sigma_0) = \sigma_0$.
  Since $\sigma_0 \sqsubseteq e_1$, we know $\pi_1(e_1) = e_1$.
  Hence $\pi_1([v_1/x]e_1) = [\pi_1(v_1)/x]\pi_1(e_1) = [v_1'/x]e_1$.
  Hence by renaming, $\langle \sigma_1'; [v_1/x]e_1 \rangle \Downarrow \langle \pi_1(\sigma_2); \pi_1(v) \rangle$.
  By induction, we have π_2 such that $\pi_2(\pi_1(\sigma_2)) = \sigma_2'$ and $\pi_2(\pi_1(v)) = v'$ and $\pi_2(\sigma_1') = \sigma_1'$.
  So we can take $\pi_2 \circ \pi_1$ as our permutation witness.

  Suppose $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1; \mathsf{inr}\, v_2 \rangle$ and $\langle \sigma_1; [v_2/x]e_2 \rangle \Downarrow \langle \sigma_2; v \rangle$.
  Suppose $\langle \sigma_0'; e \rangle \Downarrow \langle \sigma_1'; \mathsf{inr}\, v_2' \rangle$ and $\langle \sigma_1'; [v_2'/x]e_2 \rangle \Downarrow \langle \sigma_2'; v' \rangle$.
  By induction, we have π_1 such that $\pi_1(\sigma_1) = \sigma_1'$ and $\pi_1(v_2) = v_2'$ and $\pi_1(\sigma_0) = \sigma_0$.
  Since $\sigma_0 \sqsubseteq e_2$, we know $\pi_1(e_2) = e_2$.
  Hence $\pi_1([v_2/x]e_2) = [\pi_1(v_2)/x]\pi_1(e_2) = [v_2'/x]e_2$.
  Hence by renaming, $\langle \sigma_1'; [v_2/x]e_2 \rangle \Downarrow \langle \pi_1(\sigma_2); \pi_1(v) \rangle$.
  By induction, we have π_2 such that $\pi_2(\pi_1(\sigma_2)) = \sigma_2'$ and $\pi_2(\pi_1(v)) = v'$ and $\pi_2(\sigma_1') = \sigma_1'$.
  So we can take $\pi_2 \circ \pi_1$ as our permutation witness.

- Case $\langle \sigma_0; (e_1, e_2) \rangle \Downarrow \langle \sigma_2; (v_1, v_2) \rangle$:
  By inversion, $\langle \sigma_0; e_1 \rangle \Downarrow \langle \sigma_1; v_1 \rangle$ and $\langle \sigma_1; e_2 \rangle \Downarrow \langle \sigma_2; v_2 \rangle$.
  By inversion, $\langle \sigma_0; e_1 \rangle \Downarrow \langle \sigma_1'; v_1' \rangle$ and $\langle \sigma_1'; e_2 \rangle \Downarrow \langle \sigma_2'; v_2' \rangle$.
  By induction, there is a π_1 such that $\pi_1(\sigma_1) = \sigma_1'$ and $\pi_1(v_1) = v_1'$ and $\pi_1(\sigma_0) = \sigma_0$.
  Since $\sigma_0 \sqsubseteq e_2$, we know that $\pi_1(e_2) = e_2$.
  Hence by renaming, we know $\langle \sigma_1'; e_2 \rangle \Downarrow \langle \pi_1(\sigma_2); \pi_1(v_2) \rangle$.
  By induction, we have π_2 such that $\pi_2(\pi_1(\sigma_2)) = \sigma_2$ and $\pi_2(\pi_1(v_2)) = v_2'$ and $\pi_2(\sigma_1') = \sigma_1'$.
  Since $\sigma_1' \sqsubseteq v_1'$, we know that $\pi_2(v_1') = v_1'$.
  Hence we know that $\pi_2(\pi_1(v_1, v_2)) = (\pi_2(\pi_1(v_1)), \pi_2(\pi_1(v_2))) = (v_1', v_2')$.
  Hence we can take $\pi_2 \circ \pi_1$ as our permutation witness.

- Case $\langle \sigma_0; \mathsf{fst}\, e \rangle \Downarrow \langle \sigma_1; v_1 \rangle$:
  By inversion, $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1; (v_1, v_2) \rangle$ and $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1'; (v_1', v_2') \rangle$.
  By induction, there is a π_1 such that $\pi_1(\sigma_1) = \sigma_1'$ and $\pi_1((v_1, v_2)) = (v_1', v_2')$.
  Hence $\pi_1(v_1) = v_1'$ and $\pi_1(v_2) = v_2'$.
  Hence we can take π_1 as our permutation witness.

- Case $\langle \sigma; \mathsf{snd}\, e \rangle \Downarrow \langle \sigma'; v' \rangle$:
  By inversion, $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1; (v_1, v_2) \rangle$ and $\langle \sigma_0; e \rangle \Downarrow \langle \sigma_1'; (v_1', v_2') \rangle$.

By induction, there is a π_1 such that $\pi_1(\sigma_1) = \sigma_1'$ and $\pi_1((v_1, v_2)) = (v_1', v_2')$.
Hence $\pi_1(v_1) = v_1'$ and $\pi_1(v_2) = v_2'$.
Hence we can take π_1 as our permutation witness.

- Case $\langle\sigma_0; \mathsf{into}\, e\rangle \Downarrow \langle\sigma_1; \mathsf{into}\, v\rangle$:
The other derivation must be of the form $\langle\sigma_0; \mathsf{into}\, e\rangle \Downarrow \langle\sigma_1'; \mathsf{into}\, v'\rangle$.
By inversion, $\langle\sigma_0; e\rangle \Downarrow \langle\sigma_1; v\rangle$.
By inversion, $\langle\sigma_0; e\rangle \Downarrow \langle\sigma_1'; v'\rangle$.
By induction, we have a π_1 such that $\pi_1(\sigma_1) = \sigma_1'$ and $\pi(\sigma_0) = \sigma_0$ and $\pi_1(v) = v'$.
Note that $\pi_1(\mathsf{into}\, v) = \mathsf{into}\, \pi_1(v) = \mathsf{into}\, v'$.
Hence we can take π_1 as our permutation witness.
- Case $\langle\sigma_0; \mathsf{out}\, e\rangle \Downarrow \langle\sigma_1; v\rangle$:
By inversion, $\langle\sigma_0; \mathsf{out}\, e\rangle \Downarrow \langle\sigma_1; \mathsf{into}\, v\rangle$.
By inversion, $\langle\sigma_0; \mathsf{out}\, e\rangle \Downarrow \langle\sigma_1'; \mathsf{into}\, v'\rangle$.
By induction, we have a π_1 such that $\pi_1(\sigma_1) = \sigma_1'$ and $\pi(\sigma_0) = \sigma_0$ and $\pi_1(\mathsf{into}\, v) = \mathsf{into}\, v'$.
Hence note that $\pi_1(v) = v'$.
Hence we can take π_1 as our permutation witness.

2. We have derivations of $\sigma \Longrightarrow \sigma'$ and $\sigma \Longrightarrow \sigma''$:

- Case $\cdot \Longrightarrow \cdot$:
We know $\sigma' = \sigma'' = \cdot$.
We can take the identity permutation as our permutation witness.
- Case $\sigma_0, l : v\ \mathsf{now} \Longrightarrow \sigma_1$:
By inversion, we know $\sigma_0 \Longrightarrow \sigma_1$.
By inversion, we know $\sigma_0 \Longrightarrow \sigma_1'$.
By induction, we have a π_1 such that $\pi(\sigma_1) = \sigma'$ and $\pi(\sigma_0) = \sigma_0$.
We can take π_1 as our permutation witness.
- Case $\sigma_0, l : e\ \mathsf{later} \Longrightarrow \sigma_2, l : v\ \mathsf{now}$:
By inversion, we have $\sigma_0 \Longrightarrow \sigma_1$ and $\langle\sigma_1; e\rangle \Downarrow \langle\sigma_2; v\rangle$.
By inversion, we have $\sigma_0 \Longrightarrow \sigma_1'$ and $\langle\sigma_1'; e\rangle \Downarrow \langle\sigma_2'; v'\rangle$.
By induction, we have a π_1 such that $\pi_1(\sigma_1) = \sigma_1'$ and $\pi_1(\sigma_0) = \sigma_0$.
Since σ is supported, the free locations of e are within the locations of σ_0.
Hence e is supported by σ_1.
Hence $\pi_1(e) = e$.
Hence $\langle\pi_1(\sigma_1); e\rangle \Downarrow \langle\pi_1(\sigma_2); \pi_1(v)\rangle$.
By induction, there is π_2 such that $\pi_2(\pi_1(\sigma_2)) = \sigma_2'$ and $\pi_2(\pi_1(v)) = v'$.
We can take $\pi_2 \circ \pi_1$ as our permutation witness.

□

3.2 Soundness

Lemma 6 (Order Permutation)**.** *If* $\sigma' \leq \sigma$ *and* $\pi \in \text{Perm}$ *then* $\pi(\sigma') \leq \pi(\sigma)$.

Proof. Assume $\sigma' \leq \sigma$.
Then there is a σ_0 such that $\sigma' = \sigma \cdot \sigma_0$.
So $\pi(\sigma') = \pi(\sigma \cdot \sigma_0)$.
So $\pi(\sigma') = \pi(\sigma) \cdot \pi(\sigma_0)$.
Taking $\pi(\sigma_0)$ as the witness, there is a σ_1 such that $\pi(\sigma') = \pi(\sigma) \cdot \sigma_1$.
Hence $\pi(\sigma') \leq \pi(\sigma)$. □

Lemma 7 (Heap Renaming)**.** *For all* $\pi \in \text{Perm}$ *and* $\sigma \in \text{Heap}_n$, $\pi(\sigma) \in \text{Heap}_n$.

Proof. We prove this by induction on n.

- Case $n = 0$:
  Immediate, since all heaps are in Heap_0.

- Case $n = k + 1$:
  By induction hypothesis, for all $\pi \in \text{Perm}$ and $\sigma \in \text{Heap}_k$, $\pi(\sigma) \in \text{Heap}_k$.
  Assume $\pi \in \text{Perm}$ and $\sigma \in \text{Heap}_{k+1}$.
  Then we know that $\sigma \Longrightarrow \sigma'$ and $\sigma' \in \text{Heap}_k$.
  By permutation, $\pi(\sigma) \Longrightarrow \pi(\sigma')$.
  By induction, $\sigma' \in \text{Heap}_k$.
  Hence $\pi(\sigma) \in \text{Heap}_{k+1}$.

□

Lemma 8 (Kripke Monotoncity). *If ρ is a monotone environment and $w' \leq w$, then $\mathcal{V}[\![A]\!]\,\rho\,w' \supseteq \mathcal{V}[\![A]\!]\,\rho\,w$.*

Proof. This proof is by induction on the type A.
Assume $w' \leq w$, and proceed by case analysis of A:

- Case α:
  Assume ρ is a monotone environment.
  Then we know $\rho(\alpha)$ is monotone, and so $\rho(\alpha)\ w' \supseteq \rho(\alpha)\ w$.
  Hence if $v \in \rho(\alpha)\ w$ then $v \in \rho(\alpha)\ w'$.

- Case $\hat{\mu}\alpha.\ A$:
  Assume ρ is a monotone environment.
  Assume $\mathsf{into}\, v \in \mathcal{V}[\![\hat{\mu}\alpha.\ A]\!]\,\rho\,w$.
  Hence $v \in \mathcal{V}[\![A]\!]\,(\rho, \mathcal{V}[\![\bullet(\hat{\mu}\alpha.\ A)]\!]\,\rho\,w/\alpha)\ w$.

  To apply the induction hypothesis, we need to know that $\mathcal{V}[\![\bullet(\hat{\mu}\alpha.\ A)]\!]\,\rho\,w$ is monotone.
  Assume we have w and w' such that $w' \leq w$.
  Assume $l \in \mathcal{V}[\![\bullet\hat{\mu}\alpha.\ A]\!]\,\rho\,w$.
  Hence $w.\sigma = \sigma_0, l : e\ \mathsf{later}, \sigma_1$ such that $\forall \pi \in \text{Perm}, w'' \leq (w.n, \sigma_0, w.a).\ \pi(e) \in \mathcal{L}[\![\hat{\mu}\alpha.\ A]\!]\,\rho\,\pi(w'')$.
  We want to show $l \in \mathcal{V}[\![\bullet\hat{\mu}\alpha.\ A]\!]\,\rho\,w'$.
  So we want to show that $w'.\sigma = \sigma_0', l : e\ \mathsf{later}, \sigma_1'$
  such that $\forall \pi \in \text{Perm}, w'' \leq (w'.n, \sigma_0', w'.a).\ \pi(e) \in \mathcal{L}[\![\hat{\mu}\alpha.\ A]\!]\,\rho\,\pi(w'')$.
  Since $w' \leq w$, we know $w'.\sigma \leq w.\sigma$, and so $w'.\sigma = \sigma_0, l : e\ \mathsf{later}, \sigma_1, \sigma_2$.
  So we can take $\sigma_0' = \sigma_0$.
  Since $w' \leq w$, we know $\forall \pi \in \text{Perm}, w'' \leq (w'.n, \sigma_0, w'.a).\ \ \pi(e) \in \mathcal{L}[\![\hat{\mu}\alpha.\ A]\!]\,\rho\,\pi(w'')$.
  Hence $l \in \mathcal{V}[\![\bullet\hat{\mu}\alpha.\ A]\!]\,\rho\,w'$.
  Hence $\mathcal{V}[\![\bullet(\hat{\mu}\alpha.\ A)]\!]\,\rho\,w \subseteq \mathcal{V}[\![\bullet(\hat{\mu}\alpha.\ A)]\!]\,\rho\,w'$.
  Hence $\mathcal{V}[\![\bullet(\hat{\mu}\alpha.\ A)]\!]\,\rho\,w$ is monotone.

  By induction, $v \in \mathcal{V}[\![A]\!]\,(\rho, \mathcal{V}[\![\bullet(\hat{\mu}\alpha.\ A)]\!]\,\rho\,w/\alpha)\ w'$.
  Hence $\mathsf{into}\, v \in \mathcal{V}[\![\hat{\mu}\alpha.\ A]\!]\,\rho\,w$.

- Case $A + B$:
  Assume ρ is a monotone environment.
  Assume $v \in \mathcal{V}[\![A + B]\!]\,\rho\,w$.

  Suppose $v = \mathsf{inl}\, v'$ where $v' \in \mathcal{V}[\![A]\!]\,\rho\,w$.
  By induction, $v' \in \mathcal{V}[\![A]\!]\,\rho\,w'$.
  Hence $\mathsf{inl}\, v' \in \mathcal{V}[\![A + B]\!]\,\rho\,w'$.

Suppose $v = \mathsf{inr}\, v'$ where $v' \in \mathcal{V}[\![B]\!]\,\rho\, w$.
By induction, $v' \in \mathcal{V}[\![B]\!]\,\rho\, w'$.
Hence $\mathsf{inr}\, v' \in \mathcal{V}[\![A + B]\!]\,\rho\, w'$.

- Case $A \times B$:
  Assume ρ is a monotone environment.
  Assume $(v_1, v_2) \in \mathcal{V}[\![A \times B]\!]\,\rho\, w$.
  Hence $v_1 \in \mathcal{V}[\![A]\!]\,\rho\, w$ and $v_2 \in \mathcal{V}[\![B]\!]\,\rho\, w$.
  By induction, $v_1 \in \mathcal{V}[\![A]\!]\,\rho\, w'$ and $v_2 \in \mathcal{V}[\![B]\!]\,\rho\, w'$.
  Hence $(v_1, v_2) \in \mathcal{V}[\![A \times B]\!]\,\rho\, w'$.

- Case $A \to B$:
  Assume ρ is a monotone environment.
  Assume $\lambda x.\ e \in \mathcal{V}[\![A \to B]\!]\,\rho\, w$.
  We want to show $\lambda x.\ e \in \mathcal{V}[\![A \to B]\!]\,\rho\, w'$.
  Assume $\pi \in \text{Perm}$ and $w'' \leq w'$, and $e' \in \mathcal{E}[\![A]\!]\,\rho\, \pi(w'')$.
  By transitivity, $w'' \leq w$.
  Hence by hypothesis, we know $[e'/x]\pi(e) \in \mathcal{E}[\![B]\!]\,\rho\, \pi(w'')$.
  Hence $\lambda x.\ e \in \mathcal{V}[\![A \to B]\!]\,\rho\, w'$.

- Case $\bullet A$:
  Assume ρ is a monotone environment.
  Assume $l \in \mathcal{V}[\![\bullet A]\!]\,\rho\, w$.
  Hence $w.\sigma = \sigma_0, l : e\ \mathsf{later}, \sigma_1$ such that $\forall \pi \in \text{Perm}, w'' \leq (w.n, \sigma_0, w.a).\ \pi(e) \in \mathcal{L}[\![A]\!]\,\rho\, \pi(w'')$.
  We want to show $l \in \mathcal{V}[\![\bullet A]\!]\,\rho\, w'$.
  So we want to show that $w'.\sigma = \sigma_0', l : e\ \mathsf{later}, \sigma_1'$
  such that $\forall \pi \in \text{Perm}, w'' \leq (w'.n, \sigma_0', w'.a).\ \pi(e) \in \mathcal{L}[\![A]\!]\,\rho\, \pi(w'')$.
  Since $w' \leq w$, we know $w'.\sigma \leq w.\sigma$, and so $w'.\sigma = \sigma_0, l : e\ \mathsf{later}, \sigma_1, \sigma_2$.
  So we can take $\sigma_0' = \sigma_0$.
  Since $w' \leq w$, we know $\forall \pi \in \text{Perm}, w'' \leq (w'.n, \sigma_0, w'.a).\ \ \pi(e) \in \mathcal{L}[\![A]\!]\,\rho\, \pi(w'')$.
  Hence $l \in \mathcal{V}[\![\bullet A]\!]\,\rho\, w'$.

- Case $\Box A$:
  Assume ρ is a monotone environment.
  Assume $\mathsf{stable}(v) \in \mathcal{V}[\![\Box A]\!]\,\rho\, w$.
  Hence we know that $v \in \mathcal{V}[\![A]\!]\,\rho\, (w.n, \cdot, \top)$.
  Since $w' \leq w$, it follows that $(w'.n, \cdot, \top) \leq (w.n, \cdot, \top)$.
  Hence by induction, $v \in \mathcal{V}[\![A]\!]\,\rho\, (w'.n, \cdot, \top)$.
  Hence $\mathsf{stable}(v) \in \mathcal{V}[\![\Box A]\!]\,\rho\, w'$.

- Case $\mathsf{S}\, A$:
  Assume ρ is a monotone environment.
  Assume $\mathsf{cons}(v, l) \in \mathcal{V}[\![\mathsf{S}\, A]\!]\,\rho\, w$.
  Hence we know that $v \in \mathcal{V}[\![A]\!]\,\rho\, w$ and $l \in \mathcal{V}[\![\bullet(\mathsf{S}\, A)]\!]\,\rho\, w$.
  Note that $\forall \pi \in \text{Perm}, w'' \leq (w.n, \sigma_0, w.a).\ \ \pi(e) \in \mathcal{L}[\![A]\!]\,\rho\, \pi(w'')$.
  By induction, we know $v \in \mathcal{V}[\![A]\!]\,\rho\, w'$.
  Now we want to show $l \in \mathcal{V}[\![\bullet \mathsf{S}\, A]\!]\,\rho\, w'$.
  So we want to show that $w'.\sigma = \sigma_0', l : e\ \mathsf{later}, \sigma_1'$
  such that $\forall \pi \in \text{Perm}, w'' \leq (w'.n, \sigma_0', w'.a).\ \pi(e) \in \mathcal{L}[\![\mathsf{S}\, A]\!]\,\rho\, \pi(w'')$.
  Since $w' \leq w$, we know $w'.\sigma \leq w.\sigma$, and so $w'.\sigma = \sigma_0, l : e\ \mathsf{later}, \sigma_1, \sigma_2$.
  So we can take $\sigma_0' = \sigma_0$.
  Since $w' \leq w$, we know $\forall \pi \in \text{Perm}, w'' \leq (w'.n, \sigma_0, w'.a).\ \ \pi(e) \in \mathcal{L}[\![A]\!]\,\rho\, \pi(w'')$.
  Hence $l \in \mathcal{V}[\![\bullet \mathsf{S}\, A]\!]\,\rho\, w'$.

Hence $\mathsf{cons}(v, l) \in \mathcal{V}\,[\![\mathsf{S}\,A]\!]\,\rho\,w'$.

- Case alloc:
  Assume ρ is a monotone environment.
  Assume $\diamond \in \mathcal{V}\,[\![\mathsf{alloc}]\!]\,\rho\,w$.
  Then $w.a = \bot$.
  Since $w' \leq w$, it follows $w'.a = \bot$.
  Hence $\diamond \in \mathcal{V}\,[\![\mathsf{alloc}]\!]\,\rho\,w'$.

□

Lemma 9 (Renaming)**.** *We have that:*

1. *If ρ is a permutable environment and $\pi \in$ Perm and $v \in \mathcal{V}\,[\![A]\!]\,\rho\,w$ then $\pi(v) \in \mathcal{V}\,[\![A]\!]\,\rho\,\pi(w)$.*
2. *If ρ is a permutable environment and $\pi \in$ Perm and $e \in \mathcal{E}\,[\![A]\!]\,\rho\,w$ then $\pi(e) \in \mathcal{E}\,[\![A]\!]\,\rho\,\pi(w)$.*
3. *If ρ is a permutable environment and $\pi \in$ Perm and $e \in \mathcal{L}\,[\![A]\!]\,\rho\,w$ then $\pi(e) \in \mathcal{L}\,[\![A]\!]\,\rho\,\pi(w)$.*

Proof. We do these proofs by induction on the type.

1. Assume we have $\pi \in$ Perm. Now we proceed by induction on types:

   - Case α:
     Assume ρ is a permutable environment.
     Assume $v \in \rho(\alpha)\ w$.
     Hence $\pi(v) \in \pi(\rho(\alpha)\ w)$.
     Since that $\rho(\alpha)$ is permutable, $\pi(\rho(\alpha)\ \ \mathtt{world}) = \rho(\alpha)\ (\pi(w))$.
     Hence $\pi(v) \in \rho(\alpha)\ \pi(w)$.
   - Case $\hat{\mu}\alpha.\ A$:
     Assume ρ is a permutable environment.
     Assume $\mathsf{into}\,v \in \mathcal{V}\,[\![\hat{\mu}\alpha.\ A]\!]\,\rho\,w$.
     Hence $v \in \mathcal{V}\,[\![A]\!]\,(\rho, \mathcal{V}\,[\![\bullet(\hat{\mu}\alpha.\ A)]\!]\,\rho\,w/\alpha)\,w$.

     To apply the induction hypothesis, we need to show $\mathcal{V}\,[\![\bullet(\hat{\mu}\alpha.\ A)]\!]\,\rho\,w$ is permutable.
     Assume $l \in \mathcal{V}\,[\![\bullet\hat{\mu}\alpha.\ A]\!]\,\rho\,w$.
     Hence $w.\sigma = \sigma_0, l : e\ \mathsf{later}, \sigma_1$ and $\forall \pi'' \in$ Perm, $w'' \leq (w.n, \sigma_0, w.a), \pi''(e) \in \mathcal{L}\,[\![\hat{\mu}\alpha.\ A]\!]\,\rho\,\pi''(w'')$.
     We want to show $\pi(l) \in \mathcal{V}\,[\![\bullet\hat{\mu}\alpha.\ A]\!]\,\rho\,\pi(w)$.
     Note $\pi(w.\sigma) = \pi(\sigma_0), \pi(l) : \pi(e)\ \mathsf{later}, \pi(\sigma_1)$.
     It remains to show $\forall \pi' \in$ Perm, $w' \leq (w.n, \pi(\sigma_0), w.a)$, $\pi'(\pi(e)) \in \mathcal{L}\,[\![\hat{\mu}\alpha.\ A]\!]\,\rho\,\pi'(w')$.
     Assume $\pi' \in$ Perm and $w' \leq (w.n, \pi(\sigma_0), w.a)$.
     Note that $\pi^{-1}(w') \leq (w.n, \sigma_0, w.a)$.
     Instantiate π'' with $\pi' \circ \pi$ and w'' with $\pi^{-1}(w')$.
     Then we know that $(\pi' \circ \pi)(e) \in \mathcal{L}\,[\![\hat{\mu}\alpha.\ A]\!]\,\rho\,(\pi' \circ \pi)(\pi^{-1}(w'))$.
     Hence $\pi'(\pi(e)) \in \mathcal{L}\,[\![\hat{\mu}\alpha.\ A]\!]\,\rho\,\pi'(w')$.
     Hence $\forall \pi' \in$ Perm, $w' \leq (w.n, \pi(\sigma_0), w.a)$, $\pi''(\pi(e)) \in \mathcal{L}\,[\![\hat{\mu}\alpha.\ A]\!]\,\rho\ \ \pi'(w')$. Hence $\pi(l) \in \mathcal{V}\,[\![\bullet\hat{\mu}\alpha.\ A]\!]\,\rho\,\pi(w)$.

     By induction, $\pi(v) \in \mathcal{V}\,[\![A]\!]\,(\rho, \mathcal{V}\,[\![\bullet(\hat{\mu}\alpha.\ A)]\!]\,\rho\,w/\alpha)\,\pi(w)$.
     Therefore $\mathsf{into}\,\pi(v) \in \mathcal{V}\,[\![\hat{\mu}\alpha.\ A]\!]\,\rho\,\pi(w)$.
     Since $\mathsf{into}\,\pi(v) = \pi(\mathsf{into}\,v)$, we know $\pi(\mathsf{into}\,v) \in \mathcal{V}\,[\![\hat{\mu}\alpha.\ A]\!]\,\rho\,\pi(w)$.
   - Case $A + B$:
     Assume ρ is a permutable environment.
     Assume $w \in$ World and $v \in \mathcal{V}\,[\![A + B]\!]\,\rho\,w$.

Suppose $v = \mathsf{inl}\, v'$ where $v' \in \mathcal{V}[\![A]\!]\, \rho\, w$.
By induction, $\pi(v') \in \mathcal{V}[\![A]\!]\, \rho\, \pi(w)$.
Hence $\mathsf{inl}\, \pi(v') \in \mathcal{V}[\![A + B]\!]\, \rho\, \pi(w)$.
Bur $\mathsf{inl}\, \pi(v') = \pi(\mathsf{inl}\, v')$.
So $\pi(\mathsf{inl}\, v') \in \mathcal{V}[\![A + B]\!]\, \rho\, \pi(w)$.

Suppose $v = \mathsf{inr}\, v'$ where $v' \in \mathcal{V}[\![B]\!]\, \rho\, w$.
By induction, $\pi(v') \in \mathcal{V}[\![B]\!]\, \rho\, \pi(w)$.
Hence $\mathsf{inr}\, \pi(v') \in \mathcal{V}[\![A + B]\!]\, \rho\, \pi(w)$.
Bur $\mathsf{inr}\, \pi(v') = \pi(\mathsf{inr}\, v')$.
So $\pi(\mathsf{inr}\, v') \in \mathcal{V}[\![A + B]\!]\, \rho\, \pi(w)$.

- Case $A \times B$:
  Assume ρ is a permutable environment.
  Assume $w \in$ World and $(v, v') \in \mathcal{V}[\![A \times B]\!]\, \rho\, w$.
  Hence $v \in \mathcal{V}[\![A]\!]\, \rho\, w$ and $v' \in \mathcal{V}[\![B]\!]\, \rho\, w$.
  By induction $\pi(v) \in \mathcal{V}[\![A]\!]\, \rho\, \pi(w)$ and $\pi(v') \in \mathcal{V}[\![B]\!]\, \rho\, \pi(w')$.
  Hence $(\pi(v), \pi(v')) \in \mathcal{V}[\![A \times B]\!]\, \rho\, \pi(w)$.
  Bur $(\pi(v), \pi(v')) = \pi(v, v')$, so $\pi(v, v') \in \mathcal{V}[\![A \times B]\!]\, \rho\, \pi(w)$.
- Case $A \to B$:
  Assume ρ is a permutable environment.
  Assume $w \in$ World and $\lambda x.\ e \in \mathcal{V}[\![A \to B]\!]\, \rho\, w$.
  Hence for all $\pi'' \in$ Perm and $w'' \leq w$, if $e' \in \mathcal{E}[\![A]\!]\, \rho\, \pi''(w'')$, then $[e'/x]\pi''(e) \in \mathcal{E}[\![B]\!]\, \rho\, \pi''(w'')$.
  We want to show that $\pi(\lambda x.\ e) \in \mathcal{V}[\![A \to B]\!]\, \rho\, \pi(w)$.
  Note that $\pi(\lambda x.\ e) = \lambda x.\ \pi(e)$.
  Hence we want to show $\lambda x.\ \pi(e) \in \mathcal{V}[\![A \to B]\!]\, \rho\, \pi(w)$.
  Assume $\pi' \in$ Perm and $w' \leq \pi(w)$ and $e' \in \mathcal{E}[\![A]\!]\, \rho\, \pi'(w')$.
  Since $w' \leq \pi(w)$, we know that $\pi^{-1}(w') \leq w$.
  Instantiate π'' with $\pi' \circ \pi$ and w'' with $\pi^{-1}(w')$.
  Then $\pi''(w'') = \pi'(\pi(\pi^{-1}(w'))) = \pi'(w')$.
  So $e' \in \mathcal{E}[\![A]\!]\, \rho\, \pi''(w'')$.
  Therefore we know that $[e'/x]\pi''(e) \in \mathcal{E}[\![B]\!]\, \rho\, \pi''(w'')$.
  Note that $\pi''(e) = \pi'(\pi(e))$ and $\pi''(w'') = \pi'(w')$.
  Hence $[e'/x]\pi'(\pi(e)) \in \mathcal{E}[\![B]\!]\, \rho\, \pi'(w')$.
  Hence $\lambda x.\ \pi(e) \in \mathcal{V}[\![A \to B]\!]\, \rho\, \pi(w)$.
- Case $\bullet A$:
  Assume ρ is a permutable environment.
  Assume $w \in$ World and $l \in \mathcal{V}[\![\bullet A]\!]\, \rho\, w$.
  Hence $w.\sigma = \sigma_0, l : e\ \mathsf{later}, \sigma_1$ and $\forall \pi'' \in$ Perm, $w'' \leq (w.n, \sigma_0, w.a)$, $\pi''(e) \in \mathcal{L}[\![A]\!]\, \rho\, \pi''(w'')$.
  We want to show $\pi(l) \in \mathcal{V}[\![\bullet A]\!]\, \rho\, \pi(w)$.
  Note $\pi(w.\sigma) = \pi(\sigma_0), \pi(l) : \pi(e)\ \mathsf{later}, \pi(\sigma_1)$.
  It remains to show $\forall \pi' \in$ Perm, $w' \leq (w.n, \pi(\sigma_0), w.a)$, $\pi'(\pi(e)) \in \mathcal{L}[\![A]\!]\, \rho\, \pi'(w')$.
  Assume $\pi' \in$ Perm and $w' \leq (w.n, \pi(\sigma_0), w.a)$.
  Note that $\pi^{-1}(w') \leq (w.n, \sigma_0, w.a)$.
  Instantiate π'' with $\pi' \circ \pi$ and w'' with $\pi^{-1}(w')$.
  Then we know that $(\pi' \circ \pi)(e) \in \mathcal{L}[\![A]\!]\, \rho\, (\pi' \circ \pi)(\pi^{-1}(w'))$.
  Hence $\pi'(\pi(e)) \in \mathcal{L}[\![A]\!]\, \rho\, \pi'(w')$.
  Hence $\forall \pi' \in$ Perm, $w' \leq (w.n, \pi(\sigma_0), w.a)$, $\pi''(\pi(e)) \in \mathcal{L}[\![A]\!]\, \rho\, \pi'(w')$.
  Hence $\pi(l) \in \mathcal{V}[\![\bullet A]\!]\, \rho\, \pi(w)$.
- Case $\Box A$:
  Assume ρ is a permutable environment.
  Assume $w \in$ World and $\mathsf{stable}(v) \in \mathcal{V}[\![\Box A]\!]\, \rho\, w$.

Hence $v \in \mathcal{V}\llbracket A \rrbracket\, \rho\, (w.n, \cdot, w.a)$.
By induction, $\pi(v) \in \mathcal{V}\llbracket A \rrbracket\, \rho\, \pi(w.n, \cdot, w.a)$.
Note that $\pi(w.n, \cdot, w.a) = (\pi(w).n, \cdot, \pi(w).a)$.
Hence $\pi(v) \in \mathcal{V}\llbracket A \rrbracket\, \rho\, (\pi(w).n, \cdot, \pi(w).a)$.
Hence $\mathsf{stable}(\pi(v)) \in \mathcal{V}\llbracket \Box A \rrbracket\, \rho\, \pi(w)$.
Hence $\pi(\mathsf{stable}(v)) \in \mathcal{V}\llbracket \Box A \rrbracket\, \rho\, \pi(w)$.

- Case $\mathsf{S}\, A$:
  Assume ρ is a permutable environment.
  Assume $w \in$ World and $\mathsf{cons}(v, l) \in \mathcal{V}\llbracket \mathsf{S}\, A \rrbracket\, \rho\, w$.
  Hence $v \in \mathcal{V}\llbracket A \rrbracket\, \rho\, w$ and $l \in \mathcal{V}\llbracket \bullet \mathsf{S}\, A \rrbracket\, \rho\, w$.
  By induction, $\pi(v) \in \mathcal{V}\llbracket A \rrbracket\, \rho\, \pi(w)$.
  Since $l \in \mathcal{V}\llbracket \bullet \mathsf{S}\, A \rrbracket\, \rho\, w$, we know that:
  1. $w.\sigma = \sigma_0, l : e\ \mathsf{later}, \sigma_1$.
  2. $\forall \pi'' \in$ Perm, $w'' \leq (w.n, \sigma_0, w.a)$, $\pi''(e) \in \mathcal{L}\llbracket \mathsf{S}\, A \rrbracket\, \rho\, \pi''(w'')$.
  We want to show $\pi(l) \in \mathcal{V}\llbracket \bullet \mathsf{S}\, A \rrbracket\, \rho\, \pi(w)$.
  Note $\pi(w.\sigma) = \pi(\sigma_0), \pi(l) : \pi(e)\ \mathsf{later}, \pi(\sigma_1)$.
  It remains to show $\forall \pi' \in$ Perm, $w' \leq (w.n, \pi(\sigma_0), w.a)$, $\pi'(\pi(e)) \in \mathcal{L}\llbracket \mathsf{S}\, A \rrbracket\, \rho\, \pi'(w')$.
  Assume $\pi' \in$ Perm and $w' \leq (w.n, \pi(\sigma_0), w.a)$.
  Note that $\pi^{-1}(w') \leq (w.n, \sigma_0, w.a)$.
  Instantiate π'' with $\pi' \circ \pi$ and w'' with $\pi^{-1}(w')$.
  Then we know that $(\pi' \circ \pi)(e) \in \mathcal{L}\llbracket \mathsf{S}\, A \rrbracket\, \rho\, (\pi' \circ \pi)(\pi^{-1}(w'))$.
  Hence $\pi'(\pi(e)) \in \mathcal{L}\llbracket \mathsf{S}\, A \rrbracket\, \rho\, \pi'(w')$.
  Hence $\forall \pi' \in$ Perm, $w' \leq (w.n, \pi(\sigma_0), w.a)$, $\pi''(\pi(e)) \in \mathcal{L}\llbracket \mathsf{S}\, A \rrbracket\, \rho\, \pi'(w')$.
  Hence $\pi(l) \in \mathcal{V}\llbracket \bullet \mathsf{S}\, A \rrbracket\, \rho\, \pi(w)$.
  Hence $\mathsf{cons}(\pi(v), \pi(l)) \in \mathcal{V}\llbracket \mathsf{S}\, A \rrbracket\, \rho\, \pi(w)$.
  Hence $\pi(\mathsf{cons}(v, l)) \in \mathcal{V}\llbracket \mathsf{S}\, A \rrbracket\, \rho\, \pi(w)$.
- Case alloc:
  Assume ρ is a permutable environment.
  Assume $w \in$ World and $\diamond \in \mathcal{V}\llbracket \mathsf{alloc} \rrbracket\, \rho\, w$.
  Hence we know that $w.a = \top$.
  Note that $\pi(\diamond) = \diamond$.
  Note that $\pi(w).a = w.a = \top$.
  Hence $\pi(\diamond) \in \mathcal{V}\llbracket \mathsf{alloc} \rrbracket\, \rho\, \pi(w)$.

2. Assume ρ is a permutable environment.
   Assume we have $\pi \in$ Perm, and $e \in \mathcal{E}\llbracket A \rrbracket\, \rho\, w$.

   For all $\sigma' \leq w.\sigma$, there is $\sigma'' \leq \sigma'$ and v such that $\langle \sigma'; e \rangle \Downarrow \langle \sigma''; v \rangle$ and $v \in \mathcal{V}\llbracket A \rrbracket\, \rho\, (w.n, \sigma'', w.a)$.
   Note that $\pi(w) = (w.n, \pi(w.\sigma), w.a)$.
   Assume $\sigma_1 \leq \pi(w.\sigma)$.
   Then $\pi^{-1}(\sigma_1) \leq w.\sigma$.
   So there is a $\sigma_2 \leq \pi^{-1}(\sigma_1)$ and v such that $\langle \pi^{-1}(\sigma_1); e \rangle \Downarrow \langle \sigma_2; v \rangle$
   and $v \in \mathcal{V}\llbracket A \rrbracket\, \rho\, (w.n, \sigma_2, w.a)$ and $w.a = \top \implies \sigma_2 = \pi^{-1}(\sigma_1)$.
   We know that $\pi(\sigma_2) \leq \sigma_1 \leq \pi(w.\sigma)$.
   We know that $w.a = \top \implies \pi(\sigma_2) = \sigma_1$.
   By permutation, we know that $\langle \sigma_1; \pi(e) \rangle \Downarrow \langle \pi(\sigma_2); \pi(v) \rangle$.
   By renaming for values, we know that $\pi(v) \in \mathcal{V}\llbracket A \rrbracket\, \rho\, (w.n, \pi(\sigma_2), w.a)$.
   Hence $\pi(e) \in \mathcal{E}\llbracket A \rrbracket\, \rho\, (w.n, \pi(w.\sigma), w.a)$.
   Hence $\pi(e) \in \mathcal{E}\llbracket A \rrbracket\, \rho\, \pi(w)$.

3. Assume ρ is a permutable environment.

Assume we have $\pi \in \text{Perm}$ and $e \in \mathcal{L}[\![A]\!]\,\rho\,w$.
Assume $e \in \mathcal{L}[\![A]\!]\,\rho\,w$.
We have two cases:

- Case $w.n = 0$:
  In this case $\pi(e) \in \mathcal{L}[\![A]\!]\,\rho\,\pi(w)$ by definition.
- Case $w.n = k + 1$:
  In this case, we know that $w.\sigma \Longrightarrow \sigma'$ and $e \in \mathcal{E}[\![A]\!]\,\rho\,(k, \sigma', w.a)$.
  By renaming, we know that $\pi(e) \in \mathcal{E}[\![A]\!]\,\rho\,(k, \pi(\sigma'), w.a)$.
  By permutation, we know that $\pi(w.\sigma) \Longrightarrow \pi(\sigma')$.
  Hence $\pi(e) \in \mathcal{L}[\![A]\!]\,\rho\,\pi(w)$.

□

Lemma 10 (Supportedness of the Logical Relation)**.** *If ρ is a supported environment and $v \in \mathcal{V}[\![A]\!]\,\rho\,w$ then $v \sqsubseteq w.\sigma$.*

Proof. This proof is by induction on the type. Assume we have a world w.

- Case α :
  Assume we have a supported environment ρ.
  Then $\mathcal{V}[\![\alpha]\!]\,\rho\,w = \rho(\alpha)\,w$.
  Since $\rho(\alpha)\,w$ is supported, if $v \in \rho(\alpha)\,w$, then $v \sqsubseteq w.\sigma$.
- Case $\hat{\mu}\alpha.\,A$:
  Assume we have a supported environment ρ.
  Assume $\mathsf{into}\,v \in \mathcal{V}[\![\hat{\mu}\alpha.\,A]\!]\,\rho\,w$.
  Hence $v \in \mathcal{V}[\![A]\!]\,(\rho, \mathcal{V}[\![\bullet(\hat{\mu}\alpha.\,A)]\!]\,\rho\,w/\alpha)\,w$.

  To apply the induction hypothesis, we need to show that $\mathcal{V}[\![\bullet(\hat{\mu}\alpha.\,A)]\!]\,\rho\,w$ is supported.
  Assume $l \in \mathcal{V}[\![\bullet(\hat{\mu}\alpha.\,A)]\!]\,\rho\,w$.
  Hence $w.\sigma = \sigma_0, l : e\ \mathsf{later}, \sigma_1$.
  Hence $l \in \text{dom}(w.\sigma)$.
  Hence $l \sqsubseteq w.\sigma$.
- Case $A + B$:
  Assume we have a supported environment ρ.
  Now assume that $v \in \mathcal{V}[\![A + B]\!]\,\rho\,w$.
  Either $v = \mathsf{inl}\,v'$ and $v' \in \mathcal{V}[\![A]\!]\,\rho\,w$ or $v = \mathsf{inr}\,v'$ and $v' \in \mathcal{V}[\![B]\!]\,\rho\,w$.

  Suppose $v = \mathsf{inl}\,v'$ and $v' \in \mathcal{V}[\![A]\!]\,\rho\,w$.
  Then by induction, we know that $v' \sqsubseteq w.\sigma$.
  Hence $\mathsf{inl}\,v' \sqsubseteq w.\sigma$.

  Suppose $v = \mathsf{inr}\,v'$ and $v' \in \mathcal{V}[\![B]\!]\,\rho\,w$.
  Then by induction, we know that $v' \sqsubseteq w.\sigma$.
  Hence $\mathsf{inr}\,v' \sqsubseteq w.\sigma$.

  Hence $v \sqsubseteq w.\sigma$.
- Case $A \times B$:
  Assume we have a supported environment ρ.
  Now assume $(v, v') \in \mathcal{V}[\![A \times B]\!]\,\rho\,w$.
  Hence $v \in \mathcal{V}[\![A]\!]\,\rho\,w$ and $v' \in \mathcal{V}[\![B]\!]\,\rho\,w$.
  By induction, $v \sqsubseteq w.\sigma$ and $v' \sqsubseteq w.\sigma$.
  Hence $(v, v') \sqsubseteq w.\sigma$.

- Case $A \to B$:
  Assume we have a supported environment ρ.
  Now assume $\lambda x.\ e \in \mathcal{V}\llbracket A \to B \rrbracket\ \rho\ w$.
  Hence by hypothesis $\lambda x.\ e \sqsubseteq w.\sigma$.

- Case $\bullet A$:
  Assume we have a supported environment ρ.
  Now assume $l \in \mathcal{V}\llbracket \bullet A \rrbracket\ \rho\ w$.
  Hence $w.\sigma = \sigma_0, l : e\ \mathsf{later}, \sigma_1$.
  Hence $l \in \mathrm{dom}(w.\sigma)$.
  Hence $l \sqsubseteq w.\sigma$.

- Case $\mathsf{S}\,A$:
  Assume we have a supported environment ρ.
  Assume we have $\mathsf{cons}(v, l) \in \mathcal{V}\llbracket \mathsf{S}\,A \rrbracket\ \rho\ w$.
  Hence $v \in \mathcal{V}\llbracket A \rrbracket\ \rho\ w$ and $l \in \mathcal{V}\llbracket \bullet\mathsf{S}\,A \rrbracket\ \rho\ w$.

  By induction, $v \sqsubseteq w.\sigma$.
  Since $l \in \mathcal{V}\llbracket \bullet\mathsf{S}\,A \rrbracket\ \rho\ w$, we know $w.\sigma = \sigma_0, l : e\ \mathsf{later}, \sigma_1$.
  Hence $l \in \mathrm{dom}(w.\sigma)$.
  Hence $l \sqsubseteq w.\sigma$.
  Hence $\mathsf{cons}(v, l) \sqsubseteq w.\sigma$.

- Case $\Box A$:
  Assume we have a supported environment ρ.
  Assume we have a $\mathsf{stable}(v) \in \mathcal{V}\llbracket \Box A \rrbracket\ \rho\ w$.
  Hence $v \in \mathcal{V}\llbracket A \rrbracket\ \rho\ (w.n, \cdot, \top)$.
  By induction, $v \sqsubseteq \cdot$.
  Note $\sigma \leq \cdot$.
  By supportedness lemma, $v \sqsubseteq \sigma$.

- Case alloc:
  Assume $\diamond \in \mathcal{V}\llbracket \mathsf{alloc} \rrbracket\ \rho\ w$.
  By definition $\diamond \sqsubseteq \sigma$.

□

Lemma 11 (Weakening). *Assuming ρ is a type environment, we have that:*

1. *If* $\mathrm{FV}(A) \subseteq \mathrm{dom}(\rho)$ *then* $\mathcal{V}\llbracket A \rrbracket\ \rho\ w = \mathcal{V}\llbracket A \rrbracket\ (\rho, \rho')\ w$.
2. *If* $\mathrm{FV}(A) \subseteq \mathrm{dom}(\rho)$ *then* $\mathcal{E}\llbracket A \rrbracket\ \rho\ w = \mathcal{E}\llbracket A \rrbracket\ (\rho, \rho')\ w$.
3. *If* $\mathrm{FV}(A) \subseteq \mathrm{dom}(\rho)$ *then* $\mathcal{L}\llbracket A \rrbracket\ \rho\ w = \mathcal{L}\llbracket A \rrbracket\ (\rho, \rho')\ w$.

Proof. These follow by a lexicographic induction on the step index $w.n$, and the structure of A.

1. - Case α:
     Note that $\rho(\alpha) = (\rho, \rho')(\alpha)$.
   - Case $\hat{\mu}\alpha.\ A$:
     First, let's show that $\mathcal{V}\llbracket \bullet(\hat{\mu}\alpha.\ A) \rrbracket\ \rho\ w = \mathcal{V}\llbracket \bullet(\hat{\mu}\alpha.\ A) \rrbracket\ (\rho, \rho')\ w$.

     Assume that $l \in \mathcal{V}\llbracket \bullet(\hat{\mu}\alpha.\ A) \rrbracket\ \rho\ w$.
     Then $w.\sigma = \sigma_0, l : e\ \mathsf{later}, \sigma_1$ and $\forall w' \leq (w.n, \sigma_0, w.a), \pi \in \mathrm{Perm}.\pi(e) \in \mathcal{L}\llbracket \hat{\mu}\alpha.\ A \rrbracket\ \rho\ \pi(w')$.
     We want that $\forall w' \leq (w.n, \sigma_0, w.a), \pi \in \mathrm{Perm}.\pi(e) \in \mathcal{L}\llbracket \hat{\mu}\alpha.\ A \rrbracket\ (\rho, \rho')\ \pi(w')$.
     Assume $w' \leq (w.n, \sigma_0, w.a)$ and $\pi \in \mathrm{Perm}$.

We know that $\pi(e) \in \mathcal{L}\llbracket \hat{\mu}\alpha.\ A \rrbracket\ \rho\ \pi(w')$.
By renaming, $e \in \mathcal{L}\llbracket \hat{\mu}\alpha.\ A \rrbracket\ \rho\ w'$.
By mutual induction, $e \in \mathcal{L}\llbracket \hat{\mu}\alpha.\ A \rrbracket\ (\rho, \rho')\ w'$. By renaming, $\pi(e) \in \mathcal{L}\llbracket \hat{\mu}\alpha.\ A \rrbracket\ (\rho, \rho')\ \pi(w')$.
Assume that $l \in \mathcal{V}\llbracket \bullet(\hat{\mu}\alpha.\ A) \rrbracket\ (\rho, \rho')\ w$.
Then $w.\sigma = \sigma_0, l : e\ \mathsf{later}, \sigma_1$ and $\forall w' \leq (w.n, \sigma_0, w.a), \pi \in \text{Perm}.\pi(e) \in \mathcal{L}\llbracket \hat{\mu}\alpha.\ A \rrbracket\ (\rho, \rho')\ \pi(w')$.
We want that $\forall w' \leq (w.n, \sigma_0, w.a), \pi \in \text{Perm}.\pi(e) \in \mathcal{L}\llbracket \hat{\mu}\alpha.\ A \rrbracket\ \rho\ \pi(w')$.
Assume $w' \leq (w.n, \sigma_0, w.a)$ and $\pi \in \text{Perm}$.
Hence $\pi(e) \in \mathcal{L}\llbracket \hat{\mu}\alpha.\ A \rrbracket\ (\rho, \rho')\ \pi(w')$.
By renaming, $e \in \mathcal{L}\llbracket \hat{\mu}\alpha.\ A \rrbracket\ (\rho, \rho')\ w'$.
By mutual induction, $e \in \mathcal{L}\llbracket \hat{\mu}\alpha.\ A \rrbracket\ \rho\ w'$.
By renaming, $\pi(e) \in \mathcal{L}\llbracket \hat{\mu}\alpha.\ A \rrbracket\ \rho\ \pi(w')$.

Consider $\mathcal{V}\llbracket A \rrbracket\ (\rho, \mathcal{V}\llbracket \bullet(\hat{\mu}\alpha.\ A) \rrbracket\ \rho\ w/\alpha)\ w$.
By lemma, this equals $\mathcal{V}\llbracket A \rrbracket\ (\rho, \mathcal{V}\llbracket \bullet(\hat{\mu}\alpha.\ A) \rrbracket\ (\rho, \rho')\ w/\alpha)\ w$.
By induction, this equals $\mathcal{V}\llbracket A \rrbracket\ (\rho, \rho', \mathcal{V}\llbracket \bullet(\hat{\mu}\alpha.\ A) \rrbracket\ (\rho, \rho')\ w/\alpha)\ w$.

Now we will show that $\mathsf{into}\, v \in \mathcal{V}\llbracket \hat{\mu}\alpha.\ A \rrbracket\ \rho\ w$ iff $\mathsf{into}\, v \in \mathcal{V}\llbracket \hat{\mu}\alpha.\ A \rrbracket\ (\rho, \rho')\ w$.
Assume $\mathsf{into}\, v \in \mathcal{V}\llbracket \hat{\mu}\alpha.\ A \rrbracket\ \rho\ w$.
Hence $v \in \mathcal{V}\llbracket A \rrbracket\ (\rho, \mathcal{V}\llbracket \bullet(\hat{\mu}\alpha.\ A) \rrbracket\ \rho\ w/\alpha)\ w$.
Hence $v \in \mathcal{V}\llbracket A \rrbracket\ (\rho, \rho', \mathcal{V}\llbracket \bullet(\hat{\mu}\alpha.\ A) \rrbracket\ (\rho, \rho')\ w/\alpha)\ w$.
Hence $\mathsf{into}\, v \in \mathcal{V}\llbracket \hat{\mu}\alpha.\ A \rrbracket\ (\rho, \rho')\ w$.
Assume $\mathsf{into}\, v \in \mathcal{V}\llbracket \hat{\mu}\alpha.\ A \rrbracket\ (\rho, \rho')\ w$.
Hence $v \in \mathcal{V}\llbracket A \rrbracket\ (\rho, \rho', \mathcal{V}\llbracket \bullet(\hat{\mu}\alpha.\ A) \rrbracket\ (\rho, \rho')\ w/\alpha)\ w$.
Hence $v \in \mathcal{V}\llbracket A \rrbracket\ (\rho, \mathcal{V}\llbracket \bullet(\hat{\mu}\alpha.\ A) \rrbracket\ \rho\ w/\alpha)\ w$.
Hence $\mathsf{into}\, v \in \mathcal{V}\llbracket \hat{\mu}\alpha.\ A \rrbracket\ \rho\ w$.

Hence $\mathcal{V}\llbracket \hat{\mu}\alpha.\ A \rrbracket\ \rho\ w = \mathcal{V}\llbracket \hat{\mu}\alpha.\ A \rrbracket\ (\rho, \rho')\ w$.

- Case $A + B$:
  By induction, we know that $\mathcal{V}\llbracket A \rrbracket\ \rho\ w = \mathcal{V}\llbracket A \rrbracket\ (\rho, \rho')\ w$.
  By induction, we know that $\mathcal{V}\llbracket B \rrbracket\ \rho\ w = \mathcal{V}\llbracket B \rrbracket\ (\rho, \rho')\ w$.
  Now we will show for all v, $v \in \mathcal{V}\llbracket A + B \rrbracket\ \rho\ w$ iff $v \in \mathcal{V}\llbracket A + B \rrbracket\ (\rho, \rho')\ w$.

  Assume $v \in \mathcal{V}\llbracket A + B \rrbracket\ \rho\ w$.
  Then either $v = \mathsf{inl}\, v' \wedge v' \in \mathcal{V}\llbracket A \rrbracket\ \rho\ w$ or $v = \mathsf{inr}\, v' \wedge v' \in \mathcal{V}\llbracket B \rrbracket\ \rho\ w$.
  Suppose $v = \mathsf{inl}\, v' \wedge v' \in \mathcal{V}\llbracket A \rrbracket\ \rho\ w$.
  Then $v' \in \mathcal{V}\llbracket A \rrbracket\ (\rho, \rho')\ w$.
  Hence $\mathsf{inl}\, v' \in \mathcal{V}\llbracket A + B \rrbracket\ (\rho, \rho')\ w$.
  Suppose $v = \mathsf{inr}\, v' \wedge v' \in \mathcal{V}\llbracket B \rrbracket\ \rho\ w$.
  Then $v' \in \mathcal{V}\llbracket B \rrbracket\ (\rho, \rho')\ w$.
  Hence $\mathsf{inr}\, v' \in \mathcal{V}\llbracket A + B \rrbracket\ (\rho, \rho')\ w$.

  Assume $v \in \mathcal{V}\llbracket A + B \rrbracket\ (\rho, \rho')\ w$.
  Then either $v = \mathsf{inl}\, v' \wedge v' \in \mathcal{V}\llbracket A \rrbracket\ (\rho, \rho')\ w$ or $v = \mathsf{inr}\, v' \wedge v' \in \mathcal{V}\llbracket B \rrbracket\ (\rho, \rho')\ w$.
  Suppose $v = \mathsf{inl}\, v' \wedge v' \in \mathcal{V}\llbracket A \rrbracket\ (\rho, \rho')\ w$.
  Then $v' \in \mathcal{V}\llbracket A + B \rrbracket\ \rho\ w$.
  Hence $\mathsf{inl}\, v' \in \mathcal{V}\llbracket A \rrbracket\ \rho\ w$.
  Suppose $v = \mathsf{inr}\, v' \wedge v' \in \mathcal{V}\llbracket B \rrbracket\ (\rho, \rho')\ w$.
  Hence $v' \in \mathcal{V}\llbracket A + B \rrbracket\ \rho\ w$.
  Then $\mathsf{inr}\, v' \in \mathcal{V}\llbracket B \rrbracket\ \rho\ w$.

- Case $A \times B$:
  By induction, we know that $\mathcal{V}[\![A]\!]\,\rho\; w = \mathcal{V}[\![A]\!]\,(\rho, \rho')\; w$.
  By induction, we know that $\mathcal{V}[\![B]\!]\,\rho\; w = \mathcal{V}[\![B]\!]\,(\rho, \rho')\; w$.
  Now we will show for all (v, v'), $(v, v') \in \mathcal{V}[\![A + B]\!]\,\rho\; w$ iff $(v, v') \in \mathcal{V}[\![A + B]\!]\,(\rho, \rho')\; w$.

  Assume $(v, v') \in \mathcal{V}[\![A + B]\!]\,\rho\; w$.
  Hence $v \in \mathcal{V}[\![A]\!]\,\rho\; w$ and $v' \in \mathcal{V}[\![B]\!]\,\rho\; w$.
  Hence $v \in \mathcal{V}[\![A]\!]\,(\rho, \rho')\; w$.
  Hence $v' \in \mathcal{V}[\![B]\!]\,(\rho, \rho')\; w$.
  Hence $(v, v') \in \mathcal{V}[\![A \times B]\!]\,(\rho, \rho')\; w$.

  Assume $(v, v') \in \mathcal{V}[\![A + B]\!]\,(\rho, \rho')\; w$.
  Hence $v \in \mathcal{V}[\![A]\!]\,(\rho, \rho')\; w$ and $v' \in \mathcal{V}[\![B]\!]\,(\rho, \rho')\; w$.
  Hence $v \in \mathcal{V}[\![A]\!]\,\rho\; w$.
  Hence $v' \in \mathcal{V}[\![B]\!]\,\rho\; w$.
  Hence $(v, v') \in \mathcal{V}[\![A \times B]\!]\,\rho\; w$.

- Case $A \to B$:
  By induction, we know that for all $w' \leq w$, $\mathcal{E}[\![A]\!]\,\rho\; w = \mathcal{E}[\![A]\!]\,(\rho, \rho')\; w'$.
  By induction, we know that for all $w' \leq w$, $\mathcal{E}[\![A]\!]\,\rho\; w = \mathcal{E}[\![A]\!]\,(\rho, \rho')\; w'$.

  We want to show $\lambda x.\; e' \in \mathcal{V}[\![A \to B]\!]\,\rho\; w$ iff $\lambda x.\; e' \in \mathcal{V}[\![A \to B]\!]\,(\rho, \rho')\; w$.

  Assume we have $\lambda x.\; e' \in \mathcal{V}[\![A \to B]\!]\,\rho\; w$.
  We want to show $\lambda x.\; e' \in \mathcal{V}[\![A \to B]\!]\,(\rho, \rho')\; w$.
  Assume we have $w' \leq w$ and $\pi \in \text{Perm}$.
  Assume we have $e \in \mathcal{E}[\![A]\!]\,(\rho, \rho')\; \pi(w')$.
  Hence we know that $e \in \mathcal{E}[\![A]\!]\,\rho\; \pi(w')$.
  By renaming, we know that $\pi^{-1}(e) \in \mathcal{E}[\![A]\!]\,\rho\; w'$.
  Hence we know that $[\pi^{-1}(e)/x]e' \in \mathcal{E}[\![B]\!]\,\rho\; w'$.
  Hence we know that $[\pi^{-1}(e)/x]e' \in \mathcal{E}[\![B]\!]\,(\rho, \rho')\; w'$.
  By renaming, we know that $[e/x]\pi(e') \in \mathcal{E}[\![B]\!]\,(\rho, \rho')\; \pi(w')$.
  Hence $\lambda x.\; e' \in \mathcal{V}[\![A \to B]\!]\,(\rho, \rho')\; w$.

  Assume we have $\lambda x.\; e' \in \mathcal{V}[\![A \to B]\!]\,(\rho, \rho')\; w$.
  We want to show $\lambda x.\; e' \in \mathcal{V}[\![A \to B]\!]\,\rho\; w$.
  Assume we have $w' \leq w$ and $\pi \in \text{Perm}$.
  Assume we have $e \in \mathcal{E}[\![A]\!]\,\rho\; \pi(w')$.
  By renaming, we know $\pi^{-1}(e) \in \mathcal{E}[\![A]\!]\,\rho\; w'$.
  Hence we know that $\pi^{-1}(e) \in \mathcal{E}[\![A]\!]\,(\rho, \rho')\; w'$.
  Hence we know that $[\pi^{-1}(e)/x]e' \in \mathcal{E}[\![B]\!]\,(\rho, \rho')\; w'$.
  Hence we know that $[\pi^{-1}(e)/x]e' \in \mathcal{E}[\![B]\!]\,\rho\; w'$.
  By renaming, $[e/x]\pi(e') \in \mathcal{E}[\![B]\!]\,\rho\; \pi(w')$.
  Hence $\lambda x.\; e' \in \mathcal{V}[\![A \to B]\!]\,\rho\; w$.

- Case $\bullet A$:
  We will show $l \in \mathcal{V}[\![\bullet A]\!]\,\rho\; w$ iff $l \in \mathcal{V}[\![\bullet A]\!]\,(\rho, \rho')\; w$.

  Assume that $l \in \mathcal{V}[\![\bullet A]\!]\,\rho\; w$.
  Then $w.\sigma = \sigma_0, l : e\ \text{later}, \sigma_1$ and $\forall w' \leq (w.n, \sigma_0, w.a), \pi \in \text{Perm}.\pi(e) \in \mathcal{L}[\![A]\!]\,\rho\; \pi(w')$.
  We want that $\forall w' \leq (w.n, \sigma_0, w.a), \pi \in \text{Perm}.\pi(e) \in \mathcal{L}[\![A]\!]\,(\rho, \rho')\; \pi(w')$.

Assume $w' \leq (w.n, \sigma_0, w.a)$ and $\pi \in \text{Perm}$.
We know that $\pi(e) \in \mathcal{L}[\![A]\!]\, \rho\, \pi(w')$.
By mutual induction, $\pi(e) \in \mathcal{L}[\![A]\!]\, (\rho, \rho')\, \pi(w')$.
Assume that $l \in \mathcal{V}[\![\bullet A]\!]\, (\rho, \rho')\, w$.
Then $w.\sigma = \sigma_0, l : e\ \mathsf{later}, \sigma_1$ and $\forall w' \leq (w.n, \sigma_0, w.a), \pi \in \text{Perm}.\pi(e) \in \mathcal{L}[\![A]\!]\, (\rho, \rho')\, \pi(w')$.
We want that $\forall w' \leq (w.n, \sigma_0, w.a), \pi \in \text{Perm}.\pi(e) \in \mathcal{L}[\![A]\!]\, \rho\, \pi(w')$.
Assume $w' \leq (w.n, \sigma_0, w.a)$ and $\pi \in \text{Perm}$.
Hence $\pi(e) \in \mathcal{L}[\![A]\!]\, (\rho, \rho')\, \pi(w')$.
By renaming, $e \in \mathcal{L}[\![A]\!]\, (\rho, \rho')\, w'$.
By mutual induction, $e \in \mathcal{L}[\![A]\!]\, \rho\, w'$. By renaming, $\pi(e) \in \mathcal{L}[\![A]\!]\, \rho\, \pi(w')$.

- Case $\mathsf{S}\, A$:
  We will show $\mathsf{cons}(v, l) \in \mathcal{V}[\![\mathsf{S}\, A]\!]\, \rho\, w$ iff $\mathsf{cons}(v, l) \in \mathcal{V}[\![\mathsf{S}\, A]\!]\, (\rho, \rho')\, w$.

  Assume $\mathsf{cons}(v, l) \in \mathcal{V}[\![\mathsf{S}\, A]\!]\, \rho\, w$.
  Hence $v \in \mathcal{V}[\![A]\!]\, \rho\, w$ and $l \in \mathcal{V}[\![\bullet(\mathsf{S}\, A)]\!]\, \rho\, w$.
  By induction, $v \in \mathcal{V}[\![A]\!]\, (\rho, \rho')\, w$.
  Then $w.\sigma = \sigma_0, l : e\ \mathsf{later}, \sigma_1$ and $\forall w' \leq (w.n, \sigma_0, w.a), \pi \in \text{Perm}.\pi(e) \in \mathcal{L}[\![\mathsf{S}\, A]\!]\, \rho\, \pi(w')$.
  We want that $\forall w' \leq (w.n, \sigma_0, w.a), \pi \in \text{Perm}.\pi(e) \in \mathcal{L}[\![\mathsf{S}\, A]\!]\, (\rho, \rho')\, \pi(w')$.
  Assume $w' \leq (w.n, \sigma_0, w.a)$ and $\pi \in \text{Perm}$.
  We know that $\pi(e) \in \mathcal{L}[\![\mathsf{S}\, A]\!]\, \rho\, \pi(w')$.
  By renaming $e \in \mathcal{L}[\![\mathsf{S}\, A]\!]\, \rho\, w'$.
  By mutual induction, $e \in \mathcal{L}[\![\mathsf{S}\, A]\!]\, (\rho, \rho')\, w'$.
  By renaming, $\pi(e) \in \mathcal{L}[\![\mathsf{S}\, A]\!]\, (\rho, \rho')\, \pi(w')$.
  Hence $l \in \mathcal{V}[\![\bullet(\mathsf{S}\, A)]\!]\, (\rho, \rho')\, w$.
  Hence $\mathsf{cons}(v, l) \in \mathcal{V}[\![\mathsf{S}\, A]\!]\, (\rho, \rho')\, w$.

  Assume $\mathsf{cons}(v, l) \in \mathcal{V}[\![\mathsf{S}\, A]\!]\, (\rho, \rho')\, w$.
  Hence $v \in \mathcal{V}[\![A]\!]\, (\rho, \rho')\, w$ and $l \in \mathcal{V}[\![\bullet(\mathsf{S}\, A)]\!]\, (\rho, \rho')\, w$.
  By induction, $v \in \mathcal{V}[\![A]\!]\, \rho\, w$.
  Then $w.\sigma = \sigma_0, l : e\ \mathsf{later}, \sigma_1$ and $\forall w' \leq (w.n, \sigma_0, w.a), \pi \in \text{Perm}.\pi(e) \in \mathcal{L}[\![\mathsf{S}\, A]\!]\, (\rho, \rho')\, \pi(w')$.
  We want that $\forall w' \leq (w.n, \sigma_0, w.a), \pi \in \text{Perm}.\pi(e) \in \mathcal{L}[\![\mathsf{S}\, A]\!]\, \rho\, \pi(w')$.
  Assume $w' \leq (w.n, \sigma_0, w.a)$ and $\pi \in \text{Perm}$.
  We know that $\pi(e) \in \mathcal{L}[\![\mathsf{S}\, A]\!]\, (\rho, \rho')\, \pi(w')$.
  By renaming $e \in \mathcal{L}[\![\mathsf{S}\, A]\!]\, (\rho, \rho')\, w'$.
  By mutual induction, $e \in \mathcal{L}[\![\mathsf{S}\, A]\!]\, \rho\, w'$.
  By renaming, $\pi(e) \in \mathcal{L}[\![\mathsf{S}\, A]\!]\, \rho\, \pi(w')$.
  Hence $l \in \mathcal{V}[\![\bullet(\mathsf{S}\, A)]\!]\, \rho\, w$.
  Hence $\mathsf{cons}(v, l) \in \mathcal{V}[\![\mathsf{S}\, A]\!]\, \rho\, w$.
- Case $\Box A$:
  We will show $\mathsf{stable}(v) \in \mathcal{V}[\![\Box A]\!]\, \rho\, w$ iff $\mathsf{stable}(v) \in \mathcal{V}[\![\Box A]\!]\, (\rho, \rho')\, w$.

  Assume $\mathsf{stable}(v) \in \mathcal{V}[\![\Box A]\!]\, \rho\, w$.
  Then $v \in \mathcal{V}[\![A]\!]\, \rho\, (w.n, \cdot, \top)$.
  By induction, $v \in \mathcal{V}[\![A]\!]\, (\rho, \rho')\, (w.n, \cdot, \top)$.
  Hence $\mathsf{stable}(v) \in \mathcal{V}[\![\Box A]\!]\, (\rho, \rho')\, w$.

  Assume $\mathsf{stable}(v) \in \mathcal{V}[\![\Box A]\!]\, (\rho, \rho')\, w$.
  Then $v \in \mathcal{V}[\![A]\!]\, (\rho, \rho')\, (w.n, \cdot, \top)$.
  By induction, $v \in \mathcal{V}[\![A]\!]\, \rho\, (w.n, \cdot, \top)$.
  Hence $\mathsf{stable}(v) \in \mathcal{V}[\![\Box A]\!]\, \rho\, w$.

- Case alloc:
  Immediate, since $\mathcal{V}\llbracket \mathsf{alloc} \rrbracket\, \rho\; w = \mathcal{V}\llbracket \mathsf{alloc} \rrbracket\, \rho, \rho'\; w = \{\diamond \mid w.a = \bot\}$.

2. (Note that since we make a recursive call to the value relation, we can only appeal to this case on subterms or at lower step indexes in the other two mutually-inductive lemmas.)

   We will show $e \in \mathcal{E}\llbracket A \rrbracket\, \rho\; w$ iff $e \in \mathcal{E}\llbracket A \rrbracket\, (\rho, \rho')\; w$.

   Assume $e \in \mathcal{E}\llbracket A \rrbracket\, \rho\; w$.
   Then there exists a $\sigma' \leq \sigma$ and v such that $\langle w.\sigma; e\rangle \Downarrow \langle \sigma'; v\rangle$ and $v \in \mathcal{V}\llbracket A \rrbracket\, \rho\; (w.n, \sigma', w.a)$.
   By induction, $v \in \mathcal{V}\llbracket A \rrbracket\, (\rho, \rho')\; (w.n, \sigma', w.a)$.
   Hence $e \in \mathcal{E}\llbracket A \rrbracket\, (\rho, \rho')\; w$.

   Assume $e \in \mathcal{E}\llbracket A \rrbracket\, \rho\; [(\rho, \rho')]w$.
   Then there exists a $\sigma' \leq \sigma$ and v such that $\langle w.\sigma; e\rangle \Downarrow \langle \sigma'; v\rangle$ and $v \in \mathcal{V}\llbracket A \rrbracket\, (\rho, \rho')\; (w.n, \sigma', w.a)$.
   By induction, $v \in \mathcal{V}\llbracket A \rrbracket\, \rho\; (w.n, \sigma', w.a)$.
   Hence $e \in \mathcal{E}\llbracket A \rrbracket\, \rho\; w$.

3. We will show $e \in \mathcal{L}\llbracket A \rrbracket\, \rho\; w$ iff $e \in \mathcal{L}\llbracket A \rrbracket\, (\rho, \rho')\; w$.

   Consider the value of $w.n$.
   If $w.n = 0$, then the result is immediate.
   If $w.n = k + 1$, then we proceed as follows:

   Assume $e \in \mathcal{L}\llbracket A \rrbracket\, \rho\; w$.
   Then $w.\sigma \Longrightarrow \sigma'$ and $e \in \mathcal{E}\llbracket A \rrbracket\, \rho\; (k, \sigma', w.a)$.
   By induction, $e \in \mathcal{E}\llbracket A \rrbracket\, \rho, \rho'\; (k, \sigma', w.a)$.
   Hence $e \in \mathcal{L}\llbracket A \rrbracket\, (\rho, \rho')\; w$.

   Assume $e \in \mathcal{L}\llbracket A \rrbracket\, (\rho, \rho')\; w$.
   Then $w.\sigma \Longrightarrow \sigma'$ and $e \in \mathcal{E}\llbracket A \rrbracket\, (\rho, \rho')\; (k, \sigma', w.a)$.
   By induction, $e \in \mathcal{E}\llbracket A \rrbracket\, \rho\; (k, \sigma', w.a)$.
   Hence $e \in \mathcal{L}\llbracket A \rrbracket\, \rho\; w$.

□

Lemma 12 (Type Substitution)**.** *We have that:*

1. *For all type environments* ρ *and* w, $\mathcal{V}\llbracket B \rrbracket\, (\rho, \mathcal{V}\llbracket A \rrbracket\, \rho\; /\alpha)\; w = \mathcal{V}\llbracket [A/\alpha]B \rrbracket\, \rho\; w$.
2. *For all type environments* ρ *and* w, $\mathcal{E}\llbracket B \rrbracket\, (\rho, \mathcal{V}\llbracket A \rrbracket\, \rho\; /\alpha)\; w = \mathcal{E}\llbracket [A/\alpha]B \rrbracket\, \rho\; w$.
3. *For all type environments* ρ *and* w, $\mathcal{L}\llbracket B \rrbracket\, (\rho, \mathcal{V}\llbracket A \rrbracket\, \rho\; /\alpha)\; w = \mathcal{L}\llbracket [A/\alpha]B \rrbracket\, \rho\; w$.

Proof. This proof follows by a lexicographic induction on the world index $w.n$ and the structure of B.

1. - Case β :
     If $\beta = \alpha$:
     Note that $\mathcal{V}\llbracket \alpha \rrbracket\, (\rho, \mathcal{V}\llbracket A \rrbracket\, \rho\; /\alpha)\; w = (\rho, \mathcal{V}\llbracket A \rrbracket\, \rho\; /\alpha)(\alpha)\; w = \mathcal{V}\llbracket A \rrbracket\, \rho\; w$.
     Since $[A/\alpha]\alpha = A$, this is equal to $\mathcal{V}\llbracket [A/\alpha]\alpha \rrbracket\, \rho\; w$.

     If $\beta \neq \alpha$: Note that $\mathcal{V}\llbracket \beta \rrbracket\, (\rho, \mathcal{V}\llbracket A \rrbracket\, \rho\; /\alpha)\; w = \rho(\beta)$.
     Since $[A/\alpha]\beta = \beta$, this is equal to $\mathcal{V}\llbracket [A/\alpha]\beta \rrbracket\, \rho\; w = \rho(\beta)$.

- Case $\hat{\mu}\beta.\ B$:
  First, we want $\mathcal{V}\llbracket\bullet(\hat{\mu}\beta.\ B)\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha)\ w$ equals $\mathcal{V}\llbracket[A/\alpha] \bullet (\hat{\mu}\beta.\ B)\rrbracket\,\rho\ w$.
  Assume $l \in \mathcal{V}\llbracket\bullet(\hat{\mu}\beta.\ B)\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha)\ w$.
  Then $w.\sigma = \sigma_0, l : e\ \mathsf{later}, \sigma_1$
  such that $\forall w' \leq (w.n, \sigma_0, w.a), \pi \in \text{Perm}, \pi(e) \in \mathcal{L}\llbracket\hat{\mu}\beta.\ B\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha)\ \pi(w')$.
  We want $\forall w' \leq (w.n, \sigma_0, w.a), \pi \in \text{Perm}, \pi(e) \in \mathcal{L}\llbracket[A/\alpha]\hat{\mu}\beta.\ B\rrbracket\,\rho\ \pi(w')$.
  Assume $w' \leq (w.n, \sigma_0, w.a)$ and $\pi \in \text{Perm}$.
  Hence $\pi(e) \in \mathcal{L}\llbracket\hat{\mu}\beta.\ B\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha)\ \pi(w')$.
  By renaming, $e \in \mathcal{L}\llbracket\hat{\mu}\beta.\ B\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha)\ w'$.
  By induction, $e \in \mathcal{L}\llbracket[A/\alpha]\hat{\mu}\beta.\ B\rrbracket\,\rho\ w'$.
  By renaming, $\pi(e) \in \mathcal{L}\llbracket[A/\alpha]\hat{\mu}\beta.\ B\rrbracket\,\rho\ \pi(w')$.
  Hence $l \in \mathcal{V}\llbracket\bullet([A/\alpha]\hat{\mu}\beta.\ B)\rrbracket\,\rho\ w$.
  Hence $l \in \mathcal{V}\llbracket[A/\alpha] \bullet (\hat{\mu}\beta.\ B)\rrbracket\,\rho\ w$.

  Assume $l \in \mathcal{V}\llbracket[A/\alpha] \bullet (\hat{\mu}\beta.\ B)\rrbracket\,\rho\ w$.
  Hence $l \in \mathcal{V}\llbracket\bullet([A/\alpha]\hat{\mu}\beta.\ B)\rrbracket\,\rho\ w$.
  Then $w.\sigma = \sigma_0, l : e\ \mathsf{later}, \sigma_1$
  such that $\forall w' \leq (w.n, \sigma_0, w.a), \pi \in \text{Perm}, \pi(e) \in \mathcal{L}\llbracket[A/\alpha]\hat{\mu}\beta.\ B\rrbracket\,\rho\ \pi(w')$.
  We want $\forall w' \leq (w.n, \sigma_0, w.a), \pi \in \text{Perm}, \pi(e) \in \mathcal{L}\llbracket\hat{\mu}\beta.\ B\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha)\ \pi(w')$.
  Assume $w' \leq (w.n, \sigma_0, w.a)$ and $\pi \in \text{Perm}$.
  Hence $\pi(e) \in \mathcal{L}\llbracket[A/\alpha]\hat{\mu}\beta.\ B\rrbracket\,\rho\ \pi(w')$.
  By renaming, $e \in \mathcal{L}\llbracket[A/\alpha]\hat{\mu}\beta.\ B\rrbracket\,\rho\ w'$.
  By induction, $e \in \mathcal{L}\llbracket\hat{\mu}\beta.\ B\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha)\ w'$.
  By renaming, $\pi(e) \in \mathcal{L}\llbracket\hat{\mu}\beta.\ B\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha)\ \pi(w')$.
  Hence $l \in \mathcal{V}\llbracket\bullet(\hat{\mu}\beta.\ B)\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha)\ w$.

  Now consider $\mathcal{V}\llbracket B\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha, \mathcal{V}\llbracket\bullet(\hat{\mu}\beta.\ B)\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha)\ /\beta)\ w$.
  This equals $\mathcal{V}\llbracket B\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha, \mathcal{V}\llbracket\bullet[A/\alpha](\hat{\mu}\beta.\ B)\rrbracket\,\rho\ /\beta)\ w$.
  This equals $\mathcal{V}\llbracket B\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,(\rho, \mathcal{V}\llbracket\bullet[A/\alpha](\hat{\mu}\beta.\ B)\rrbracket\,\rho\ /\beta)\ w/\alpha, \mathcal{V}\llbracket\bullet[A/\alpha](\hat{\mu}\beta.\ B)\rrbracket\,\rho\ /\beta)\ w$.
  By induction, this equals $\mathcal{V}\llbracket[A/\alpha]B\rrbracket\,(\rho, \mathcal{V}\llbracket\bullet[A/\alpha](\hat{\mu}\beta.\ B)\rrbracket\,\rho\ /\beta)\ w$.
  This equals $\mathcal{V}\llbracket[A/\alpha]B\rrbracket\,(\rho, \mathcal{V}\llbracket\bullet(\hat{\mu}\beta.\ [A/\alpha]B)\rrbracket\,\rho\ /\beta)\ w$.

  Now we will show that $\mathsf{into}\, v \in \mathcal{V}\llbracket\hat{\mu}\beta.\ B\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha)\ w$ iff $\mathsf{into}\, v \in \mathcal{V}\llbracket[A/\alpha]\hat{\mu}\beta.\ B\rrbracket\,\rho\ w$.
  Assume $\mathsf{into}\, v \in \mathcal{V}\llbracket\hat{\mu}\beta.\ B\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha)\ w$.
  Then $v \in \mathcal{V}\llbracket B\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha, \mathcal{V}\llbracket\bullet(\hat{\mu}\beta.\ B)\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha)\ /\beta)\ w$.
  Then $v \in \mathcal{V}\llbracket[A/\alpha]B\rrbracket\,(\rho, \mathcal{V}\llbracket\bullet(\hat{\mu}\beta.\ [A/\alpha]B)\rrbracket\,\rho\ /\beta)\ w$.
  Then $\mathsf{into}\, v \in \mathcal{V}\llbracket\hat{\mu}\beta.\ [A/\alpha]B\rrbracket\,\rho\ w$.
  Then $\mathsf{into}\, v \in \mathcal{V}\llbracket[A/\alpha]\hat{\mu}\beta.\ B\rrbracket\,\rho\ w$.

  Assume $\mathsf{into}\, v \in \mathcal{V}\llbracket[A/\alpha]\hat{\mu}\beta.\ B\rrbracket\,\rho\ w$.
  Then $\mathsf{into}\, v \in \mathcal{V}\llbracket\hat{\mu}\beta.\ [A/\alpha]B\rrbracket\,\rho\ w$.
  Then $v \in \mathcal{V}\llbracket B\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha, \mathcal{V}\llbracket\bullet(\hat{\mu}\beta.\ B)\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha)\ /\beta)\ w$.
  Then $v \in \mathcal{V}\llbracket[A/\alpha]B\rrbracket\,(\rho, \mathcal{V}\llbracket\bullet(\hat{\mu}\beta.\ [A/\alpha]B)\rrbracket\,\rho\ /\beta)\ w$.
  Then $\mathsf{into}\, v \in \mathcal{V}\llbracket\hat{\mu}\beta.\ B\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha)\ w$.
- Case $B + C$:
  We will show that $v \in \mathcal{V}\llbracket B + C\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha)\ w$ iff $v \in \mathcal{V}\llbracket[A/\alpha](B + C)\rrbracket\,\rho\ w$.

  Assume $v \in \mathcal{V}\llbracket B + C\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha)\ w$.
  Then either $v = \mathsf{inl}\, v' \wedge v' \in \mathcal{V}\llbracket B\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha)\ w$ or $v = \mathsf{inr}\, v' \wedge v' \in \mathcal{V}\llbracket C\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha)\ w$.
  Suppose $v = \mathsf{inl}\, v'$ and $v' \in \mathcal{V}\llbracket B\rrbracket\,(\rho, \mathcal{V}\llbracket A\rrbracket\,\rho\ /\alpha)\ w$.
  Then by induction $v' \in \mathcal{V}\llbracket[A/\alpha]B\rrbracket\,\rho\ w$.

Hence $\mathsf{inl}\, v' \in \mathcal{V}\,[\![[A/\alpha]B + [A/\alpha]C]\!]\,\rho\, w$.
Hence $\mathsf{inl}\, v' \in \mathcal{V}\,[\![[A/\alpha](B + C)]\!]\,\rho\, w$.
Suppose $v = \mathsf{inr}\, v'$ and $v' \in \mathcal{V}\,[\![C]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$.
Then by induction $v' \in \mathcal{V}\,[\![[A/\alpha]C]\!]\,\rho\, w$.
Hence $\mathsf{inr}\, v' \in \mathcal{V}\,[\![[A/\alpha]B + [A/\alpha]C]\!]\,\rho\, w$.
Hence $\mathsf{inr}\, v' \in \mathcal{V}\,[\![[A/\alpha](B + C)]\!]\,\rho\, w$.

Assume $v \in \mathcal{V}\,[\![[A/\alpha](B + C)]\!]\,\rho\, w$.
Then $v \in \mathcal{V}\,[\![[A/\alpha]B + [A/\alpha]C]\!]\,\rho\, w$.
Either $v = \mathsf{inl}\, v' \wedge v' \in \mathcal{V}\,[\![[A/\alpha]B]\!]\,\rho\, w$ or $v = \mathsf{inr}\, v' \wedge v' \in \mathcal{V}\,[\![[A/\alpha]C]\!]\,\rho\, w$.
Suppose $v = \mathsf{inl}\, v'$ and $v' \in \mathcal{V}\,[\![[A/\alpha]B]\!]\,\rho\, w$.
Then by induction, $v' \in \mathcal{V}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$.
Hence $\mathsf{inl}\, v' \in \mathcal{V}\,[\![B + C]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$.
Suppose $v = \mathsf{inr}\, v'$ and $v' \in \mathcal{V}\,[\![[A/\alpha]C]\!]\,\rho\, w$.
Then by induction, $v' \in \mathcal{V}\,[\![C]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$.
Hence $\mathsf{inr}\, v' \in \mathcal{V}\,[\![B + C]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$.

- Case $B \times C$:
  We will show that $(v, v') \in \mathcal{V}\,[\![B \times C]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$ iff $(v, v') \in \mathcal{V}\,[\![[A/\alpha](B \times C)]\!]\,\rho\, w$.
  Assume $(v, v') \in \mathcal{V}\,[\![B \times C]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$.
  Then $v \in \mathcal{V}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$.
  Then $v' \in \mathcal{V}\,[\![C]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$.
  By induction, $v \in \mathcal{V}\,[\![[A/\alpha]B]\!]\,\rho\, w$.
  By induction, $v' \in \mathcal{V}\,[\![[A/\alpha]C]\!]\,\rho\, w$.
  Hence $(v, v') \in \mathcal{V}\,[\![[A/\alpha]B \times [A/\alpha]C]\!]\,\rho\, w$.
  Hence $(v, v') \in \mathcal{V}\,[\![[A/\alpha](B \times C)]\!]\,\rho\, w$.

  Assume $(v, v') \in \mathcal{V}\,[\![[A/\alpha](B \times C)]\!]\,\rho\, w$.
  Hence $(v, v') \in \mathcal{V}\,[\![[A/\alpha]B \times [A/\alpha]C]\!]\,\rho\, w$.
  Hence $v \in \mathcal{V}\,[\![[A/\alpha]B]\!]\,\rho\, w$.
  Hence $v' \in \mathcal{V}\,[\![[A/\alpha]C]\!]\,\rho\, w$.
  By induction, $v \in \mathcal{V}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$.
  By induction, $v' \in \mathcal{V}\,[\![C]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$.
  Hence $(v, v') \in \mathcal{V}\,[\![B \times C]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$.

- Case $B \to C$:
  We will show that $\lambda x.\ e' \in \mathcal{V}\,[\![B \to C]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$ iff $\lambda x.\ e' \in \mathcal{V}\,[\![[A/\alpha](B \to C)]\!]\,\rho\, w$.

  Assume $\lambda x.\ e' \in \mathcal{V}\,[\![B \to C]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$.
  Assume $w' \leq w$, $\pi \in \mathrm{Perm}$ and $e \in \mathcal{E}\,[\![[A/\alpha]B]\!]\,\rho\, \pi(w')$.
  By renaming $\pi^{-1}(e) \in \mathcal{E}\,[\![[A/\alpha]B]\!]\,\rho\, w'$.
  By mutual induction, $\pi^{-1}(e) \in \mathcal{E}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\, w'/\alpha)\, w'$.
  By renaming, $e \in \mathcal{E}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\, w'/\alpha)\, \pi(w')$.
  Hence $[e/x]\pi(e') \in \mathcal{E}\,[\![C]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\, w'/\alpha)\, \pi(w')$.
  By renaming $\pi^{-1}([e/x]\pi(e')) \in \mathcal{E}\,[\![C]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\, w'/\alpha)\, w'$.
  By mutual induction, $\pi^{-1}([e/x]\pi(e')) \in \mathcal{E}\,[\![[A/\alpha]C]\!]\,\rho\, w'$.
  By renaming, $[e/x]\pi(e') \in \mathcal{E}\,[\![[A/\alpha]C]\!]\,\rho\, \pi(w')$.
  Hence $\lambda x.\ e' \in \mathcal{V}\,[\![[A/\alpha]B \to [A/\alpha]C]\!]\,\rho\, w$.
  Hence $\lambda x.\ e' \in \mathcal{V}\,[\![[A/\alpha](B \to C)]\!]\,\rho\, w$.

  Note $\mathcal{V}\,[\![[A/\alpha](B \to C)]\!]\,\rho\, w = \mathcal{V}\,[\![[A/\alpha]B \to [A/\alpha]C]\!]\,\rho\, w$.
  Assume $\lambda x.\ e' \in \mathcal{V}\,[\![[A/\alpha](B \to C)]\!]\,\rho\, w$.

Assume $w' \leq w, \pi \in \text{Perm}$ and $e \in \mathcal{E}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\ /\alpha)\ \pi(w')$.
By renaming, $\pi^{-1}(e) \in \mathcal{E}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\ /\alpha)\ w'$.
By mutual induction, $\pi^{-1}(e) \in \mathcal{E}\,[\![[A/\alpha]B]\!]\,\rho\ w'$.
By renaming, $e \in \mathcal{E}\,[\![[A/\alpha]B]\!]\,\rho\ \pi(w')$.
Hence $[e/x]\pi(e') \in \mathcal{E}\,[\![[A/\alpha]B]\!]\,\rho\ \pi(w')$.
By renaming, $\pi^{-1}([e/x]\pi(e')) \in \mathcal{E}\,[\![[A/\alpha]B]\!]\,\rho\ w'$.
By mutual induction, $\pi^{-1}([e/x]\pi(e')) \in \mathcal{E}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\ /\alpha)\ w'$.
By renaming, $[e/x]\pi(e') \in \mathcal{E}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\ /\alpha)\ \pi(w')$.
Hence $\lambda x.\ e' \in \mathcal{V}\,[\![B \to C]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\ /\alpha)\ w'$.

- Case $\bullet B$:
  We will show that $l \in \mathcal{V}\,[\![\bullet B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\ /\alpha)\ w$ iff $l \in \mathcal{V}\,[\![\bullet [A/\alpha]B]\!]\,\rho\ w$.

  Assume $l \in \mathcal{V}\,[\![\bullet B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\ /\alpha)\ w$.
  Then $w.\sigma = \sigma_0, l : e\ \mathsf{later}, \sigma_1$ and $\forall w' \leq (w.n, \sigma_0, w.a), \pi \in \text{Perm}, \pi(e) \in \mathcal{L}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\ /\alpha)\ \pi(w')$.
  We want to show that $\forall w' \leq (w.n, \sigma_0, w.a), \pi \in \text{Perm}, \pi(e) \in \mathcal{L}\,[\![[A/\alpha]B]\!]\,\rho\ \pi(w')$.
  Assume $w' \leq (w.n, \sigma_0, w.a)$ and $\pi \in \text{Perm}$.
  Hence $e \in \mathcal{L}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\ /\alpha)\ w'$.
  By mutual induction, $e \in \mathcal{L}\,[\![[A/\alpha]B]\!]\,\rho\ w'$.
  By renaming, $\pi(e) \in \mathcal{L}\,[\![[A/\alpha]B]\!]\,\rho\ \pi(w')$.
  Hence $\forall w' \leq (w.n, \sigma_0, w.a), \pi \in \text{Perm}, \pi(e) \in \mathcal{L}\,[\![[A/\alpha]B]\!]\,\rho\ \pi(w')$.
  Hence $l \in \mathcal{V}\,[\![\bullet [A/\alpha]B]\!]\,\rho\ w$.

  Assume $l \in \mathcal{V}\,[\![\bullet [A/\alpha]B]\!]\,\rho\ w$.
  Then $w.\sigma = \sigma_0, l : e\ \mathsf{later}, \sigma_1$ and $\forall w' \leq (w.n, \sigma_0, w.a), \pi \in \text{Perm}, \pi(e) \in \mathcal{L}\,[\![[A/\alpha]B]\!]\,\rho\ \pi(w')$.
  Assume $w' \leq (w.n, \sigma_0, w.a)$ and $\pi \in \text{Perm}$.
  Hence $e \in \mathcal{L}\,[\![[A/\alpha]B]\!]\,\rho\ w'$.
  By mutual induction, $e \in \mathcal{L}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\ /\alpha)\ w'$.
  By renaming, $\pi(e) \in \mathcal{L}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\ /\alpha)\ \pi(w')$.
  Hence $\forall w' \leq (w.n, \sigma_0, w.a), \pi \in \text{Perm}, \pi(e) \in \mathcal{L}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\ /\alpha)\ \pi(w')$.
  Hence $l \in \mathcal{V}\,[\![\bullet B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\ /\alpha)\ w$.

- Case $\mathsf{S}\,B$:
  We will show that $\mathsf{cons}(v, l) \in \mathcal{V}\,[\![\mathsf{S}\,B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\ /\alpha)\ w$ iff $\mathsf{cons}(v, l) \in \mathcal{V}\,[\![[A/\alpha](\mathsf{S}\,B)]\!]\,\rho\ w$.

  Assume $\mathsf{cons}(v, l) \in \mathcal{V}\,[\![\mathsf{S}\,B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\ /\alpha)\ w$.
  Then $v \in \mathcal{V}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\ /\alpha)\ w$.
  Then $l \in \mathcal{V}\,[\![\bullet \mathsf{S}\,B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\ /\alpha)\ w$.
  By induction, $v \in \mathcal{V}\,[\![[A/\alpha]B]\!]\,\rho\ w$.
  We know that $w.\sigma = \sigma_0, l : e\ \mathsf{later}, \sigma_1$
  and for all $w' \leq (w.n, \sigma_0, w.a)$ and $\pi \in \text{Perm}$, $\pi(e) \in \mathcal{L}\,[\![\mathsf{S}\,B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\ /\alpha)\ \pi(w')$.
  Assume $w' \leq (w.n, \sigma_0, w.a)$ and $\pi \in \text{Perm}$.
  Then we know $e \in \mathcal{L}\,[\![\mathsf{S}\,B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\ /\alpha)\ w'$.
  By mutual induction, $e \in \mathcal{L}\,[\![[A/\alpha]\mathsf{S}\,B]\!]\,\rho\ w'$.
  By renaming, $\pi(e) \in \mathcal{L}\,[\![[A/\alpha]\mathsf{S}\,B]\!]\,\rho\ \pi(w')$.
  Hence for all $w' \leq (w.n, \sigma_0, w.a)$ and $\pi \in \text{Perm}$, $\pi(e) \in \mathcal{L}\,[\![[A/\alpha]\mathsf{S}\,B]\!]\,\rho\ \pi(w')$.
  Hence $l \in \mathcal{V}\,[\![\bullet([A/\alpha]\mathsf{S}\,B)]\!]\,\rho\ w$.
  Hence $l \in \mathcal{V}\,[\![[A/\alpha]\bullet(\mathsf{S}\,B)]\!]\,\rho\ w$.
  Hence $\mathsf{cons}(v, l) \in \mathcal{V}\,[\![[A/\alpha](\mathsf{S}\,B)]\!]\,\rho\ w$.

  Assume $\mathsf{cons}(v, l) \in \mathcal{V}\,[\![[A/\alpha](\mathsf{S}\,B)]\!]\,\rho\ w$.
  So $\mathsf{cons}(v, l) \in \mathcal{V}\,[\![\mathsf{S}\,([A/\alpha]B)]\!]\,\rho\ w$.

So $v \in \mathcal{V}\,[\![[A/\alpha]B]\!]\,\rho\, w$ and $l \in \mathcal{V}\,[\![\bullet \mathsf{S}\,([A/\alpha]B)]\!]\,\rho\, w$.
By induction, $v \in \mathcal{V}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$.
We know $w.\sigma = \sigma_0, l : e\ \mathsf{later}, \sigma_1$,
and for all $w' \leq (w.n, \sigma_0, w.a)$ and $\pi \in \text{Perm}$, $\pi(e) \in \mathcal{L}\,[\![[A/\alpha](\mathsf{S}\,B)]\!]\,\rho\,\pi(w')$.
Assume $w' \leq (w.n, \sigma_0, w.a)$ and $\pi \in \text{Perm}$.
Hence $e \in \mathcal{L}\,[\![[A/\alpha](\mathsf{S}\,B)]\!]\,\rho\, w'$.
By induction, $e \in \mathcal{L}\,[\![\mathsf{S}\,B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w'$.
By renaming, $\pi(e) \in \mathcal{L}\,[\![\mathsf{S}\,B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\,\pi(w')$.
Hence for all $w' \leq (w.n, \sigma_0, w.a)$ and $\pi \in \text{Perm}$, $\pi(e) \in \mathcal{L}\,[\![\mathsf{S}\,B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\,\pi(w')$.
Hence $l \in \mathcal{V}\,[\![\bullet(\mathsf{S}\,B)]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$.
Hence $\mathsf{cons}(v, l) \in \mathcal{V}\,[\![\mathsf{S}\,B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$.

- Case $\square B$:
  We will show that $\mathsf{stable}(v) \in \mathcal{V}\,[\![\square B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$ iff $\mathsf{stable}(v) \in \mathcal{V}\,[\![[A/\alpha](\square B)]\!]\,\rho\, w$.

  Assume $\mathsf{stable}(v) \in \mathcal{V}\,[\![\square B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$.
  Then we know that $v \in \mathcal{V}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\,(w.n, \cdot, \top)$.
  Hence by induction, $v \in \mathcal{V}\,[\![[A/\alpha]B]\!]\,\rho\,(w.n, \cdot, \top)$.
  Hence $\mathsf{stable}(v) \in \mathcal{V}\,[\![\square([A/\alpha]B)]\!]\,\rho\, w$.
  So $\mathsf{stable}(v) \in \mathcal{V}\,[\![[A/\alpha](\square B)]\!]\,\rho\, w$.

  Assume $\mathsf{stable}(v) \in \mathcal{V}\,[\![[A/\alpha](\square B)]\!]\,\rho\, w$.
  So $\mathsf{stable}(v) \in \mathcal{V}\,[\![\square([A/\alpha]B)]\!]\,\rho\, w$.
  Then we know that $v \in \mathcal{V}\,[\![[A/\alpha]B]\!]\,\rho\,(w.n, \cdot, \top)$.
  By induction, $v \in \mathcal{V}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$.
  Hence $\mathsf{stable}(v) \in \mathcal{V}\,[\![\square B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$.

- Case alloc:

$$\begin{aligned}
\mathcal{V}\,[\![\mathsf{alloc}]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w &= \{\diamond \mid w.a = \bot\} \\
&= \mathcal{V}\,[\![\mathsf{alloc}]\!]\,\rho\, w \\
&= \mathcal{V}\,[\![[A/\alpha]\mathsf{alloc}]\!]\,\rho\, w
\end{aligned}$$

2. Note that we have to take the same care as in the weakening lemma.

   We will show that $e \in \mathcal{E}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$ iff $e \in \mathcal{E}\,[\![[A/\alpha]B]\!]\,\rho\, w$.

   Assume $e \in \mathcal{E}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$.
   Then $e \sqsubseteq w.\sigma$. Then for all $\sigma \leq w.\sigma$, there is a $\sigma' \leq \sigma$ and v such that $\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle$,
   and $v \in \mathcal{V}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$.
   Assume $\sigma \leq w.\sigma$.
   Then there is a $\sigma' \leq \sigma$ and v such that $\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle$, and $v \in \mathcal{V}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$.
   By induction, $v \in \mathcal{V}\,[\![[A/\alpha]B]\!]\,\rho\, w$.
   So for all $\sigma \leq w.\sigma$, there is a $\sigma' \leq \sigma$ and v such that $\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle$, and $v \in \mathcal{V}\,[\![[A/\alpha]B]\!]\,\rho\, w$.
   Hence $e \in \mathcal{E}\,[\![[A/\alpha]B]\!]\,\rho\, w$.

   Assume $e \in \mathcal{E}\,[\![[A/\alpha]B]\!]\,\rho\, w$.
   Then for all $\sigma \leq w.\sigma$, there is a $\sigma' \leq \sigma$ and v such that $\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle$,
   and $v \in \mathcal{V}\,[\![[A/\alpha]B]\!]\,\rho\, w$.
   Assume $\sigma \leq w.\sigma$.
   Then there is a $\sigma' \leq \sigma$ and v such that $\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle$, and $v \in \mathcal{V}\,[\![[A/\alpha]B]\!]\,\rho\, w$.
   By induction $v \in \mathcal{V}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\, w$.

So for all $\sigma \leq w.\sigma$, there is a $\sigma' \leq \sigma$ and v such that $\langle \sigma; e \rangle \Downarrow \langle \sigma'; v \rangle$, and $v \in \mathcal{V}\,[\![(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)]\!]\,\rho\,Bw$.
Hence $e \in \mathcal{E}\,[\![(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)]\!]\,\rho\,Bw$.

3. We will show that $e \in \mathcal{L}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\;w$ iff $e \in \mathcal{L}\,[\![[A/\alpha]B]\!]\,\rho\;w$.

   If $w.n = 0$, then the result is immediate.
   If $w.n = k + 1$, then we proceed as follows.

   Assume $e \in \mathcal{L}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\;w$.
   Then $e \sqsubseteq w.\sigma$ and $w.\sigma \Longrightarrow \sigma'$ and $e \in \mathcal{E}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\;(k, \sigma', w.a)$.
   By mutual induction, $e \in \mathcal{E}\,[\![[A/\alpha]B]\!]\,\rho\;(k, \sigma', w.a)$.
   Hence $e \in \mathcal{L}\,[\![[A/\alpha]B]\!]\,\rho\;w$.

   Assume $e \in \mathcal{L}\,[\![[A/\alpha]B]\!]\,\rho\;w$.
   Then $e \sqsubseteq w.\sigma$ and $w.\sigma \Longrightarrow \sigma'$ and $e \in \mathcal{E}\,[\![[A/\alpha]B]\!]\,\rho\;(k, \sigma', w.a)$.
   By induction, $e \in \mathcal{E}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\;(k, \sigma', w.a)$.
   Hence $e \in \mathcal{L}\,[\![B]\!]\,(\rho, \mathcal{V}\,[\![A]\!]\,\rho\,/\alpha)\;w$.

□

Lemma 13 (Value Inclusion)**.** *If* $v \in \mathcal{V}\,[\![A]\!]\,\rho\;w$ *then* $v \in \mathcal{E}\,[\![A]\!]\,\rho\;w$*.*

Proof. Assume $v \in \mathcal{V}\,[\![A]\!]\,\rho\;w$.
We want to show that for all $\sigma' \leq w.\sigma$, there exists a σ'' such that $\langle \sigma'; v \rangle \Downarrow \langle \sigma''; v \rangle$ and $v \in \mathcal{V}\,[\![A]\!]\,\rho\;(w.n, \sigma'', w.a)$.
Assume $\sigma' \leq w.\sigma$.
By rule, we know that $\langle \sigma'; v \rangle \Downarrow \langle \sigma'; v \rangle$.
Take $\sigma'' = \sigma'$.
Note that $(w.n, \sigma', w.a) \leq w$.
By Kripke monotoncity, $v \in \mathcal{V}\,[\![A]\!]\,\rho\;(w.n, \sigma', w.a)$ □

Lemma 14 (Kripke Monotoncity for Environments)**.** *If* $w' \leq w$*, then* $\mathrm{Env}(\Gamma)\;w' \supseteq \mathrm{Env}(\Gamma)\;w$*.*

Proof. We proceed by induction on Γ.
Assume we have $w' \leq w$, and $\gamma \in \mathrm{Env}(\Gamma)\;w$.

- Case $\Gamma = \cdot$:
  Immediate
- Case $\Gamma = \Gamma', x : A\ \mathsf{now}$:
  Hence $\gamma = (\gamma', e/x) \in \mathrm{Env}(\Gamma', x : X\ \mathsf{now})\;w$.
  Hence $\gamma' \in \mathrm{Env}(\Gamma')\;w$ and $\forall \pi \in \mathrm{Perm}, w'' \leq w.\ \pi(e) \in \mathcal{V}\,[\![A]\!]\,\rho\;\pi(w'')$.
  By induction $\gamma' \in \mathrm{Env}(\Gamma')\;w'$.
  Since $w' \leq w$, it follows $\forall \pi \in \mathrm{Perm}, w'' \leq w'.\ \pi(e) \in \mathcal{V}\,[\![A]\!]\,\rho\;\pi(w'')$.
  Hence $(\gamma', e/x) \in \mathrm{Env}(\Gamma', x : X\ \mathsf{now})\;w'$.
- Case $\Gamma = \Gamma', x : A\ \mathsf{later}$:
  Hence $\gamma = (\gamma', e/x) \in \mathrm{Env}(\Gamma', x : X\ \mathsf{later})\;w$.
  Hence $\gamma' \in \mathrm{Env}(\Gamma')\;w$ and $\forall \pi \in \mathrm{Perm}, w'' \leq w.\ \pi(e) \in \mathcal{L}\,[\![A]\!]\,\rho\;\pi(w'')$.
  By induction $\gamma' \in \mathrm{Env}(\Gamma')\;w'$.
  Since $w' \leq w$, it follows $\forall \pi \in \mathrm{Perm}, w'' \leq w'.\ \pi(e) \in \mathcal{L}\,[\![A]\!]\,\rho\;\pi(w'')$.
  Hence $(\gamma', e/x) \in \mathrm{Env}(\Gamma', x : X\ \mathsf{later})\;w'$.

- Case $\Gamma = \Gamma', x : A\ \mathsf{stable}$:
  Hence $\gamma = (\gamma', e/x) \in \mathrm{Env}(\Gamma', x : X\ \mathsf{stable})\ w$.
  Hence $\gamma' \in \mathrm{Env}(\Gamma')\ w$ and $\forall \pi \in \mathrm{Perm}, w'' \leq w.\ \pi(e) \in \mathcal{E}\,[\![A]\!]\,\rho\ \pi(w''.n, \cdot, \top)$.
  By induction $\gamma' \in \mathrm{Env}(\Gamma')\ w'$.
  Since $w' \leq w$, it follows $\forall \pi \in \mathrm{Perm}, w'' \leq w'.\ \pi(e) \in \mathcal{E}\,[\![A]\!]\,\rho\ \pi(w''.n, \cdot, \top)$.
  Hence $(\gamma', e/x) \in \mathrm{Env}(\Gamma', x : X\ \mathsf{stable})\ w'$.

□

Lemma 15 (Renaming for Environments)**.** *If* $\pi \in \mathrm{Perm}$ *and* $\gamma \in \mathrm{Env}(A)\ w$ *then* $\pi(\gamma) \in \mathrm{Env}(A)\ \pi(w)$.

Proof. We proceed by induction on Γ.
Assume we have $\pi \in \mathrm{Perm}$ and $\gamma \in \mathrm{Env}(\Gamma)\ w$.

- Case $\Gamma = \cdot$:
  Immediate

- Case $\Gamma = \Gamma', x : A\ \mathsf{now}$:
  Hence $\gamma = (\gamma', e/x) \in \mathrm{Env}(\Gamma', x : A\ \mathsf{now})\ w$.
  So $\gamma' \in \mathrm{Env}(\Gamma')\ w$ and $\forall w' \leq w.\ e \in \mathcal{E}\,[\![A]\!]\,\rho\ w'$.
  By induction, $\pi(\gamma') \in \mathrm{Env}(\Gamma')\ \pi(w)$.
  Assume $w'' \leq \pi(w)$.
  Then $\pi^{-1}(w'') \leq w$.
  Hence $e \in \mathcal{E}\,[\![A]\!]\,\rho\ \pi^{-1}(w'')$.
  By renaming lemma, $\pi(e) \in \mathcal{E}\,[\![A]\!]\,\rho\ \pi(\pi^{-1}(w''))$.
  Hence $\pi(e) \in \mathcal{E}\,[\![A]\!]\,\rho\ w''$.
  Hence $\forall w'' \leq \pi(w).\ \pi(e) \in \mathcal{E}\,[\![A]\!]\,\rho\ w''$.
  Hence $(\pi(\gamma'), \pi(e)/x) \in \mathrm{Env}(\Gamma', x : A\ \mathsf{now})\ \pi(w)$.
  Hence $\pi(\gamma) \in \mathrm{Env}(\Gamma)\ \pi(w)$.

- Case $\Gamma = \Gamma', x : A\ \mathsf{later}$:
  Hence $\gamma = (\gamma', e/x) \in \mathrm{Env}(\Gamma', x : A\ \mathsf{later})\ w$.
  So $\gamma' \in \mathrm{Env}(\Gamma')\ w$ and $\forall w' \leq w.\ e \in \mathcal{L}\,[\![A]\!]\,\rho\ w'$.
  By induction, $\pi(\gamma') \in \mathrm{Env}(\Gamma')\ \pi(w)$.
  Assume $w'' \leq \pi(w)$.
  Then $\pi^{-1}(w'') \leq w$.
  Hence $e \in \mathcal{L}\,[\![A]\!]\,\rho\ \pi^{-1}(w'')$.
  By renaming lemma, $\pi(e) \in \mathcal{L}\,[\![A]\!]\,\rho\ \pi(\pi^{-1}(w''))$.
  Hence $\pi(e) \in \mathcal{L}\,[\![A]\!]\,\rho\ w''$.
  Hence $\forall w'' \leq \pi(w).\ \pi(e) \in \mathcal{L}\,[\![A]\!]\,\rho\ w''$.
  Hence $(\pi(\gamma'), \pi(e)/x) \in \mathrm{Env}(\Gamma', x : A\ \mathsf{later})\ \pi(w)$.
  Hence $\pi(\gamma) \in \mathrm{Env}(\Gamma)\ \pi(w)$.

- Case $\Gamma = \Gamma', x : A\ \mathsf{stable}$:
  Hence $\gamma = (\gamma', e/x) \in \mathrm{Env}(\Gamma', x : A\ \mathsf{stable})\ w$.
  So $\gamma' \in \mathrm{Env}(\Gamma')\ w$ and $\forall w' \leq w.\ e \in \mathcal{E}\,[\![A]\!]\,\rho\ (w'.n, \cdot, \top)$.
  By induction, $\pi(\gamma') \in \mathrm{Env}(\Gamma')\ \pi(w)$.
  Assume $w'' \leq \pi(w)$.
  Then $\pi^{-1}(w'') \leq w$.
  Hence $e \in \mathcal{E}\,[\![A]\!]\,\rho\ (\pi^{-1}(w'').n, \cdot, \top)$.
  By renaming lemma, $\pi(e) \in \mathcal{E}\,[\![A]\!]\,\rho\ \pi(\pi^{-1}(w'').n, \cdot, \top)$.
  But $\pi(\pi^{-1}(w'').n, \cdot, \top) = (w''.n, \cdot, \top)$.
  Hence $\pi(e) \in \mathcal{E}\,[\![A]\!]\,\rho\ (w''.n, \cdot, \top)$.

Hence $\forall w'' \leq \pi(w).\ \pi(e) \in \mathcal{E}\,[\![A]\!]\,\rho\ (w''.n, \cdot, \top)$.
Hence $(\pi(\gamma'), \pi(e)/x) \in \mathrm{Env}(\Gamma', x : A\ \mathsf{stable})\ \pi(w)$.
Hence $\pi(\gamma) \in \mathrm{Env}(\Gamma)\ \pi(w)$.

□

Lemma 16 (Environment Shift). *Suppose* $\gamma \in \mathrm{Env}(\Gamma)\ w$. *Then:*

1. $\gamma^{\square}_{\Gamma} \in \mathrm{Env}(\Gamma^{\square})\ (w.n, \cdot, \top)$.
2. *If* $w = (n+1, \sigma, a)$ *and* $\sigma \Longrightarrow \sigma'$, *then* $\gamma^{\bullet}_{\Gamma} \in \mathrm{Env}(\Gamma^{\bullet})\ (n, \sigma', a)$.

Proof. We proceed as follows:

1. We prove this by induction on Γ.
   Assume $\gamma \in \mathrm{Env}(\Gamma)\ w$.
   Now analyse Γ:

   - Case $\Gamma = \cdot$:
     Immediate.
   - Case $\Gamma = \Gamma', x : A\ \mathsf{now}$:
     Then we know $\gamma = (\gamma', e/x) \in \mathrm{Env}(\Gamma', x : A\ \mathsf{now})\ w$.
     So $\gamma' \in \mathrm{Env}(\Gamma')\ w$.
     By induction, $\gamma'^{\square}_{\Gamma'} \in \mathrm{Env}(\Gamma'^{\square})\ (w.n, \cdot, \top)$.
     By definition $(\Gamma', x : A\ \mathsf{now})^{\square} = \Gamma'^{\square}$.
     By definition $(\gamma', e/x)^{\square}_{\Gamma} = \gamma'^{\square}_{\Gamma'}$.
     Hence $\gamma^{\square}_{\Gamma} \in \mathrm{Env}(\Gamma^{\square})\ (w.n, \cdot, \top)$.
   - Case $\Gamma = \Gamma', x : A\ \mathsf{later}$:
     Then we know $\gamma = (\gamma', e/x) \in \mathrm{Env}(\Gamma', x : A\ \mathsf{later})\ w$.
     So $\gamma' \in \mathrm{Env}(\Gamma')\ w$.
     By induction, $\gamma'^{\square}_{\Gamma'} \in \mathrm{Env}(\Gamma'^{\square})\ (w.n, \cdot, \top)$.
     By definition $(\Gamma', x : A\ \mathsf{later})^{\square} = \Gamma'^{\square}$.
     By definition $(\gamma', e/x)^{\square}_{\Gamma} = \gamma'^{\square}_{\Gamma'}$.
     Hence $\gamma^{\square}_{\Gamma} \in \mathrm{Env}(\Gamma^{\square})\ (w.n, \cdot, \top)$.
   - Case $\Gamma = \Gamma', x : A\ \mathsf{stable}$:
     Then we know $\gamma = (\gamma', e/x) \in \mathrm{Env}(\Gamma', x : A\ \mathsf{stable})\ w$.
     So $\gamma' \in \mathrm{Env}(\Gamma')\ w$ and $\forall w' \leq w.\ e \in \mathcal{E}\,[\![A]\!]\,\rho\ (w'.n, \cdot, \top)$.
     By induction, $\gamma'^{\square}_{\Gamma'} \in \mathrm{Env}(\Gamma'^{\square})\ (w.n, \cdot, \top)$.
     Assume $w' \leq (w.n, \cdot, \top)$.
     Note that if $w' \leq (w.n, \cdot, \top)$, then $(w'.n, w.\sigma, w.a) \leq w$.
     Hence $e \in \mathcal{E}\,[\![A]\!]\,\rho\ (w'.n, \cdot, \top)$.
     Hence $\forall w' \leq (w.n, \cdot, \top).\ e \in \mathcal{E}\,[\![A]\!]\,\rho\ (w'.n, \cdot, \top)$.
     Hence $(\gamma'^{\square}_{\Gamma'}, e/x) \in \mathrm{Env}(\Gamma'^{\square}, x : A\ \mathsf{stable})\ (w.n, \cdot, \top)$.
     By definition $(\Gamma', x : A\ \mathsf{stable})^{\square} = \Gamma'^{\square}, x : A\ \mathsf{stable}$.
     By definition $(\gamma', e/x)^{\square}_{\Gamma', x:A\ \mathsf{stable}} = (\gamma'^{\square}_{\Gamma'}, e/x)$.
     Hence $\gamma^{\square}_{\Gamma} \in \mathrm{Env}(\Gamma^{\square})\ (w.n, \cdot, \top)$.

2. We proceed by induction on Γ.
   Assume $w = (n+1, \sigma, a)$ and $\sigma \Longrightarrow \sigma'$. Let $w' = (n, \sigma', a)$.
   Now analyze Γ.

   - Case $\Gamma = \cdot$:
     Immediate.

- Case $\Gamma = \Gamma, x : A$ later:
  Then $\gamma = (\gamma', e/x) \in \text{Env}(\Gamma)\ w$.
  Hence $\gamma' \in \text{Env}(\Gamma')\ w$ and $\forall w'' \leq (n+1, \sigma, a).\ e \in \mathcal{L}\,[\![A]\!]\,\rho\ w''$.
  By induction, $\gamma'^{\bullet}_{\Gamma'} \in \text{Env}(\Gamma'^{\bullet})\ w'$.
  By instantiation, $e \in \mathcal{L}\,[\![A]\!]\,\rho\ (n+1, \sigma, a)$.
  By definition, $\forall w'' \leq (n, \sigma', a).\ e \in \mathcal{E}\,[\![A]\!]\,\rho\ w''$.
  Hence $\forall w'' \leq w'.\ e \in \mathcal{E}\,[\![A]\!]\,\rho\ w''$.
  Hence $(\gamma'^{\bullet}_{\Gamma'}, e/x) \in \text{Env}(\Gamma'^{\bullet}, x : A\ \mathsf{now})\ w'$.
  By definition, $(\Gamma', x : A\ \mathsf{later})^{\bullet} = \Gamma'^{\bullet}, x : A\ \mathsf{now}$.
  By definition, $(\gamma', e/x)^{\bullet}_{\Gamma', x:A\ \mathsf{later}} = (\gamma'^{\bullet}_{\Gamma'}, e/x)$.
  Hence $\gamma^{\bullet}_{\Gamma} \in \text{Env}(\Gamma^{\bullet})\ w'$.
- Case $\Gamma = \Gamma, x : A$ now:
  Then $\gamma = (\gamma', e/x) \in \text{Env}(\Gamma)\ w$.
  Hence $\gamma' \in \text{Env}(\Gamma')\ w$ and $\forall w'' \leq (n+1, \sigma, a).\ e \in \mathcal{L}\,[\![A]\!]\,\rho\ w''$.
  By induction, $\gamma'^{\bullet}_{\Gamma'} \in \text{Env}(\Gamma'^{\bullet})\ w'$.
  By definition, $(\Gamma', x : A\ \mathsf{now})^{\bullet} = \Gamma'^{\bullet}$.
  By definition, $(\gamma', e/x)^{\bullet}_{\Gamma', x:A\ \mathsf{now}} = \gamma'^{\bullet}_{\Gamma'}$.
  Hence $\gamma^{\bullet}_{\Gamma} \in \text{Env}(\Gamma^{\bullet})\ w'$.
- Case $\Gamma = \Gamma, x : A$ stable:
  Then $\gamma = (\gamma', e/x) \in \text{Env}(\Gamma)\ w$.
  Hence $\gamma' \in \text{Env}(\Gamma')\ w$ and $\forall w'' \leq (n+1, \sigma, a).\ e \in \mathcal{E}\,[\![A]\!]\,\rho\ (w''.n, \cdot, \top)$.
  By induction, $\gamma'^{\bullet}_{\Gamma'} \in \text{Env}(\Gamma'^{\bullet})\ w'$.
  Assume $w'' \leq w'$.
  Note that $(n, \sigma, a) \leq (n+1, \sigma, a)$.
  Since $w'' \leq w'$, then $(w''.n, \sigma, a) \leq (n+1, \sigma, a)$.
  Hence $e \in \mathcal{E}\,[\![A]\!]\,\rho\ (w''.n, \cdot, \top)$.
  Hence $\forall w'' \leq w'.\ e \in \mathcal{E}\,[\![A]\!]\,\rho\ (w''.n, \cdot, \top)$.
  Hence $(\gamma'^{\bullet}_{\Gamma'}, e/x) \in \text{Env}(\Gamma'^{\bullet}, x : A\ \mathsf{stable})\ w'$.
  By definition, $(\Gamma', x : A\ \mathsf{stable})^{\bullet} = \Gamma'^{\bullet}, x : A\ \mathsf{now}$.
  By definition, $(\gamma', e/x)^{\bullet}_{\Gamma', x:A\ \mathsf{stable}} = (\gamma'^{\bullet}_{\Gamma'}, e/x)$.
  Hence $\gamma^{\bullet}_{\Gamma} \in \text{Env}(\Gamma^{\bullet})\ w'$.

□

Lemma 17 (Stability). *If* A stable *and* $v \in \mathcal{V}\,[\![A]\!]\,\rho\ w$, *then* $v \in \mathcal{V}\,[\![A]\!]\,\rho\ (w.n, \cdot, \top)$.

Proof. This follows from an induction on the derivation of A stable.

- Case $\square A$:
  Assume $\mathsf{stable}(v) \in \mathcal{V}\,[\![\square A]\!]\,\rho\ w$.
  Then $v \in \mathcal{V}\,[\![A]\!]\,\rho\ (w.n, \cdot, \top)$.
  Hence $\mathsf{stable}(v) \in \mathcal{V}\,[\![\square A]\!]\,\rho\ (w.n, \cdot, \top)$.
- Case $A + B$:
  By inversion, A stable and B stable.
  Assume $v \in \mathcal{V}\,[\![A + B]\!]\,\rho\ w$.
  Either $v = \mathsf{inl}\,v' \wedge v' \in \mathcal{V}\,[\![A]\!]\,\rho\ w$ or $v = \mathsf{inr}\,v' \wedge v' \in \mathcal{V}\,[\![B]\!]\,\rho\ w$.

  Suppose $v = \mathsf{inl}\,v'$ and $v' \in \mathcal{V}\,[\![A]\!]\,\rho\ w$.
  By induction, $v' \in \mathcal{V}\,[\![A]\!]\,\rho\ (w.n, \cdot, \top)$.
  Hence $\mathsf{inl}\,v' \in \mathcal{V}\,[\![A + B]\!]\,\rho\ (w.n, \cdot, \top)$.

  Suppose $v = \mathsf{inr}\,v'$ and $v' \in \mathcal{V}\,[\![B]\!]\,\rho\ w$.
  By induction, $v' \in \mathcal{V}\,[\![B]\!]\,\rho\ (w.n, \cdot, \top)$.
  Hence $\mathsf{inr}\,v' \in \mathcal{V}\,[\![A + B]\!]\,\rho\ (w.n, \cdot, \top)$.

- Case $A \times B$:
  By inversion, A stable and B stable.
  Assume $(v, v') \in \mathcal{V}[\![A \times B]\!]\, \rho\, w$.
  Then $v \in \mathcal{V}[\![A]\!]\, \rho\, w$ and $v' \in \mathcal{V}[\![B]\!]\, \rho\, w$.
  By induction, $v \in \mathcal{V}[\![A]\!]\, \rho\, (w.n, \cdot, \top)$.
  By induction, $v' \in \mathcal{V}[\![B]\!]\, \rho\, (w.n, \cdot, \top)$.
  Hence $(v, v') \in \mathcal{V}[\![A \times B]\!]\, \rho\, (w.n, \cdot, \top)$.

$\square$

Theorem 1 (Fundamental Property)**.** *The following properties hold:*

1. *If* $\Gamma \vdash e : A\ \mathsf{later}$ *and* $\gamma \in \mathrm{Env}(\Gamma)\ w$, *then* $\gamma(e) \in \mathcal{L}[\![A]\!] \cdot w$.
2. *If* $\Gamma \vdash e : A\ \mathsf{stable}$ *and* $\gamma \in \mathrm{Env}(\Gamma)\ w$, *then* $\gamma(e) \in \mathcal{E}[\![A]\!] \cdot (w.n, \cdot, \top)$.
3. *If* $\Gamma \vdash e : A\ \mathsf{now}$ *and* $\gamma \in \mathrm{Env}(\Gamma)\ w$, *then* $\gamma(e) \in \mathcal{E}[\![A]\!] \cdot w$.

Proof. We proceed as follows:

1. Assume $\Gamma \vdash e : A\ \mathsf{later}$ and $\gamma \in \mathrm{Env}(\Gamma)\ w$.
   Now consider what the world is:
   - $w = (0, \sigma, a)$.
     In this case, $\gamma(e) \in \mathcal{L}[\![A]\!] \cdot (0, \sigma, a)$ by definition of $\mathcal{L}[\![A]\!] \cdot w$.
   - $w = (n+1, \sigma, a)$.
     We know $\sigma \in \mathrm{Heap}_{n+1}$, so $\sigma \Longrightarrow \sigma'$ such that $\sigma' \in \mathrm{Heap}_n$.
     Hence $w' \triangleq (n, \sigma', a) \in \mathrm{World}$.
     By inversion, we have $\Gamma^{\bullet} \vdash e : A\ \mathsf{now}$.
     By environment shift, we have $\gamma^{\bullet}_{\Gamma} \in \mathrm{Env}(\Gamma)\ w'$.
     Assume $w'' \le w'$.
     By Kripke monotoncity, $\gamma^{\bullet}_{\Gamma} \in \mathrm{Env}(\Gamma)\ w''$.
     Hence by fundamental theorem, $\gamma^{\bullet}_{\Gamma}(e) \in \mathcal{E}[\![A]\!] \cdot w''$.
     But by definition, $\gamma^{\bullet}_{\Gamma}(e) = \gamma(e)$, so $\gamma(e) \in \mathcal{E}[\![A]\!] \cdot w''$.
     Hence for all $w'' \le w'$, we have $\gamma(e) \in \mathcal{E}[\![A]\!] \cdot w''$.
     Hence $\gamma(e) \in \mathcal{L}[\![A]\!] \cdot w$.
2. Assume $\Gamma \vdash e : A\ \mathsf{stable}$ and $\gamma \in \mathrm{Env}(\Gamma)\ w$.
   By inversion, we know that $\Gamma^{\square} \vdash e : A\ \mathsf{now}$.
   By environment shift, we know that $\gamma^{\square}_{\Gamma} \in \mathrm{Env}(\Gamma^{\square})\ (w.n, \cdot, \top)$.
   By the the fundamental property, $\gamma^{\square}_{\Gamma}(e) \in \mathcal{E}[\![A]\!] \cdot (w.n, \cdot, \top)$.
   Note $\gamma^{\square}_{\Gamma}(e) = \gamma(e)$.
   Hence $\gamma(e) \in \mathcal{E}[\![A]\!] \cdot (w.n, \cdot, \top)$.
3. We proceed by induction on the typing derivation:
   - Case HYP:
     Assume $\gamma \in \mathrm{Env}(\Gamma)\ w$.
     We know $\Gamma \vdash x : A\ \mathsf{now}$.
     By inversion, we know that $x : A\ \mathsf{now} \in \Gamma$ or $x : A\ \mathsf{stable}$.
     - Suppose $x : A\ \mathsf{now} \in \Gamma$:
       Then by definition of $\mathrm{Env}(\Gamma)\ w$, $\forall w' \le w.\ \gamma(x) \in \mathcal{E}[\![A]\!] \cdot w'$.
       Hence $\gamma(x) \in \mathcal{E}[\![A]\!] \cdot w$.

– Suppose $x : A$ stable $\in \Gamma$:
Then by definition of $\mathrm{Env}(\Gamma)\ w$, $\forall w' \leq w.\ \gamma(x) \in \mathcal{E}[\![A]\!] \cdot (w'.n, \cdot, \top)$.
Hence $\langle \cdot; e\rangle \Downarrow \langle \cdot; v\rangle$ such that $v \in \mathcal{V}[\![A]\!] \cdot (w.n, \cdot, \top)$.
Note that $w \leq (w.n, \cdot, \top)$.
We want to show $e \in \mathcal{E}[\![A]\!] \cdot w$.
Assume $\sigma' \leq w.store$.
Note $(w.n, \sigma', w.a) \leq w$.
By uniformity, we know that $\langle \sigma'; e\rangle \Downarrow \langle \sigma'; v\rangle$.
Hence by Kripke monotoncity, $v \in \mathcal{V}[\![A]\!] \cdot (w.n, \sigma', w.a)$.
Take the existential witness σ'' to be σ' itself.
Vacuously, if $w.a = \top$, then $\sigma' = \sigma''$.
Hence $e \in \mathcal{E}[\![A]\!] \cdot w$.

- Case FIX:
Assume $\gamma \in \mathrm{Env}(\Gamma)\ w$.
We know $\Gamma \vdash \mathsf{fix}\, x.\ e : A$ now.
By inversion, we know that $\Gamma^{\square}, x : A$ later $\vdash e : A$ now.
Now, we know that $w = (n, \sigma, a)$.
By nested induction, we show for all $m \leq n, w' \leq (m, \cdot, \top).\ \gamma(\mathsf{fix}\, x.\ e) \in \mathcal{E}[\![A]\!] \cdot w'$.

– $m = 0$:
By Kripke monotonicity, we know that $\gamma \in \mathrm{Env}(\Gamma)\ (0, \sigma, a)$.
By environment shift, we know that $\gamma^{\square}_{\Gamma} \in \mathrm{Env}(\Gamma^{\square})\ (0, \cdot, \top)$.
Assume that $w' \leq (0, \cdot, \top)$.
By Kripke monotonicity, we know that $\gamma \in \mathrm{Env}(\Gamma)\ w'$.
Assume that $w'' \leq (0, \cdot, \top)$.
Note that $w''.n = 0$.
Then by definition, $\gamma(\mathsf{fix}\, x.\ e) \in \mathcal{L}[\![A]\!] \cdot w''$.
Hence $\forall w'' \leq w', \gamma(\mathsf{fix}\, x.\ e) \in \mathcal{L}[\![A]\!] \cdot w''$.
Hence we know that $(\gamma^{\square}_{\Gamma}, \gamma(\mathsf{fix}\, x.\ e)/x) \in \mathrm{Env}(\Gamma^{\square}, x : A\ \text{later})\ w'$.
By the fundamental lemma, we know $(\gamma^{\square}_{\Gamma}, \gamma(\mathsf{fix}\, x.\ e)/x)(e) \in \mathcal{E}[\![A]\!] \cdot w'$.
Note that $(\gamma^{\square}_{\Gamma}, \gamma(\mathsf{fix}\, x.\ e)/x)(e) = (\gamma, \gamma(\mathsf{fix}\, x.\ e)/x)(e)$.
Hence there is $v \in \mathcal{V}[\![A]\!] \cdot w'$ s.t. $\langle \cdot; (\gamma, \gamma(\mathsf{fix}\, x.\ e)/x)(e)\rangle \Downarrow \langle \cdot; v\rangle$.
Hence there is $v \in \mathcal{V}[\![A]\!] \cdot w'$ s.t. $\langle \cdot; \gamma([\mathsf{fix}\, x.\ e/x]e)\rangle \Downarrow \langle \cdot; v\rangle$.
Hence there is $v \in \mathcal{V}[\![A]\!] \cdot w'$ s.t. $\langle \cdot; \gamma(\mathsf{fix}\, x.\ e)\rangle \Downarrow \langle \cdot; v\rangle$.
Hence $\gamma(\mathsf{fix}\, x.\ e) \in \mathcal{E}[\![A]\!] \cdot w'$.
Hence $\forall w' \leq (0, \cdot, \top).\ \gamma(\mathsf{fix}\, x.\ e) \in \mathcal{E}[\![A]\!] \cdot w'$.

– $m = k + 1$:
By induction, we know that for all $i \leq k, w' \leq (i, \cdot, \top), \gamma(\mathsf{fix}\, x.\ e) \in \mathcal{E}[\![A]\!] \cdot w'$.
Now, we want to show that $\forall w' \leq (k + 1, \cdot, \top).\ \gamma(\mathsf{fix}\, x.\ e) \in \mathcal{L}[\![A]\!] \cdot w'$.
Assume $w' \leq (k + 1, \cdot, \top)$.
If $w'.n = 0$, the result is immediate.
If $w'.n = j + 1$, then we want to show
there is a σ'' such that $w'.store \Longrightarrow \sigma''$ and $\sigma'' \in \mathrm{Heap}_j$ and
that for all $w'' \leq (j, \sigma'', \top).\ \gamma(\mathsf{fix}\, x.\ e) \in \mathcal{E}[\![A]\!] \cdot w''$.
Since $w'.\sigma \in \mathrm{Heap}_{j+1}$, we know there is $w'.store \Longrightarrow \sigma''$ such that $\sigma'' \in \mathrm{Heap}_j$.
We want to show for all $w'' \leq (j, \sigma'', \top).\ \gamma(\mathsf{fix}\, x.\ e) \in \mathcal{E}[\![A]\!] \cdot w''$.
Assume $w'' \leq (j, \sigma'', \top)$.
Note that since $w'.n = j + 1 \leq k + 1$, we know that $j \leq k$.
Hence $w'' \leq (j, \sigma'', \top) \leq (j, \cdot, \top)$.
Hence by the induction hypothesis, $\gamma(\mathsf{fix}\, x.\ e) \in \mathcal{E}[\![A]\!] \cdot w''$.
Hence for all $w'' \leq (j, \sigma'', \top).\ \gamma(\mathsf{fix}\, x.\ e) \in \mathcal{E}[\![A]\!] \cdot w''$.
Hence $\forall w' \leq (k + 1, \cdot, \top).\ \gamma(\mathsf{fix}\, x.\ e) \in \mathcal{L}[\![A]\!] \cdot w'$.

Since $k + 1 \leq w.n$, by Kripke monotonicity, $\gamma \in \text{Env}(\Gamma)\ (k + 1, \sigma, a)$.
By environment shift, $\gamma^{\square}_{\Gamma} \in \text{Env}(\Gamma^{\square})\ (k + 1, \cdot, \top)$.
By definition, $(\gamma^{\square}_{\Gamma}, \gamma(\mathsf{fix}\, x.\ e)/x) \in \text{Env}(\Gamma^{\square}, x : A\ \mathsf{later})\ (k + 1, \cdot, \top)$.
By the fundamental property, $(\gamma^{\square}_{\Gamma}, \gamma(\mathsf{fix}\, x.\ e)/x)(e) \in \mathcal{E}\,[\![A]\!] \cdot (k + 1, \cdot, \top)$.
Note $(\gamma^{\square}_{\Gamma}, \gamma(\mathsf{fix}\, x.\ e)/x)(e) = (\gamma, \gamma(\mathsf{fix}\, x.\ e)/x)(e)$.
Hence $(\gamma, \gamma(\mathsf{fix}\, x.\ e)/x)(e) \in \mathcal{E}\,[\![A]\!] \cdot (k + 1, \cdot, \top)$.
So for every σ, there is a $v \in \mathcal{V}\,[\![A]\!] \cdot (k + 1, \cdot, \top)$ s.t. $\langle \sigma; (\gamma, \gamma(\mathsf{fix}\, x.\ e)/x)(e)\rangle \Downarrow \langle \sigma; v\rangle$.
Hence it follows that for every σ, there is a $v \in \mathcal{V}\,[\![A]\!] \cdot (k + 1, \cdot, \top)$ s.t. $\langle \sigma; \gamma(\mathsf{fix}\, x.\ e)\rangle \Downarrow \langle \sigma; v\rangle$.
Hence $\gamma(\mathsf{fix}\, x.\ e) \in \mathcal{E}\,[\![A]\!] \cdot (k + 1, \cdot, \top)$

Therefore we know that $\gamma(\mathsf{fix}\, x.\ e) \in \mathcal{E}\,[\![A]\!] \cdot (n, \cdot, \top)$.
We want to show $\gamma(\mathsf{fix}\, x.\ e) \in \mathcal{E}\,[\![A]\!] \cdot (n, \sigma, a)$.
Assume we have $\sigma' \leq \sigma$.
Note that $\sigma' \leq \cdot$.
Hence we know that $\langle \sigma'; \gamma(\mathsf{fix}\, x.\ e)\rangle \Downarrow \langle \sigma'; v\rangle$ such that $v \in \mathcal{V}\,[\![A]\!] \cdot (n, \cdot, \top)$.
By Kripke monotonicity, we know that $v \in \mathcal{V}\,[\![A]\!] \cdot (n, \sigma', a)$.
Take $\sigma'' = \sigma'$.
Hence $\sigma'' \leq \sigma'$, such that $\langle \sigma'; \gamma(\mathsf{fix}\, x.\ e)\rangle \Downarrow \langle \sigma''; v\rangle$ and $v \in \mathcal{V}\,[\![A]\!] \cdot (n, \sigma'', a)$.
Hence $\gamma(\mathsf{fix}\, x.\ e) \in \mathcal{E}\,[\![A]\!] \cdot (n, \sigma, a)$.

- Case +LI:
Assume $\gamma \in \text{Env}(\Gamma)\ w$.
We know $\Gamma \vdash \mathsf{inl}\, e : A + B\ \mathsf{now}$.
By inversion, $\Gamma \vdash e : A\ \mathsf{now}$.
By fundamental lemma, $\gamma(e) \in \mathcal{E}\,[\![A]\!] \cdot w$.
Hence there is a v and $\sigma' \leq w.\sigma$ such that $\langle w.\sigma; e\rangle \Downarrow \langle \sigma'; v\rangle$ such that $v \in \mathcal{V}\,[\![A]\!] \cdot (w.n, \sigma', w.a)$.
Hence $\mathsf{inl}\, v \in \mathcal{V}\,[\![A + B]\!] \cdot (w.n, \sigma', w.a)$.
Note that by rule $\langle w.\sigma; \mathsf{inl}\, e\rangle \Downarrow \langle \sigma'; \mathsf{inl}\, v\rangle$.
Hence $\mathsf{inl}\, e \in \mathcal{E}\,[\![A + B]\!] \cdot w$.

- Case +RI:
Assume $\gamma \in \text{Env}(\Gamma)\ w$.
We know $\Gamma \vdash \mathsf{inr}\, e : A + B\ \mathsf{now}$.
By inversion, $\Gamma \vdash e : B\ \mathsf{now}$.
By fundamental lemma, $\gamma(e) \in \mathcal{E}\,[\![B]\!] \cdot w$.
Hence there is a v and $\sigma' \leq w.\sigma$ such that $\langle w.\sigma; e\rangle \Downarrow \langle \sigma'; v\rangle$ such that $v \in \mathcal{V}\,[\![B]\!] \cdot (w.n, \sigma', w.a)$.
Hence $\mathsf{inr}\, v \in \mathcal{V}\,[\![A + B]\!] \cdot (w.n, \sigma', w.a)$.
Note that by rule $\langle w.\sigma; \mathsf{inr}\, e\rangle \Downarrow \langle \sigma'; \mathsf{inr}\, v\rangle$.
Hence $\mathsf{inr}\, e \in \mathcal{E}\,[\![A + B]\!] \cdot w$.

- Case +E:
Assume $\gamma \in \text{Env}(\Gamma)\ w$.
We know that $\Gamma \vdash \mathsf{case}(e, \mathsf{inl}\, x \to e_1, \mathsf{inr}\, y \to e_2) : C\ \mathsf{now}$.
By inversion, $\Gamma \vdash e : A + B\ \mathsf{now}$ and $\Gamma, x : A\ \mathsf{now} \vdash e_1 : C\ \mathsf{now}$ and $\Gamma, y : B\ \mathsf{now} \vdash e_2 : C\ \mathsf{now}$.
By induction, $\gamma(e) \in \mathcal{E}\,[\![A + B]\!] \cdot w$.
Hence there is a v and $\sigma' \leq w.\sigma$ such that $v \in \mathcal{V}\,[\![A + B]\!] \cdot (w.n, \sigma', w.a)$.
Suppose $v = \mathsf{inl}\, v'$ and $v' \in \mathcal{V}\,[\![A]\!] \cdot (w.n, \sigma', w.a)$.
By weakening, we know that $\gamma \in \text{Env}(\Gamma)\ (w.n, \sigma', w.a)$.
By value inclusion, we know that $v' \in \mathcal{E}\,[\![A]\!] \cdot (w.n, \sigma', w.a)$.
Hence $(\gamma, v'/x) \in \text{Env}(\Gamma, x : A\ \mathsf{now})\ (w.n, \sigma', w.a)$.
By induction, $(\gamma, v'/x)e_1 \in \mathcal{E}\,[\![C]\!] \cdot (w.n, \sigma', w.a)$.
Note that $(\gamma, v'/x)e_1 = [v'/x]\gamma(e_1)$, so $[v'/x]\gamma(e_1) \in \mathcal{E}\,[\![C]\!] \cdot (w.n, \sigma', w.a)$.
Therefore there is a v'' and $\sigma'' \leq \sigma'$ such that $\langle \sigma'; [v'/x]\gamma(e_1)\rangle \Downarrow \langle \sigma''; v''\rangle$ and $v'' \in \mathcal{V}\,[\![C]\!] \cdot (w.n, \sigma'', w.a)$.

Note $\langle w.\sigma; \mathsf{case}(\gamma(e), \mathsf{inl}\, x \to \gamma(e_1), \mathsf{inr}\, y \to \gamma(e_2))\rangle \Downarrow \langle \sigma''; v''\rangle$.
Hence $\mathsf{case}(\gamma(e), \mathsf{inl}\, x \to \gamma(e_1), \mathsf{inr}\, y \to \gamma(e_2)) \in \mathcal{E}\,[\![C]\!] \cdot w$.
Note that $\mathsf{case}(\gamma(e), \mathsf{inl}\, x \to \gamma(e_1), \mathsf{inr}\, y \to \gamma(e_2)) = \gamma(\mathsf{case}(e, \mathsf{inl}\, x \to e_1, \mathsf{inr}\, y \to e_2))$.
Hence $\gamma(\mathsf{case}(e, \mathsf{inl}\, x \to e_1, \mathsf{inr}\, y \to e_2)) \in \mathcal{E}\,[\![C]\!] \cdot w$.
Suppose $v = \mathsf{inr}\, v'$ and $v' \in \mathcal{V}\,[\![B]\!] \cdot (w.n, \sigma', w.a)$.
By weakening, we know that $\gamma \in \mathrm{Env}(\Gamma)\ (w.n, \sigma', w.a)$.
By value inclusion, we know that $v' \in \mathcal{E}\,[\![B]\!] \cdot (w.n, \sigma', w.a)$.
Hence $(\gamma, v'/y) \in \mathrm{Env}(\Gamma, x : B\ \mathsf{now})\ (w.n, \sigma', w.a)$.
By induction, $(\gamma, v'/y)e_2 \in \mathcal{E}\,[\![C]\!] \cdot (w.n, \sigma', w.a)$.
Note that $(\gamma, v'/y)e_2 = [v'/y]\gamma(e_2)$, so $[v'/y]\gamma(e_2) \in \mathcal{E}\,[\![C]\!] \cdot (w.n, \sigma', w.a)$.
Therefore there is a v'' and $\sigma'' \leq \sigma'$ such that $\langle \sigma'; [v'/y]\gamma(e_2)\rangle \Downarrow \langle \sigma''; v''\rangle$ and $v'' \in \mathcal{V}\,[\![C]\!] \cdot (w.n, \sigma'', w.a)$.
Note $\langle w.\sigma; \mathsf{case}(\gamma(e), \mathsf{inl}\, x \to \gamma(e_1), \mathsf{inr}\, y \to \gamma(e_2))\rangle \Downarrow \langle \sigma''; v''\rangle$.
Hence $\mathsf{case}(\gamma(e), \mathsf{inl}\, x \to \gamma(e_1), \mathsf{inr}\, y \to \gamma(e_2)) \in \mathcal{E}\,[\![C]\!] \cdot w$.
Note that $\mathsf{case}(\gamma(e), \mathsf{inl}\, x \to \gamma(e_1), \mathsf{inr}\, y \to \gamma(e_2)) = \gamma(\mathsf{case}(e, \mathsf{inl}\, x \to e_1, \mathsf{inr}\, y \to e_2))$.
Hence $\gamma(\mathsf{case}(e, \mathsf{inl}\, x \to e_1, \mathsf{inr}\, y \to e_2)) \in \mathcal{E}\,[\![C]\!] \cdot w$.

- Case $\times$I:
  Assume $\gamma \in \mathrm{Env}(\Gamma)\ w$.
  We know $\Gamma \vdash (e_1, e_2) : A \times B\ \mathsf{now}$.
  By inversion, we know $\Gamma \vdash e_1 : A\ \mathsf{now}$ and $\Gamma \vdash e_2 : B\ \mathsf{now}$.
  By induction, we know that $\gamma(e_1) \in \mathcal{E}\,[\![A]\!] \cdot w$.
  Hence there is a v_1 and $\sigma' \leq w.\sigma$ such that $\langle w.\sigma; \gamma(e_1)\rangle \Downarrow \langle \sigma'; v_1\rangle$ and $v_1 \in \mathcal{V}\,[\![A]\!] \cdot (w.n, \sigma', w.a)$.
  By weakening, $\gamma \in \mathrm{Env}(\Gamma)\ (w.n, \sigma', w.a)$.
  By induction, $\gamma(e_2) \in \mathcal{E}\,[\![B]\!] \cdot (w.n, \sigma', w.a)$.
  Hence there is a v_2 and $\sigma'' \leq \sigma'$ such that $\langle \sigma'; \gamma(e_2)\rangle \Downarrow \langle \sigma''; v_2\rangle$ and $v_2 \in \mathcal{V}\,[\![B]\!] \cdot (w.n, \sigma'', w.a)$.
  By weakening, $v_1 \in \mathcal{V}\,[\![A]\!] \cdot (w.n, \sigma'', w.a)$.
  Hence $(v_1, v_2) \in \mathcal{V}\,[\![A \times B]\!] \cdot (w.n, \sigma'', w.a)$.
  By rule $\langle w.\sigma; (\gamma(e_1), \gamma(e_2))\rangle \Downarrow \langle \sigma''; (v_1, v_2)\rangle$.
  Hence $(\gamma(e_1), \gamma(e_2)) \in \mathcal{E}\,[\![A \times B]\!] \cdot w$.
  Note $(\gamma(e_1), \gamma(e_2)) = \gamma((e_1, e_2))$.
  Hence $\gamma((e_1, e_2)) \in \mathcal{E}\,[\![A \times B]\!] \cdot w$.
- Case $\times$LE:
  Assume $\gamma \in \mathrm{Env}(\Gamma)\ w$.
  We know $\Gamma \vdash \mathsf{fst}\, e : A\ \mathsf{now}$.
  By inversion, $\Gamma \vdash e : A \times B\ \mathsf{now}$.
  By induction, $\gamma(e) \in \mathcal{E}\,[\![A \times B]\!] \cdot w$.
  Hence there is a (v_1, v_2) and $\sigma' \leq \sigma$ such that $\langle w.\sigma; e\rangle \Downarrow \langle \sigma'; (v_1, v_2)\rangle$ and $(v_1, v_2) \in \mathcal{V}\,[\![A \times B]\!] \cdot (w.n, \sigma', w.a)$.
  Hence $v_1 \in \mathcal{V}\,[\![A]\!] \cdot (w.n, \sigma', w.a)$.
  By rule, $\langle w.\sigma; \mathsf{fst}\, \gamma(e)\rangle \Downarrow \langle \sigma'; v_1\rangle$.
  Hence $\mathsf{fst}\, \gamma(e) \in \mathcal{E}\,[\![A]\!] \cdot w$.
  Note $\mathsf{fst}\, \gamma(e) = \gamma(\mathsf{fst}\, e)$.
  Hence $\gamma(\mathsf{fst}\, e) \in \mathcal{E}\,[\![A]\!] \cdot w$.
- Case $\times$RE:
  Assume $\gamma \in \mathrm{Env}(\Gamma)\ w$.
  We know $\Gamma \vdash \mathsf{snd}\, e : B\ \mathsf{now}$.
  By inversion, $\Gamma \vdash e : A \times B\ \mathsf{now}$.
  By induction, $\gamma(e) \in \mathcal{E}\,[\![A \times B]\!] \cdot w$.
  Hence there is a (v_1, v_2) and $\sigma' \leq \sigma$ such that $\langle w.\sigma; e\rangle \Downarrow \langle \sigma'; (v_1, v_2)\rangle$ and $(v_1, v_2) \in \mathcal{V}\,[\![A \times B]\!] \cdot (w.n, \sigma', w.a)$.
  Hence $v_1 \in \mathcal{V}\,[\![B]\!] \cdot (w.n, \sigma', w.a)$.
  By rule, $\langle w.\sigma; \mathsf{snd}\, \gamma(e)\rangle \Downarrow \langle \sigma'; v_1\rangle$.
  Hence $\mathsf{snd}\, \gamma(e) \in \mathcal{E}\,[\![B]\!] \cdot w$.
  Note $\mathsf{snd}\, \gamma(e) = \gamma(\mathsf{snd}\, e)$.

Hence $\gamma(\mathsf{snd}\, e) \in \mathcal{E}\,[\![B]\!] \cdot w$.

- Case $\bullet$I:
  Assume $\gamma \in \text{Env}(\Gamma)\ w$.
  We know $\Gamma \vdash \delta_{e'}(e) : \bullet A\ \mathsf{now}$.
  By inversion, we know $\Gamma \vdash e : A\ \mathsf{later}$ and $\Gamma \vdash e' : \mathsf{alloc}\ \mathsf{now}$.
  By fundamental lemma, we know $\gamma(e') \in \mathcal{E}\,[\![\mathsf{alloc}]\!] \cdot w$.
  Hence there is a $\sigma' \leq w.\sigma$ such that $\langle w.\sigma; e'\rangle \Downarrow \langle \sigma'; \diamond\rangle$ and $\diamond \in \mathcal{V}\,[\![\mathsf{alloc}]\!] \cdot (w.n, \sigma', w.a)$.
  Hence we know that $w.a = \bot$.
  Let $w' = (w.n, \sigma', w.a)$, and note that $w' \leq w$.
  Assume $\pi \in \text{Perm}$ and $w'' \leq w'$.
  By Kripke monotonicity and renaming, $\pi(\gamma) \in \text{Env}(\Gamma)\ \pi(w'')$.
  By fundamental lemma, $(\pi(\gamma))(e) \in \mathcal{L}\,[\![A]\!] \cdot \pi(w')$.
  Since e has no free locations in it, $(\pi(\gamma))(e) = \pi(\gamma(e))$.
  Hence $\forall \pi \in \text{Perm}$ and $w'' \leq w'$, we know $\pi(\gamma(e)) \in \mathcal{L}\,[\![A]\!] \cdot \pi(w'')$.
  Choose $l \notin \text{dom}(\sigma')$ and let $\sigma'' = w'.\sigma, l : e\ \mathsf{later}$
  Now, we will show that $\sigma'' \in \text{Heap}_{w.n}$.
  - Suppose $w.n = 0$:
    Then $\sigma'' \in \text{Heap}_0$ immediately.
  - Suppose $w.n = k+1$:
    Then we know that there is a $\hat{\sigma}'$ such that $w'.\sigma \Longrightarrow \hat{\sigma}'$ and $\hat{\sigma}' \in \text{Heap}_k$.
    Due to the permutability and renaming properties, we can assume $l \notin \text{dom}(\hat{\sigma}')$.
    Since $\gamma(e) \in \mathcal{L}\,[\![A]\!] \cdot w'$, we know $\langle \hat{\sigma}'; \gamma(e)\rangle \Downarrow \langle \hat{\sigma}''; v\rangle$, with $\hat{\sigma}'' \leq \hat{\sigma}'$.
    Due to the permutability and renaming properties, we can assume $l \notin \text{dom}(\hat{\sigma}'')$.
    Therefore $(\hat{\sigma}'', l : v\ \mathsf{now}) \in \text{Heap}_k$.
    Note $\sigma'' \Longrightarrow (\hat{\sigma}'', l : v\ \mathsf{now})$.
    So $\sigma'' \in \text{Heap}_{k+1}$

  Since $\sigma'' \in \text{Heap}_{w.n}$, it follows $\sigma'' \leq w'.\sigma$.

  Let $w'' = (w.n, \sigma'', \bot)$.
  Now we will show that $l \in \mathcal{V}\,[\![\bullet A]\!] \cdot w''$.
  Note that $w''.\sigma = \sigma', l : e\ \mathsf{later}$, and that $(w''.n, \sigma', w''.a) = w'$. Note that $\forall \pi \in \text{Perm}$ and $w'' \leq w'$, we know $\pi(\gamma(e)) \in \mathcal{L}\,[\![A]\!] \cdot \pi(w'')$.
  Hence $l \in \text{Env}(\bullet A)\ w'$.
  Hence $\delta_{\gamma(e')}(\gamma(e)) \in \mathcal{E}\,[\![\bullet A]\!] \cdot w$.
  Hence $\gamma(\delta_{e'}(e)) \in \mathcal{E}\,[\![\bullet A]\!] \cdot w$.

- Case $\bullet$E:
  Assume $\gamma \in \text{Env}(\Gamma)\ w$.
  We know $\Gamma \vdash \mathsf{let}\ \delta(x) = e\ \mathsf{in}\ e' : C\ \mathsf{now}$.
  By inversion, $\Gamma \vdash e : \bullet A\ \mathsf{now}$.
  By the fundamental theorem, $\gamma(e) \in \mathcal{E}\,[\![\bullet A]\!] \cdot w$.
  Hence there is v, $\sigma' \leq w.\sigma$ such that $\langle w.\sigma; e\rangle \Downarrow \langle w'.\sigma; l\rangle$,
  where $w' = (w.n, \sigma', w.a) \leq w$ and $l \in \mathcal{V}\,[\![\bullet A]\!] \cdot w$.
  Therefore $\sigma' = \sigma_0, l : e_0\ \mathsf{later}, \sigma_1$ and $\forall w'' \leq (w.n, \sigma_0, w.a).\ e_0 \in \mathcal{L}\,[\![A]\!] \cdot w''$.

  Now, we will show that $!l \in \mathcal{L}\,[\![A]\!] \cdot w'$.
  If $w.n = 0$, then this is immediate.
  So suppose $w.n = k+1$.
  Then, we know that $\sigma' \Longrightarrow \hat{\sigma}'$.
  Hence $\sigma_0, l : e\ \mathsf{later}, \sigma_1 \Longrightarrow \hat{\sigma}_0, l : v\ \mathsf{now}, \hat{\sigma}_1$
  where $\sigma_0 \Longrightarrow \hat{\sigma}_0$ and $\langle \hat{\sigma}_0; e_0\rangle \Downarrow \langle \hat{\sigma}_0'; v\rangle$.

Since $e_0 \in \mathcal{L}\,[\![A]\!] \cdot (w.n, \sigma_0, w.a)$, we know that $\sigma_0 \Longrightarrow \underline{\hat{\sigma}_0}$ and $e_0 \in \mathcal{E}\,[\![A]\!] \cdot (k, \underline{\hat{\sigma}_0}, w.a)$.
Therefore $\langle \underline{\hat{\sigma}_0}; e_0 \rangle \Downarrow \left\langle \underline{\hat{\sigma}'_0}; \underline{v} \right\rangle$ and $\underline{v} \in \mathcal{V}\,[\![A]\!] \cdot (k, \underline{\hat{\sigma}'_0}, w.a)$.
There is a permutation π such that $\pi(\sigma_0) = \sigma_0$, and $v = \pi(\underline{v})$ and $\hat{\sigma}'_0 = \pi(\underline{\hat{\sigma}'_0})$.
Hence by renaming $v \in \mathcal{V}\,[\![A]\!] \cdot (k, \hat{\sigma}'_0, w.a)$.
Assume $w'' \leq (k, \hat{\sigma}', w.a)$.
Note that for all $\sigma'' \leq \hat{\sigma}'_0$, we have $\langle \sigma''; !l \rangle \Downarrow \langle \sigma''; v \rangle$.
Hence $\langle w''.\sigma; !l \rangle \Downarrow \langle w''.\sigma; v \rangle$.
By Kripke monotonicity, $v \in \mathcal{V}\,[\![A]\!] \cdot w''$.
Hence for all $w'' \leq (k, \hat{\sigma}', w.a)$, we know $!l \in \mathcal{E}\,[\![A]\!] \cdot w''$.
Hence $!l \in \mathcal{L}\,[\![A]\!] \cdot w'$.

By Kripke monotonicity, $\gamma \in \mathrm{Env}(\Gamma)\ w'$.
Hence $(\gamma, !l/x) \in \mathrm{Env}(\Gamma, x : A\ \mathsf{later})\ w'$.
By fundamental property, $(\gamma, !l/x)(e) \in \mathcal{E}\,[\![C]\!] \cdot w'$.
Hence we have v', w'' such that $\langle w'.\sigma; (\gamma, !l/x)(e) \rangle \Downarrow \langle w''.\sigma; v' \rangle$ and $v' \in \mathcal{V}\,[\![C]\!] \cdot w''$.
Hence $\langle w.\sigma; \gamma(\mathsf{let}\ \delta(x) = e\ \mathsf{in}\ e') \rangle \Downarrow \langle w''.\sigma; v' \rangle$ and $v' \in \mathcal{V}\,[\![C]\!] \cdot w''$.
Hence $\gamma(\mathsf{let}\ \delta(x) = e\ \mathsf{in}\ e') \in \mathcal{E}\,[\![C]\!] \cdot w$.

- Case $\Box$I:
  Assume $\gamma \in \mathrm{Env}(\Gamma)\ w$.
  We know $\Gamma \vdash \mathsf{stable}(e) : \Box A\ \mathsf{now}$.
  By inversion, we know $\Gamma^{\Box} \vdash e : A\ \mathsf{now}$.
  By induction, for all $w', \gamma' \in \mathrm{Env}(\Gamma^{\Box})\ w'$, we have $\gamma'(e) \in \mathcal{E}\,[\![A]\!] \cdot w'$.
  By environment shift, we know $\gamma^{\Box}_{\Gamma} \in \mathrm{Env}(\Gamma^{\Box})\ (w.n, \cdot, \top)$.
  Hence, there is a $v \in \mathcal{V}\,[\![A]\!] \cdot (w.n, \cdot, \top)$ such that $\langle \cdot; \gamma^{\Box}_{\Gamma}(e) \rangle \Downarrow \langle \cdot; v \rangle$.
  But note that $\gamma^{\Box}_{\Gamma}(e) = \gamma(e)$.
  Note $\mathsf{stable}(v) \in \mathcal{V}\,[\![\Box A]\!] \cdot w$.
  By uniformity, $\langle w.\sigma; \gamma(e) \rangle \Downarrow \langle w.\sigma; v \rangle$.
  Hence $\langle \sigma; \gamma(\mathsf{stable}(e)) \rangle \Downarrow \langle \sigma; \mathsf{stable}(v) \rangle$ and $\mathsf{stable}(v) \in \mathcal{V}\,[\![\Box A]\!] \cdot w$.
  Hence $\gamma(\mathsf{stable}(e)) \in \mathcal{E}\,[\![\Box A]\!] \cdot w$.
- Case $\Box$E:
  Assume $\gamma \in \mathrm{Env}(\Gamma)\ w$.
  We know $\Gamma \vdash \mathsf{let}\ \mathsf{stable}(x) = e\ \mathsf{in}\ e' : C\ \mathsf{now}$.
  By inversion, we know $\Gamma \vdash e : \Box A\ \mathsf{now}$ and $\Gamma, x : A\ \mathsf{stable} \vdash e' : C\ \mathsf{now}$.
  By induction, we know that $\gamma(e) \in \mathcal{E}\,[\![\Box A]\!] \cdot w$.
  Hence there is a $\sigma' \leq w.\sigma$ s.t. $\langle w.\sigma; \gamma(e) \rangle \Downarrow \langle \sigma'; \mathsf{stable}(v) \rangle$
  and $\mathsf{stable}(v) \in \mathcal{V}\,[\![\Box A]\!] \cdot (w.n, \sigma', w.a)$ and $\sigma' = w.\sigma$ if $w.a = \top$
  Let $w' = (w.n, \sigma', w.a)$.
  By Kripke monotonicity, $\gamma \in \mathrm{Env}(\Gamma)\ w'$.
  By definition, $v \in \mathcal{V}\,[\![A]\!] \cdot (w'.n, \cdot, \top)$.
  Assume $w'' \leq w'$. Then by Kripke monotonicity, $v \in \mathcal{V}\,[\![A]\!] \cdot (w''.n, \cdot, \top)$.
  Hence $(\gamma, v/x) \in \mathrm{Env}(\Gamma, x : A\ \mathsf{stable})\ w'$.
  Hence $(\gamma, v/x)(e') \in \mathcal{E}\,[\![C]\!] \cdot w'$.
  Therefore $\sigma'' \leq w'.\sigma$ s.t. $\langle w'.\sigma; \gamma(e) \rangle \Downarrow \langle w.\sigma''; v'' \rangle$
  and $v'' \in \mathcal{V}\,[\![C]\!] \cdot (w'.n, \sigma'', w'.a)$ and $\sigma'' = w'.\sigma$ if $w.a = \top$
  Note that $w'.n = w.n$ and $w'.a = w.a$.
  Hence we know that $\sigma'' \leq w.\sigma$ and $\sigma'' = w.\sigma$ if $w.a = \top$.
  By rule, we know that $\langle w.\sigma; \gamma(\mathsf{let}\ \mathsf{stable}(x) = e\ \mathsf{in}\ e') \rangle \Downarrow \langle \sigma''; v'' \rangle$.
  Hence $\gamma(\mathsf{let}\ \mathsf{stable}(x) = e\ \mathsf{in}\ e') \in \mathcal{E}\,[\![C]\!] \cdot w$.
- Case PROMOTE:
  Assume $\gamma \in \mathrm{Env}(\Gamma)\ w$.

We know $\Gamma \vdash \mathsf{promote}(e) : \Box A\ \mathsf{now}$.
By inversion, $\Gamma \vdash e : A\ \mathsf{now}$ and A stable.
By induction, $\gamma(e) \in \mathcal{E}\,[\![A]\!] \cdot w$.
Hence there are $v, \sigma' \leq w.\sigma$ such that $\langle w.\sigma; \gamma(e)\rangle \Downarrow \langle \sigma'; v\rangle$
and $v \in \mathcal{V}\,[\![A]\!] \cdot (w.n, \sigma', w.a)$ and $w.a = \top \implies \sigma' = w.\sigma$.
Since A is a stable type, $v \in \mathcal{V}\,[\![A]\!] \cdot (w.n, \cdot, \top)$.
Hence $\mathsf{stable}(v) \in \mathcal{V}\,[\![\Box A]\!] \cdot (w.n, \sigma', w.a)$.
By rule $\langle w.\sigma; \gamma(\mathsf{promote}(e))\rangle \Downarrow \langle \sigma'; \mathsf{stable}(v)\rangle$.
Hence $\gamma(\mathsf{promote}(e)) \in \mathcal{E}\,[\![\Box A]\!] \cdot w$.

- Case $\to$I:
  Assume $\gamma \in \mathrm{Env}(\Gamma)\ w$.
  We know $\Gamma \vdash \lambda x.\ e : A \to B\ \mathsf{now}$.
  By inversion, $\Gamma, x : A\ \mathsf{now} \vdash e : B\ \mathsf{now}$.
  It suffices to show $\gamma(\lambda x.\ e) \in \mathcal{V}\,[\![A \to B]\!] \cdot w$.
  This is equivalent to showing $\lambda x.\ \gamma(e) \in \mathcal{V}\,[\![A \to B]\!] \cdot w$.
  Assume $\pi \in \mathrm{Perm}$ and $w' \leq w$ and $e_0 \in \mathcal{E}\,[\![A]\!] \cdot \pi(w')$.
  Then, by Kripke monotonicity, $\gamma \in \mathrm{Env}(\Gamma)\ w'$.
  Then, by environment renaming, $\pi(\gamma) \in \mathrm{Env}(\Gamma)\ \pi(w')$.
  Then we know $(\pi(\gamma), e_0/x) \in \mathrm{Env}(\Gamma, x : A\ \mathsf{now})\ \pi(w')$.
  By induction, we know $(\pi(\gamma), e_0/x)(e) \in \mathcal{E}\,[\![B]\!] \cdot \pi(w')$.
  Since e has no free location variables, $[e_0/x](\pi(\gamma(e)) \in \mathcal{E}\,[\![B]\!] \cdot \pi(w')$.
  Hence $\lambda x.\ \gamma(e) \in \mathcal{V}\,[\![A \to B]\!] \cdot w$.

- Case $\to$E:
  Assume $\gamma \in \mathrm{Env}(\Gamma)\ w$.
  We know $\Gamma \vdash e\ e' : B\ \mathsf{now}$.
  By inversion, $\Gamma \vdash e : A \to B\ \mathsf{now}$ and $\Gamma \vdash e' : A\ \mathsf{now}$.
  By induction, $\gamma(e) \in \mathcal{E}\,[\![A \to B]\!] \cdot w$.
  Hence there is a $\sigma' \leq w.\sigma$ such that $\langle w.\sigma; \gamma(e)\rangle \Downarrow \langle \sigma'; \lambda x.\ e_1\rangle$
  and $\lambda x.\ e_1 \in \mathcal{V}\,[\![A \to B]\!] \cdot (w.n, \sigma', w.a)$ and $w.a = \top$ implies $\sigma' = w.\sigma$.
  Note that $w' \triangleq (w.n, \sigma', w.a) \leq w$.
  By Kripke monotonicity, $\gamma \in \mathrm{Env}(\Gamma)\ w'$.
  Hence $\gamma(e') \in \mathcal{E}\,[\![A]\!] \cdot w'$.
  Hence there is a $\sigma'' \leq w'.\sigma$ such that $\langle w'.\sigma; \gamma(e)\rangle \Downarrow \langle \sigma''; v\rangle$
  and $v \in \mathcal{V}\,[\![A]\!] \cdot (w'.n, \sigma'', w'.a)$ and $w'.a = \top$ implies $\sigma'' = w'.\sigma$.
  Note that $w'' \triangleq (w'.n, \sigma'', w'.a) \leq w'$.
  Hence $[v/x]e_1 \in \mathcal{E}\,[\![B]\!] \cdot w''$.
  Hence there is a $\sigma''' \leq w''.\sigma$ such that $\langle w''.\sigma; (\gamma, v/x)e_1\rangle \Downarrow \langle \sigma'''; v'\rangle$
  and $v' \in \mathcal{V}\,[\![B]\!] \cdot (w''.n, \sigma''', w''.a)$ and $w''.a = \top$ implies $\sigma''' = w''.\sigma$.
  By rule $\langle w.\sigma; \gamma(e\ e')\rangle \Downarrow \langle \sigma'''; v'\rangle$.
  Note that $w''' \triangleq (w''.n, \sigma''', w''.a) \leq w''$.
  Note that $w'''.n = w.n$ and $w'''.\sigma \leq w.\sigma$ and $w'''.a = w.a$ and $w.a = \top$ implies $\sigma''' = w.\sigma$.
  Hence $\gamma(e\ e') \in \mathcal{E}\,[\![B]\!] \cdot w$.

- Case SI:
  Assume $\gamma \in \mathrm{Env}(\Gamma)\ w$.
  We know $\Gamma \vdash \mathsf{cons}(e, e') : \mathsf{S}\,A\ \mathsf{now}$.
  By inversion, $\Gamma \vdash e : A\ \mathsf{now}$ and $\Gamma \vdash e' : \bullet(\mathsf{S}\,A)\ \mathsf{now}$.
  By induction, $\gamma(e) \in \mathcal{E}\,[\![A]\!] \cdot w$.
  Hence there is a $\sigma' \leq w.\sigma$ and v s.t. $\langle w.\sigma; \gamma(e)\rangle \Downarrow \langle \sigma'; v\rangle$
  and $v \in \mathcal{V}\,[\![A]\!] \cdot (w.n, \sigma', w.a)$ and $w.a = \top \implies \sigma' = w.\sigma$.
  Note that $w' \triangleq (w.n, \sigma', w.a) \leq w$.
  By Kripke monotonicity, $\gamma \in \mathrm{Env}(\Gamma)\ w'$.

By induction, $\gamma(e') \in \mathcal{E}\,[\![\bullet(\mathsf{S}\,A)]\!] \cdot w'$.
Hence there is a $\sigma'' \leq w'.\sigma$ and v s.t. $\langle w'.\sigma; \gamma(e)\rangle \Downarrow \langle \sigma''; l\rangle$
and $l \in \mathcal{V}\,[\![A]\!] \cdot (w'.n, \sigma'', w'.a)$ and $w'.a = \top \implies \sigma'' = w'.\sigma$.
Hence by rule, $\langle w.\sigma; \mathsf{cons}(e, e')\rangle \Downarrow \langle \sigma''; \mathsf{cons}(v, l)\rangle$.
Note that $w'' \triangleq (w'.n, \sigma'', w'.a) \leq w'$.
Hence $w'' \leq w$.
Furthermore, $w''.n = w.n$ and $w''.a = w.a$, and so if $w.a = \top$ then $\sigma'' = w.\sigma$.
Hence $\gamma(\mathsf{cons}(e, e')) \in \mathcal{E}\,[\![\mathsf{S}\,A]\!] \cdot w$.

- Case SE:
  Assume $\gamma \in \mathrm{Env}(\Gamma)\ w$.
  We know $\Gamma \vdash \mathsf{let\ cons}(x, xs) = e \mathsf{\ in\ } e' : C \mathsf{\ now}$.
  By inversion, $\Gamma \vdash e : \mathsf{S}\,A \mathsf{\ now}$ and $\Gamma, x : A \mathsf{\ now}, xs : \bullet(\mathsf{S}\,A) \mathsf{\ now} \vdash e' : C \mathsf{\ now}$.
  By induction, $\gamma(e) \in \mathcal{E}\,[\![A]\!] \cdot w$.
  Hence there is a $\sigma' \leq w.\sigma$ and v s.t. $\langle w.\sigma; \gamma(e)\rangle \Downarrow \langle \sigma'; \mathsf{cons}(v, l)\rangle$
  and $\mathsf{cons}(v, l) \in \mathcal{V}\,[\![A]\!] \cdot (w.n, \sigma', w.a)$ and $w.a = \top \implies \sigma' = w.\sigma$.
  Note that $w' \triangleq (w.n, \sigma', w.a) \leq w$.
  By definition, $v \in \mathcal{V}\,[\![A]\!] \cdot w'$ and $l \in \mathcal{V}\,[\![\bullet(\mathsf{S}\,A)]\!] \cdot w'$.
  By Kripke monotonicity, $\gamma \in \mathrm{Env}(\Gamma)\ w'$.
  Assume $w'' \leq w'$.
  By Kripke monotonicity, $v \in \mathcal{V}\,[\![A]\!] \cdot w''$ and $l \in \mathcal{V}\,[\![\bullet\mathsf{S}\,A]\!] \cdot w''$.
  Hence $v \in \mathcal{E}\,[\![A]\!] \cdot w''$ and $l \in \mathcal{E}\,[\![\bullet\mathsf{S}\,A]\!] \cdot w''$.
  Hence for all $w'' \leq w'$, $v \in \mathcal{E}\,[\![A]\!] \cdot w''$.
  Hence for all $w'' \leq w'$, $l \in \mathcal{E}\,[\![\bullet\mathsf{S}\,A]\!] \cdot w''$.
  Hence $(\gamma, v/x, l/xs) \in \mathrm{Env}(\Gamma, x : A \mathsf{\ now}, xs : \bullet(\mathsf{S}\,A) \mathsf{\ now})\ w'$.
  By induction, $(\gamma, v/x, l/xs)(e') \in \mathcal{E}\,[\![C]\!] \cdot w'$.
  Hence there is a $\sigma'' \leq w'.\sigma$ and v' s.t. $\langle w'.\sigma; \gamma(e)\rangle \Downarrow \langle \sigma''; v'\rangle$
  and $v' \in \mathcal{V}\,[\![C]\!] \cdot (w'.n, \sigma'', w'.a)$ and $w'.a = \top \implies \sigma'' = w.\sigma'$.
  Note that $w'' \triangleq (w'.n, \sigma'', w'.a) \leq w'$.
  Hence $w'' \leq w$.
  Furthermore, $w''.n = w.n$ and $w''.a = w.a$, and so if $w.a = \top$ then $\sigma'' = w.\sigma$.
  By rule $\langle w.\sigma; \gamma(\mathsf{let\ cons}(x, xs) = e \mathsf{\ in\ } e')\rangle \Downarrow \langle \sigma''; v'\rangle$.
  Hence $\gamma(\mathsf{let\ cons}(x, xs) = e \mathsf{\ in\ } e') \in \mathcal{E}\,[\![C]\!] \cdot w$.

- Case μI:
  Assume w and $\gamma \in \mathrm{Env}(\Gamma)\ w$.
  Assume $\Gamma \vdash \mathsf{into}\, e : \hat{\mu}\alpha.\ A \mathsf{\ now}$.
  By inversion, we know $\Gamma \vdash e : [\bullet(\hat{\mu}\alpha.\ A)/\alpha]A \mathsf{\ now}$.
  By induction, we know that $\gamma(e) \in \mathcal{E}\,[\![[\bullet(\hat{\mu}\alpha.\ A)/\alpha]A]\!] \cdot w$.
  Hence for all $\sigma \leq w.\sigma$, there exists v and $\sigma' \leq \sigma$
  such that $\langle \sigma; \gamma(e)\rangle \Downarrow \langle \sigma'; v\rangle$ and $v \in \mathcal{V}\,[\![[\bullet(\hat{\mu}\alpha.\ A)/\alpha]A]\!] \cdot w$.

  Assume $\sigma \leq w.\sigma$.
  Then there exists v and $\sigma' \leq \sigma$ such that $\langle \sigma; \gamma(e)\rangle \Downarrow \langle \sigma'; v\rangle$ and $v \in \mathcal{V}\,[\![[\bullet(\hat{\mu}\alpha.\ A)/\alpha]A]\!] \cdot w$.
  By substitution lemma, $v \in \mathcal{V}\,[\![A]\!]\ (\mathcal{V}\,[\![\bullet(\hat{\mu}\alpha.\ A)]\!] \cdot w/\alpha)\ w$.
  Hence $\mathsf{into}\, v \in \mathcal{V}\,[\![\hat{\mu}\alpha.\ A]\!]\ \rho\ w$.
  So $\langle \sigma; \gamma(e)\rangle \Downarrow \langle \sigma'; v\rangle$ and $\mathsf{into}\, v \in \mathcal{V}\,[\![\hat{\mu}\alpha.\ A]\!] \cdot w$.
  So for all $\sigma \leq w.\sigma$, there exists v and $\sigma' \leq \sigma$
  such that $\langle \sigma; \mathsf{into}\,\gamma(e)\rangle \Downarrow \langle \sigma'; v\rangle$ and $\mathsf{into}\, v \in \mathcal{V}\,[\![\hat{\mu}\alpha.\ A]\!] \cdot w$.
  Hence $\mathsf{into}\,\gamma(e) \in \mathcal{E}\,[\![\hat{\mu}\alpha.\ A]\!] \cdot w$.
  Hence $\gamma(\mathsf{into}\, e) \in \mathcal{E}\,[\![\hat{\mu}\alpha.\ A]\!] \cdot w$.

- Case μE:
  Assume w and $\gamma \in \mathrm{Env}(\Gamma)\ w$.

Assume $\Gamma \vdash \mathsf{out}\, e : [\bullet(\hat{\mu}\alpha.\ A)/\alpha]A\ \mathsf{now}$.
Hence $\Gamma \vdash e : \hat{\mu}\alpha.\ A\ \mathsf{now}$.
Hence $\gamma(e) \in \mathcal{E}\,[\![\hat{\mu}\alpha.\ A]\!] \cdot w$.
So for all $\sigma \leq w.\sigma$, there exists v and σ'
such that $\langle\sigma; e\rangle \Downarrow \langle\sigma'; \mathsf{into}\, v\rangle$ and $\mathsf{into}\, v \in \mathcal{V}\,[\![\hat{\mu}\alpha.\ A]\!] \cdot w$.

We want to show $\gamma(\mathsf{out}\, e) \in \mathcal{E}\,[\![[\bullet(\hat{\mu}\alpha.\ A)/\alpha]A]\!] \cdot w$.
Assume $\sigma \leq w.\sigma$.
So there are v and σ' such that $\langle\sigma; e\rangle \Downarrow \langle\sigma'; \mathsf{into}\, v\rangle$ and $\mathsf{into}\, v \in \mathcal{V}\,[\![\hat{\mu}\alpha.\ A]\!] \cdot w$.
Hence $v \in \mathcal{V}\,[\![A]\!]\,(\mathcal{V}\,[\![\bullet(\hat{\mu}\alpha.\ A)]\!] \cdot w/\alpha)\ w$.
By substitution $v \in \mathcal{V}\,[\![[\bullet(\hat{\mu}\alpha.\ A)/\alpha]A]\!] \cdot w$.
Hence $\langle\sigma; \mathsf{out}\,\gamma(e)\rangle \Downarrow \langle\sigma'; v\rangle$ and $v \in \mathcal{V}\,[\![[\bullet(\hat{\mu}\alpha.\ A)/\alpha]A]\!] \cdot w$.
So $\mathsf{out}\,\gamma(e) \in \mathcal{E}\,[\![[\bullet(\hat{\mu}\alpha.\ A)/\alpha]A]\!] \cdot w$.
So $\gamma(\mathsf{out}\, e) \in \mathcal{E}\,[\![[\bullet(\hat{\mu}\alpha.\ A)/\alpha]A]\!] \cdot w$.

$\square$