Lemmas and Proofs for "Complete and Easy Bidirectional Typechecking for Higher-Rank Polymorphism"

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Proposition 2 (Substitution). *If* $\Psi \vdash A$ *and* $\Psi, \alpha, \Psi' \vdash B$ *then* $\Psi, \Psi' \vdash [A/\alpha]B$.

A.2 Reflexivity

Lemma 3 (Reflexivity of Declarative Subtyping). *Subtyping is reflexive: if* $\Psi \vdash A$ *then* $\Psi \vdash A \leq A$.

A.3 Subtyping Implies Well-Formedness

Lemma 4 (Well-Formedness). *If* $\Psi \vdash A \leq B$ *then* $\Psi \vdash A$ *and* $\Psi \vdash B$.

A.4 Substitution

Lemma 5 (Substitution). *If* $\Psi \vdash \tau$ *and* $\Psi, \alpha, \Psi' \vdash A \leq B$ *then* $\Psi, [\tau/\alpha]\Psi' \vdash [\tau/\alpha]A \leq [\tau/\alpha]B$.

A.5 Transitivity

Lemma 6 (Transitivity of Declarative Subtyping). *If* $\Psi \vdash A \leq B$ *and* $\Psi \vdash B \leq C$ *then* $\Psi \vdash A \leq C$.

A.6 Invertibility of $\leq\forall\mathsf{R}$

Lemma 7 (Invertibility).
If $\mathcal{D}$ *derives* $\Psi \vdash A \leq \forall\beta.\, B$ *then* $\mathcal{D}'$ *derives* $\Psi, \beta \vdash A \leq B$ *where* $\mathcal{D}' < \mathcal{D}$.

A.7 Non-Circularity and Equality

Definition 1 (Subterm Occurrence).
Let $A \preceq B$ *iff* A *is a subterm of* B.
Let $A \prec B$ *iff* A *is a proper subterm of* B *(that is,* $A \preceq B$ *and* $A \neq B$*).*
Let $A \precsim B$ *iff* A *occurs in* B *inside an arrow, that is, there exist* B_1*,* B_2 *such that* $(B_1 \to B_2) \preceq B$ *and* $A \preceq B_k$ *for some* $k \in \{1, 2\}$.

Lemma 8 (Occurrence).

(i) If $\Psi \vdash A \leq \tau$ *then* $\tau \not\precsim A$.

(ii) If $\Psi \vdash \tau \leq B$ *then* $\tau \not\precsim B$.

Lemma 9 (Monotype Equality). *If* $\Psi \vdash \sigma \leq \tau$ *then* $\sigma = \tau$.

Definition 2 (Contextual Size). *The size of* A *with respect to a context* Γ*, written* $|\Gamma \vdash A|$*, is defined by*

$$\begin{array}{lcl}
|\Gamma \vdash \alpha| & = & 1 \\
|\Gamma[\hat{\alpha}] \vdash \hat{\alpha}| & = & 1 \\
|\Gamma[\hat{\alpha} = \tau] \vdash \hat{\alpha}| & = & 1 + |\Gamma[\hat{\alpha} = \tau] \vdash \tau| \\
|\Gamma \vdash \forall\alpha.\, A| & = & 1 + |\Gamma, \alpha \vdash A| \\
|\Gamma \vdash A \to B| & = & 1 + |\Gamma \vdash A| + |\Gamma \vdash B|
\end{array}$$

B Type Assignment

Theorem 1 (Completeness of Bidirectional Typing). *If* $\Psi \vdash e : A$ *then there exists* e' *such that* $\Psi \vdash e' \Rightarrow A$ *and* $|e'| = e$.

Lemma 10 (Subtyping Coercion). *If* $\Psi \vdash A \leq B$ *then there exists* f *which is* $\beta\eta$*-equal to the identity such that* $\Psi \vdash f : A \to B$.

Lemma 11 (Application Subtyping). *If* $\Psi \vdash A \bullet e \Rrightarrow C$ *then there exists* B *such that* $\Psi \vdash A \leq B \to C$ *and* $\Psi \vdash e \Leftarrow B$ *by a smaller derivation.*

Theorem 2 (Soundness of Bidirectional Typing). *We have that:*

- *If* $\Psi \vdash e \Leftarrow A$*, then there is an* e' *such that* $\Psi \vdash e' : A$ *and* $e' =_{\beta\eta} |e|$.
- *If* $\Psi \vdash e \Rightarrow A$*, then there is an* e' *such that* $\Psi \vdash e' : A$ *and* $e' =_{\beta\eta} |e|$.

C Robustness of Typing

Theorem 3 (Substitution).
Assume $\Psi \vdash e \Rightarrow A$.

- *If* $\Psi, x : A \vdash e' \Leftarrow C$ *then* $\Psi \vdash [e/x]e' \Leftarrow C$.
- *If* $\Psi, x : A \vdash e' \Rightarrow C$ *then* $\Psi \vdash [e/x]e' \Rightarrow C$.
- *If* $\Psi, x : A \vdash B \bullet e' \Rrightarrow C$ *then* $\Psi \vdash B \bullet [e/x]e' \Rrightarrow C$.

Lemma 12 (Type Substitution).
Assume $\Psi \vdash \tau$.

- *If* $\Psi, \alpha, \Psi' \vdash e' \Leftarrow C$ *then* $\Psi, [\tau/\alpha]\Psi' \vdash [\tau/\alpha]e' \Leftarrow [\tau/\alpha]C$.
- *If* $\Psi, \alpha, \Psi' \vdash e' \Rightarrow C$ *then* $\Psi, [\tau/\alpha]\Psi' \vdash [\tau/\alpha]e' \Rightarrow [\tau/\alpha]C$.
- *If* $\Psi, \alpha, \Psi' \vdash B \bullet e' \Rrightarrow C$ *then* $\Psi, [\tau/\alpha]\Psi' \vdash [\tau/\alpha]B \bullet [\tau/\alpha]e' \Rrightarrow [A/\alpha]C$.

Moreover, the resulting derivation contains no more applications of typing rules than the given one. (Internal subtyping derivations, however, may grow.)

Definition 3 (Context Subtyping). *We define the judgment* $\Psi' \leq \Psi$ *with the following rules:*

$$\frac{}{\cdot \leq \cdot}\ \textsf{CtxSubEmpty} \qquad \frac{\Psi' \leq \Psi}{\Psi', \alpha \leq \Psi, \alpha}\ \textsf{CtxSubUvar} \qquad \frac{\Psi' \leq \Psi \qquad \Psi \vdash A' \leq A}{\Psi', x : A' \leq \Psi, x : A}\ \textsf{CtxSubVar}$$

Lemma 13 (Subsumption). *Suppose* $\Psi' \leq \Psi$. *Then:*

(i) *If* $\Psi \vdash e \Leftarrow A$ *and* $\Psi \vdash A \leq A'$ *then* $\Psi' \vdash e \Leftarrow A'$.

(ii) *If* $\Psi \vdash e \Rightarrow A$ *then there exists* A' *such that* $\Psi \vdash A' \leq A$ *and* $\Psi' \vdash e \Rightarrow A'$.

(iii) *If* $\Psi \vdash C \bullet e \Rrightarrow A$ *and* $\Psi \vdash C' \leq C$
then there exists A' *such that* $\Psi \vdash A' \leq A$ *and* $\Psi' \vdash C' \bullet e \Rrightarrow A'$.

Theorem 4 (Inverse Substitution). *Assume* $\Psi \vdash e \Leftarrow A$. *Then:*

(i) *If* $\Psi \vdash [(e : A)/x]e' \Leftarrow C$ *then* $\Psi, x : A \vdash e' \Leftarrow C$.

(ii) *If* $\Psi \vdash [(e : A)/x]e' \Rightarrow C$ *then* $\Psi, x : A \vdash e' \Rightarrow C$.

(iii) *If* $\Psi \vdash B \bullet [(e : A)/x]e' \Rrightarrow C$ *then* $\Psi, x : A \vdash B \bullet e' \Rrightarrow C$.

Theorem 5 (Annotation Removal). *We have that:*

- *If* $\Psi \vdash ((\lambda x. e) : A) \Leftarrow C$ *then* $\Psi \vdash \lambda x. e \Leftarrow C$.
- *If* $\Psi \vdash (() : A) \Leftarrow C$ *then* $\Psi \vdash () \Leftarrow C$.
- *If* $\Psi \vdash e_1 (e_2 : A) \Rightarrow C$ *then* $\Psi \vdash e_1 e_2 \Rightarrow C$.
- *If* $\Psi \vdash (x : A) \Rightarrow A$ *then* $\Psi \vdash x \Rightarrow B$ *and* $\Psi \vdash B \leq A$.
- *If* $\Psi \vdash ((e_1 e_2) : A) \Rightarrow A$ *then* $\Psi \vdash e_1 e_2 \Rightarrow B$ *and* $\Psi \vdash B \leq A$.
- *If* $\Psi \vdash ((e : B) : A) \Rightarrow A$ *then* $\Psi \vdash (e : B) \Rightarrow B$ *and* $\Psi \vdash B \leq A$.
- *If* $\Psi \vdash ((\lambda x. e) : \sigma \to \tau) \Rightarrow \sigma \to \tau$ *then* $\Psi \vdash \lambda x. e \Rightarrow \sigma \to \tau$.

Theorem 6 (Soundness of Eta).
If $\Psi \vdash \lambda x. e\ x \Leftarrow A$ *and* $x \notin FV(e)$, *then* $\Psi \vdash e \Leftarrow A$.

D Properties of Context Extension

D.1 Syntactic Properties

Lemma 14 (Declaration Preservation). *If* $\Gamma \longrightarrow \Delta$, *and* u *is a variable or marker* $\blacktriangleright_{\hat{\alpha}}$ *declared in* Γ, *then* u *is declared in* Δ.

Lemma 15 (Declaration Order Preservation). *If* $\Gamma \longrightarrow \Delta$ *and* u *is declared to the left of* v *in* Γ, *then* u *is declared to the left of* v *in* Δ.

Lemma 16 (Reverse Declaration Order Preservation). *If* $\Gamma \longrightarrow \Delta$ *and* u *and* v *are both declared in* Γ *and* u *is declared to the left of* v *in* Δ, *then* u *is declared to the left of* v *in* Γ.

Lemma 17 (Substitution Extension Invariance). *If* $\Theta \vdash A$ *and* $\Theta \longrightarrow \Gamma$ *then* $[\Gamma]A = [\Gamma]([\Theta]A)$ *and* $[\Gamma]A = [\Theta]([\Gamma]A)$.

Lemma 18 (Extension Equality Preservation).
If $\Gamma \vdash A$ *and* $\Gamma \vdash B$ *and* $[\Gamma]A = [\Gamma]B$ *and* $\Gamma \longrightarrow \Delta$, *then* $[\Delta]A = [\Delta]B$.

Lemma 19 (Reflexivity). *If* Γ *is well-formed, then* $\Gamma \longrightarrow \Gamma$.

Lemma 20 (Transitivity). *If* $\Gamma \longrightarrow \Delta$ *and* $\Delta \longrightarrow \Theta$, *then* $\Gamma \longrightarrow \Theta$.

Definition 4 (Softness). *A context* Θ *is* soft *iff it consists only of* $\hat{\alpha}$ *and* $\hat{\alpha} = \tau$ *declarations.*

Lemma 21 (Right Softness). *If* $\Gamma \longrightarrow \Delta$ *and* Θ *is soft (and* (Δ, Θ) *is well-formed) then* $\Gamma \longrightarrow \Delta, \Theta$.

Lemma 22 (Evar Input).
If $\Gamma, \hat{\alpha} \longrightarrow \Delta$ *then* $\Delta = (\Delta_0, \Delta_{\hat{\alpha}}, \Theta)$ *where* $\Gamma \longrightarrow \Delta_0$, *and* $\Delta_{\hat{\alpha}}$ *is either* $\hat{\alpha}$ *or* $\hat{\alpha} = \tau$, *and* Θ *is soft.*

Lemma 23 (Extension Order).

(i) *If* $\Gamma_L, \alpha, \Gamma_R \longrightarrow \Delta$ *then* $\Delta = (\Delta_L, \alpha, \Delta_R)$ *where* $\Gamma_L \longrightarrow \Delta_L$.
Moreover, if Γ_R *is soft then* Δ_R *is soft.*

(ii) *If* $\Gamma_L, \blacktriangleright_{\hat{\alpha}}, \Gamma_R \longrightarrow \Delta$ *then* $\Delta = (\Delta_L, \blacktriangleright_{\hat{\alpha}}, \Delta_R)$ *where* $\Gamma_L \longrightarrow \Delta_L$.
Moreover, if Γ_R *is soft then* Δ_R *is soft.*

(iii) *If* $\Gamma_L, \hat{\alpha}, \Gamma_R \longrightarrow \Delta$ *then* $\Delta = \Delta_L, \Theta, \Delta_R$ *where* $\Gamma_L \longrightarrow \Delta_L$ *and* Θ *is either* $\hat{\alpha}$ *or* $\hat{\alpha} = \tau$ *for some* τ.

(iv) *If* $\Gamma_L, \hat{\alpha} = \tau, \Gamma_R \longrightarrow \Delta$ *then* $\Delta = \Delta_L, \hat{\alpha} = \tau', \Delta_R$ *where* $\Gamma_L \longrightarrow \Delta_L$ *and* $[\Delta_L]\tau = [\Delta_L]\tau'$.

(v) *If* $\Gamma_L, x : A, \Gamma_R \longrightarrow \Delta$ *then* $\Delta = (\Delta_L, x : A', \Delta_R)$ *where* $\Gamma_L \longrightarrow \Delta_L$ *and* $[\Delta_L]A = [\Delta_L]A'$.
Moreover, Γ_R *is soft if and only if* Δ_R *is soft.*

Lemma 24 (Extension Weakening). *If* $\Gamma \vdash A$ *and* $\Gamma \longrightarrow \Delta$ *then* $\Delta \vdash A$.

Lemma 25 (Solution Admissibility for Extension). *If* $\Gamma_L \vdash \tau$ *then* $\Gamma_L, \hat{\alpha}, \Gamma_R \longrightarrow \Gamma_L, \hat{\alpha} = \tau, \Gamma_R$.

Lemma 26 (Solved Variable Addition for Extension). *If* $\Gamma_L \vdash \tau$ *then* $\Gamma_L, \Gamma_R \longrightarrow \Gamma_L, \hat{\alpha} = \tau, \Gamma_R$.

Lemma 27 (Unsolved Variable Addition for Extension). *We have that* $\Gamma_L, \Gamma_R \longrightarrow \Gamma_L, \hat{\alpha}, \Gamma_R$.

Lemma 28 (Parallel Admissibility).
If $\Gamma_L \longrightarrow \Delta_L$ *and* $\Gamma_L, \Gamma_R \longrightarrow \Delta_L, \Delta_R$ *then:*

(i) $\Gamma_L, \hat{\alpha}, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha}, \Delta_R$

(ii) *If* $\Delta_L \vdash \tau'$ *then* $\Gamma_L, \hat{\alpha}, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} = \tau', \Delta_R$.

(iii) *If* $\Gamma_L \vdash \tau$ *and* $\Delta_L \vdash \tau'$ *and* $[\Delta_L]\tau = [\Delta_L]\tau'$, *then* $\Gamma_L, \hat{\alpha} = \tau, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} = \tau', \Delta_R$.

Lemma 29 (Parallel Extension Solution).
If $\Gamma_L, \hat{\alpha}, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} = \tau', \Delta_R$ *and* $\Gamma_L \vdash \tau$ *and* $[\Delta_L]\tau = [\Delta_L]\tau'$ *then* $\Gamma_L, \hat{\alpha} = \tau, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} = \tau', \Delta_R$.

Lemma 30 (Parallel Variable Update).
If $\Gamma_L, \hat{\alpha}, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} = \tau_0, \Delta_R$ *and* $\Gamma_L \vdash \tau_1$ *and* $\Delta_L \vdash \tau_2$ *and* $[\Delta_L]\tau_0 = [\Delta_L]\tau_1 = [\Delta_L]\tau_2$
then $\Gamma_L, \hat{\alpha} = \tau_1, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} = \tau_2, \Delta_R$.

D.2 Instantiation Extends

Lemma 31 (Instantiation Extension).
If $\Gamma \vdash \hat{\alpha} :\leqq \tau \dashv \Delta$ *or* $\Gamma \vdash \tau \leqq: \hat{\alpha} \dashv \Delta$ *then* $\Gamma \longrightarrow \Delta$.

D.3 Subtyping Extends

Lemma 32 (Subtyping Extension).
If $\Gamma \vdash A <: B \dashv \Delta$ *then* $\Gamma \longrightarrow \Delta$.

E Decidability of Instantiation

Lemma 33 (Left Unsolvedness Preservation).
If $\underbrace{\Gamma_0, \hat{\alpha}, \Gamma_1}_{\Gamma} \vdash \hat{\alpha} :\leqq A \dashv \Delta$ *or* $\underbrace{\Gamma_0, \hat{\alpha}, \Gamma_1}_{\Gamma} \vdash A \leqq: \hat{\alpha} \dashv \Delta$, *and* $\hat{\beta} \in \mathsf{unsolved}(\Gamma_0)$, *then* $\hat{\beta} \in \mathsf{unsolved}(\Delta)$.

Lemma 34 (Left Free Variable Preservation). *If* $\overbrace{\Gamma_0, \hat{\alpha}, \Gamma_1}^{\Gamma} \vdash \hat{\alpha} :\leqq A \dashv \Delta$ *or* $\overbrace{\Gamma_0, \hat{\alpha}, \Gamma_1}^{\Gamma} \vdash A \leqq: \hat{\alpha} \dashv \Delta$, *and* $\Gamma \vdash B$ *and* $\hat{\alpha} \notin FV([\Gamma]B)$ *and* $\hat{\beta} \in \mathsf{unsolved}(\Gamma_0)$ *and* $\hat{\beta} \notin FV([\Gamma]B)$, *then* $\hat{\beta} \notin FV([\Delta]B)$.

Lemma 35 (Instantiation Size Preservation). *If* $\overbrace{\Gamma_0, \hat{\alpha}, \Gamma_1}^{\Gamma} \vdash \hat{\alpha} :\leqq A \dashv \Delta$ *or* $\overbrace{\Gamma_0, \hat{\alpha}, \Gamma_1}^{\Gamma} \vdash A \leqq: \hat{\alpha} \dashv \Delta$, *and* $\Gamma \vdash B$ *and* $\hat{\alpha} \notin FV([\Gamma]B)$, *then* $|[\Gamma]B| = |[\Delta]B|$, *where* $|C|$ *is the plain size of the term* C.

This lemma lets us show decidability by taking the size of the type argument as the induction metric.

Theorem 7 (Decidability of Instantiation). *If* $\Gamma = \Gamma_0[\hat{\alpha}]$ *and* $\Gamma \vdash A$ *such that* $[\Gamma]A = A$ *and* $\hat{\alpha} \notin FV(A)$, *then:*

(1) Either there exists Δ *such that* $\Gamma[\hat{\alpha}] \vdash \hat{\alpha} :\leqq A \dashv \Delta$, *or not.*

(2) Either there exists Δ *such that* $\Gamma[\hat{\alpha}] \vdash A \leqq: \hat{\alpha} \dashv \Delta$, *or not.*

F Decidability of Algorithmic Subtyping

F.1 Lemmas for Decidability of Subtyping

Lemma 36 (Monotypes Solve Variables). *If* $\Gamma \vdash \hat{\alpha} :\leqq \tau \dashv \Delta$ *or* $\Gamma \vdash \tau \leqq: \hat{\alpha} \dashv \Delta$*, then if* $[\Gamma]\tau = \tau$ *and* $\hat{\alpha} \notin FV([\Gamma]\tau)$*, then* $|\mathsf{unsolved}(\Gamma)| = |\mathsf{unsolved}(\Delta)| + 1$*.*

Lemma 37 (Monotype Monotonicity). *If* $\Gamma \vdash \tau_1 <: \tau_2 \dashv \Delta$ *then* $|\mathsf{unsolved}(\Delta)| \leq |\mathsf{unsolved}(\Gamma)|$*.*

Lemma 38 (Substitution Decreases Size). *If* $\Gamma \vdash A$ *then* $|\Gamma \vdash [\Gamma]A| \leq |\Gamma \vdash A|$*.*

Lemma 39 (Monotype Context Invariance).
If $\Gamma \vdash \tau <: \tau' \dashv \Delta$ *where* $[\Gamma]\tau = \tau$ *and* $[\Gamma]\tau' = \tau'$ *and* $|\mathsf{unsolved}(\Gamma)| = |\mathsf{unsolved}(\Delta)|$ *then* $\Gamma = \Delta$*.*

F.2 Decidability of Subtyping

Theorem 8 (Decidability of Subtyping).
Given a context Γ *and types* A, B *such that* $\Gamma \vdash A$ *and* $\Gamma \vdash B$ *and* $[\Gamma]A = A$ *and* $[\Gamma]B = B$*, it is decidable whether there exists* Δ *such that* $\Gamma \vdash A <: B \dashv \Delta$*.*

G Decidability of Typing

Theorem 9 (Decidability of Typing).

(i) Checking: *Given an algorithmic context* Γ*, a term* e*, and a type* B *such that* $\Gamma \vdash B$*,*
it is decidable whether there is a context Δ *such that*
$\Gamma \vdash e \Leftarrow B \dashv \Delta$*.*

(ii) Synthesis: *Given an algorithmic context* Γ *and a term* e*,*
it is decidable whether there exist a type A *and a context* Δ *such that*
$\Gamma \vdash e \Rightarrow A \dashv \Delta$*.*

(iii) Application: *Given an algorithmic context* Γ*, a term* e*, and a type* A *such that* $\Gamma \vdash A$*,*
it is decidable whether there exist a type C *and a context* Δ *such that*
$\Gamma \vdash A \bullet e \Rrightarrow C \dashv \Delta$*.*

H Soundness of Subtyping

Definition 5 (Filling). *The* filling *of a context* $|\Gamma|$ *solves all unsolved variables:*

$$
\begin{array}{lcl}
|\cdot| & = & \cdot \\
|\Gamma, x : A| & = & |\Gamma|, x : A \\
|\Gamma, \alpha| & = & |\Gamma|, \alpha \\
|\Gamma, \hat{\alpha} = \tau| & = & |\Gamma|, \hat{\alpha} = \tau \\
|\Gamma, \blacktriangleright_{\hat{\alpha}}| & = & |\Gamma|, \blacktriangleright_{\hat{\alpha}} \\
|\Gamma, \hat{\alpha}| & = & |\Gamma|, \hat{\alpha} = 1
\end{array}
$$

H.1 Lemmas for Soundness

Lemma 40 (Uvar Preservation).
If $\alpha \in \Omega$ *and* $\Delta \longrightarrow \Omega$ *then* $\alpha \in [\Omega]\Delta$*.*

Proof. By induction on Ω, following the definition of context application. □

Lemma 41 (Variable Preservation).
If $(x : A) \in \Delta$ *or* $(x : A) \in \Omega$ *and* $\Delta \longrightarrow \Omega$ *then* $(x : [\Omega]A) \in [\Omega]\Delta$*.*

Lemma 42 (Substitution Typing). *If* $\Gamma \vdash A$ *then* $\Gamma \vdash [\Gamma]A$*.*

Lemma 43 (Substitution for Well-Formedness). *If* $\Omega \vdash A$ *then* $[\Omega]\Omega \vdash [\Omega]A$.

Lemma 44 (Substitution Stability).
For any well-formed complete context (Ω, Ω_Z)*, if* $\Omega \vdash A$ *then* $[\Omega]A = [\Omega, \Omega_Z]A$.

Lemma 45 (Context Partitioning).
If $\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta \longrightarrow \Omega, \blacktriangleright_{\hat{\alpha}}, \Omega_Z$ *then there is a* Ψ *such that* $[\Omega, \blacktriangleright_{\hat{\alpha}}, \Omega_Z](\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta) = [\Omega]\Delta, \Psi$.

Lemma 46 (Softness Goes Away).
If $\Delta, \Theta \longrightarrow \Omega, \Omega_Z$ *where* $\Delta \longrightarrow \Omega$ *and* Θ *is soft, then* $[\Omega, \Omega_Z](\Delta, \Theta) = [\Omega]\Delta$.

Proof. By induction on Θ, following the definition of $[\Omega]\Gamma$. □

Lemma 47 (Filling Completes). *If* $\Gamma \longrightarrow \Omega$ *and* (Γ, Θ) *is well-formed, then* $\Gamma, \Theta \longrightarrow \Omega, |\Theta|$.

Proof. By induction on Θ, following the definition of $|-|$ and applying the rules for $\longrightarrow$. □

Lemma 48 (Completing Stability).
If $\Gamma \longrightarrow \Omega$ *then* $[\Omega]\Gamma = [\Omega]\Omega$.

Lemma 49 (Finishing Types).
If $\Omega \vdash A$ *and* $\Omega \longrightarrow \Omega'$ *then* $[\Omega]A = [\Omega']A$.

Lemma 50 (Finishing Completions).
If $\Omega \longrightarrow \Omega'$ *then* $[\Omega]\Omega = [\Omega']\Omega'$.

Lemma 51 (Confluence of Completeness).
If $\Delta_1 \longrightarrow \Omega$ *and* $\Delta_2 \longrightarrow \Omega$ *then* $[\Omega]\Delta_1 = [\Omega]\Delta_2$.

H.2 Instantiation Soundness

Theorem 10 (Instantiation Soundness).
Given $\Delta \longrightarrow \Omega$ *and* $[\Gamma]B = B$ *and* $\hat{\alpha} \notin FV(B)$*:*

(1) If $\Gamma \vdash \hat{\alpha} :\leqq B \dashv \Delta$ *then* $[\Omega]\Delta \vdash [\Omega]\hat{\alpha} \leq [\Omega]B$.

(2) If $\Gamma \vdash B \leqq: \hat{\alpha} \dashv \Delta$ *then* $[\Omega]\Delta \vdash [\Omega]B \leq [\Omega]\hat{\alpha}$.

H.3 Soundness of Subtyping

Theorem 11 (Soundness of Algorithmic Subtyping).
If $\Gamma \vdash A <: B \dashv \Delta$ *where* $[\Gamma]A = A$ *and* $[\Gamma]B = B$ *and* $\Delta \longrightarrow \Omega$ *then* $[\Omega]\Delta \vdash [\Omega]A \leq [\Omega]B$.

Corollary 52 (Soundness, Pretty Version). *If* $\Psi \vdash A <: B \dashv \Delta$*, then* $\Psi \vdash A \leq B$.

I Typing Extension

Lemma 53 (Typing Extension).
If $\Gamma \vdash e \Leftarrow A \dashv \Delta$ *or* $\Gamma \vdash e \Rightarrow A \dashv \Delta$ *or* $\Gamma \vdash A \bullet e \Rrightarrow C \dashv \Delta$ *then* $\Gamma \longrightarrow \Delta$.

J Soundness of Typing

Theorem 12 (Soundness of Algorithmic Typing). *Given* $\Delta \longrightarrow \Omega$*:*

(i) If $\Gamma \vdash e \Leftarrow A \dashv \Delta$ *then* $[\Omega]\Delta \vdash e \Leftarrow [\Omega]A$.

(ii) If $\Gamma \vdash e \Rightarrow A \dashv \Delta$ *then* $[\Omega]\Delta \vdash e \Rightarrow [\Omega]A$.

(iii) If $\Gamma \vdash A \bullet e \Rrightarrow C \dashv \Delta$ *then* $[\Omega]\Delta \vdash [\Omega]A \bullet e \Rrightarrow [\Omega]C$.

K Completeness of Subtyping

K.1 Instantiation Completeness

Theorem 13 (Instantiation Completeness).
Given $\Gamma \longrightarrow \Omega$ *and* $A = [\Gamma]A$ *and* $\hat{\alpha} \in \mathsf{unsolved}(\Gamma)$ *and* $\hat{\alpha} \notin FV(A)$*:*

(1) If $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq [\Omega]A$
then there are Δ, Ω' *such that* $\Omega \longrightarrow \Omega'$ *and* $\Delta \longrightarrow \Omega'$ *and* $\Gamma \vdash \hat{\alpha} :\leqq A \dashv \Delta$.

(2) If $[\Omega]\Gamma \vdash [\Omega]A \leq [\Omega]\hat{\alpha}$
then there are Δ, Ω' *such that* $\Omega \longrightarrow \Omega'$ *and* $\Delta \longrightarrow \Omega'$ *and* $\Gamma \vdash A \leqq: \hat{\alpha} \dashv \Delta$.

K.2 Completeness of Subtyping

Theorem 14 (Generalized Completeness of Subtyping). *If* $\Gamma \longrightarrow \Omega$ *and* $\Gamma \vdash A$ *and* $\Gamma \vdash B$ *and* $[\Omega]\Gamma \vdash [\Omega]A \leq [\Omega]B$ *then there exist* Δ *and* Ω' *such that* $\Delta \longrightarrow \Omega'$ *and* $\Omega \longrightarrow \Omega'$ *and* $\Gamma \vdash [\Gamma]A <: [\Gamma]B \dashv \Delta$.

Corollary 54 (Completeness of Subtyping). *If* $\Psi \vdash A \leq B$ *then there is a* Δ *such that* $\Psi \vdash A <: B \dashv \Delta$.

L Completeness of Typing

Theorem 15 (Completeness of Algorithmic Typing). *Given* $\Gamma \longrightarrow \Omega$ *and* $\Gamma \vdash A$*:*

(i) If $[\Omega]\Gamma \vdash e \Leftarrow [\Omega]A$
then there exist Δ *and* Ω'
such that $\Delta \longrightarrow \Omega'$ *and* $\Omega \longrightarrow \Omega'$ *and* $\Gamma \vdash e \Leftarrow [\Gamma]A \dashv \Delta$.

(ii) If $[\Omega]\Gamma \vdash e \Rightarrow A$
then there exist Δ, Ω'*, and* A'
such that $\Delta \longrightarrow \Omega'$ *and* $\Omega \longrightarrow \Omega'$ *and* $\Gamma \vdash e \Rightarrow A' \dashv \Delta$ *and* $A = [\Omega']A'$.

(iii) If $[\Omega]\Gamma \vdash [\Omega]A \bullet e \Rrightarrow C$
then there exist Δ, Ω'*, and* C'
such that $\Delta \longrightarrow \Omega'$ *and* $\Omega \longrightarrow \Omega'$ *and* $\Gamma \vdash [\Gamma]A \bullet e \Rrightarrow C' \dashv \Delta$ *and* $C = [\Omega']C'$.

Proofs

In the rest of this document, we prove the results stated above, with the same sectioning.

A′ Declarative Subtyping

Proposition 1 (Weakening). *If* $\Psi \vdash A$ *then* $\Psi, \Psi' \vdash A$ *by a derivation of the same size.*

Proposition 2 (Substitution). *If* $\Psi \vdash A$ *and* $\Psi, \alpha, \Psi' \vdash B$ *then* $\Psi, \Psi' \vdash [A/\alpha]B$.

The proofs of these two propositions are routine inductions.

A′.1 Properties of Well-Formedness

A′.2 Reflexivity

Lemma 3 (Reflexivity of Declarative Subtyping). *Subtyping is reflexive: if* $\Psi \vdash A$ *then* $\Psi \vdash A \leq A$.

Proof. By induction on A.

- **Case** $A = 1$: Apply rule $\leq$Unit.
- **Case** $A = \alpha$: Apply rule $\leq$Var.
- **Case** $A = A_1 \rightarrow A_2$:

  $\Psi \vdash A_1 \leq A_1$ By i.h.
  $\Psi \vdash A_2 \leq A_2$ By i.h.
  $\Psi \vdash A_1 \rightarrow A_2 \leq A_1 \rightarrow A_2$ By $\leq\rightarrow$

- **Case** $A = \forall\alpha.\, A_0$:

  $\Psi, \alpha \vdash A_0 \leq A_0$ By i.h.
  $\Psi, \alpha \vdash \alpha$ By DeclUvarWF
  $\Psi, \alpha \vdash [\alpha/\alpha]A_0 \leq A_0$ By def. of substitution
  $\Psi, \alpha \vdash \forall\alpha.\, A_0 \leq A_0$ By $\leq\forall$L
  $\Psi \vdash \forall\alpha.\, A_0 \leq \forall\alpha.\, A_0$ By $\leq\forall$R

□

A′.3 Subtyping Implies Well-Formedness

Lemma 4 (Well-Formedness). *If* $\Psi \vdash A \leq B$ *then* $\Psi \vdash A$ *and* $\Psi \vdash B$.

Proof. By induction on the given derivation. All 5 cases are straightforward. □

A′.4 Substitution

Lemma 5 (Substitution). *If* $\Psi \vdash \tau$ *and* $\Psi, \alpha, \Psi' \vdash A \leq B$ *then* $\Psi, [\tau/\alpha]\Psi' \vdash [\tau/\alpha]A \leq [\tau/\alpha]B$.

Proof. By induction on the given derivation.

- **Case** $$\frac{\beta \in (\Psi, \alpha, \Psi')}{\Psi, \alpha, \Psi' \vdash \beta \leq \beta}\leq\text{Var}$$

  It is given that $\Psi \vdash \tau$.

  Either $\beta = \alpha$ or $\beta \neq \alpha$. In the former case: We need to show $\Psi, \Psi' \vdash [\tau/\alpha]\alpha \leq [\tau/\alpha]\alpha$, that is, $\Psi, \Psi' \vdash \tau \leq \tau$, which follows by Lemma 3 (Reflexivity of Declarative Subtyping). In the latter case: We need to show $\Psi, \Psi' \vdash [\tau/\alpha]\beta \leq [\tau/\alpha]\beta$, that is, $\Psi, \Psi' \vdash \beta \leq \beta$. Since $\beta \in (\Psi, \alpha, \Psi')$ and $\beta \neq \alpha$, we have $\beta \in (\Psi, \Psi')$, so applying $\leq$Var gives the result.

- **Case**

$$\frac{}{\Psi, \alpha, \Psi' \vdash 1 \leq 1} \leq\mathsf{Unit}$$

For all τ, substituting $[\tau/\alpha]1 = 1$, and applying $\leq$Unit gives the result.

- **Case** $\dfrac{\Psi, \alpha, \Psi' \vdash B_1 \leq A_1 \qquad \Psi, \alpha, \Psi' \vdash A_2 \leq B_2}{\Psi, \alpha, \Psi' \vdash A_1 \to A_2 \leq B_1 \to B_2} \leq\to$

$\Psi, \alpha, \Psi' \vdash B_1 \leq A_1$ — Subderivation
$\Psi, \Psi' \vdash [\tau/\alpha]B_1 \leq [\tau/\alpha]A_1$ — By i.h.

$\Psi, \alpha, \Psi' \vdash A_2 \leq B_2$ — Subderivation
$\Psi, \Psi' \vdash [\tau/\alpha]A_2 \leq [\tau/\alpha]B_2$ — By i.h.

$\Psi, \Psi' \vdash ([\tau/\alpha]A_1) \to ([\tau/\alpha]A_2) \leq ([\tau/\alpha]B_1) \to ([\tau/\alpha]B_2)$ — By $\leq\to$
☞ $\Psi, \Psi' \vdash [\tau/\alpha](A_1 \to A_2) \leq [\tau/\alpha](B_1 \to B_2)$ — By definition of subst.

- **Case** $\dfrac{\Psi, \alpha, \Psi' \vdash \sigma \qquad \Psi, \alpha, \Psi' \vdash [\sigma/\beta]A_0 \leq B}{\Psi, \alpha, \Psi' \vdash \forall\beta.\, A_0 \leq B} \leq\forall\mathsf{L}$

$\Psi, \alpha, \Psi' \vdash [\sigma/\beta]A_0 \leq B$ — Subderivation
$\Psi, \Psi' \vdash [\tau/\alpha][\sigma/\beta]A_0 \leq [\tau/\alpha]B$ — By i.h.
$\Psi, \Psi' \vdash \big[[\tau/\alpha]\sigma \,/\, \beta\big][\tau/\alpha]A_0 \leq [\tau/\alpha]B$ — By distributivity of substitution

$\Psi, \alpha, \Psi' \vdash \sigma$ — Premise
$\Psi \vdash \tau$ — Given
$\Psi, \Psi' \vdash [\tau/\alpha]\sigma$ — By Proposition 2

$\Psi, \Psi' \vdash \forall\beta.\, [\tau/\alpha]A_0 \leq [\tau/\alpha]B$ — By $\leq\forall$L
☞ $\Psi, \Psi' \vdash [\tau/\alpha]\,\big(\forall\beta.\, A_0\big) \leq [\tau/\alpha]B$ — By definition of substitution

- **Case** $\dfrac{\Psi, \alpha, \Psi', \beta \vdash A \leq B_0}{\Psi, \alpha, \Psi' \vdash A \leq \forall\beta.\, B_0} \leq\forall\mathsf{R}$

$\Psi, \alpha, \Psi', \beta \vdash A \leq B_0$ — Subderivation
$\Psi, \Psi', \beta \vdash [\tau/\alpha]A \leq [\tau/\alpha]B_0$ — By i.h.
$\Psi, \Psi' \vdash [\tau/\alpha]A \leq \forall\beta.\, [\tau/\alpha]B_0$ — By $\leq\forall$R
☞ $\Psi, \Psi' \vdash [\tau/\alpha]A \leq [\tau/\alpha](\forall\beta.\, B_0)$ — By definition of substitution ☐

A′.5 Transitivity

To prove transitivity, we use a metric that adapts ideas from a proof of cut elimination by Pfenning (1995).

Lemma 6 (Transitivity of Declarative Subtyping)**.** *If* $\Psi \vdash A \leq B$ *and* $\Psi \vdash B \leq C$ *then* $\Psi \vdash A \leq C$.

Proof. By induction with the following metric:

$$\langle \#\forall(B), \quad \mathcal{D}_1 + \mathcal{D}_2 \rangle$$

where $\langle \dots \rangle$ denotes lexicographic order, the first part $\#\forall(B)$ is the number of quantifiers in B, and the second part is the (simultaneous) size of the derivations $\mathcal{D}_1 :: \Psi \vdash A \leq B$ and $\mathcal{D}_2 :: \Psi \vdash B \leq C$. We need to consider the number of quantifiers first in one case: when $\leq\forall$R concluded $\mathcal{D}_1$ and $\leq\forall$L concluded $\mathcal{D}_2$, because in that case, the derivations on which the i.h. must be applied are not necessarily smaller.

- **Case** $\dfrac{\alpha \in \Psi}{\Psi \vdash \alpha \leq \alpha} \leq\mathsf{Var} \qquad \dfrac{\alpha \in \Psi}{\Psi \vdash \alpha \leq \alpha} \leq\mathsf{Var}$

Apply rule $\leq$Var.

- **Case** $\leq$Unit / $\leq$Unit: Similar to the $\leq$Var / $\leq$Var case.

- **Case** $\dfrac{\Psi \vdash B_1 \leq A_1 \qquad \Psi \vdash A_2 \leq B_2}{\Psi \vdash A_1 \to A_2 \leq B_1 \to B_2} \leq\to \qquad \dfrac{\Psi \vdash C_1 \leq B_1 \qquad \Psi \vdash B_2 \leq C_2}{\Psi \vdash B_1 \to B_2 \leq C_1 \to C_2} \leq\to$

  By i.h. on the 3rd and 1st subderivations, $\Psi \vdash C_1 \leq A_1$.
  By i.h. on the 2nd and 4th subderivations, $\Psi \vdash A_2 \leq C_2$.
  By $\leq\to$, $\Psi \vdash A_1 \to A_2 \leq C_1 \to C_2$.

If $\leq\forall$L concluded $\mathcal{D}_1$:

- **Case** $\dfrac{\Psi \vdash \tau \qquad \Psi \vdash [\tau/\alpha]A_0 \leq B}{\Psi \vdash \forall\alpha.\, A_0 \leq B} \leq\forall\mathsf{L}$

	$\Psi \vdash \tau$	Premise
	$\Psi \vdash [\tau/\alpha]A_0 \leq B$	Subderivation
	$\Psi \vdash B \leq C$	Given ($\mathcal{D}_2$)
	$\Psi \vdash [\tau/\alpha]A_0 \leq C$	By i.h.
☞	$\Psi \vdash \forall\alpha.\, A_0 \leq C$	By $\leq\forall$L

If $\leq\forall$R concluded $\mathcal{D}_2$:

- **Case** $\dfrac{\Psi, \beta \vdash B \leq C}{\Psi \vdash B \leq \forall\beta.\, C} \leq\forall\mathsf{R}$

	$\Psi \vdash \tau$	Premise
	$\Psi, \beta \vdash B \leq C$	Subderivation
	$\Psi \vdash A \leq B$	Given ($\mathcal{D}_1$)
	$\Psi, \beta \vdash A \leq B$	By Proposition 1
	$\Psi, \beta \vdash A \leq C$	By i.h.
☞	$\Psi \vdash A \leq \forall\beta.\, C$	By $\leq\forall$L

The only remaining possible case is $\leq\forall$R / $\leq\forall$L.

- **Case** $\dfrac{\Psi, \beta \vdash A \leq B_0}{\Psi \vdash A \leq \forall\beta.\, B_0} \leq\forall\mathsf{R} \qquad \dfrac{\Psi \vdash \tau \qquad \Psi \vdash [\tau/\beta]B_0 \leq C}{\Psi \vdash \forall\beta.\, B_0 \leq C} \leq\forall\mathsf{L}$

	$\Psi, \beta \vdash A \leq B_0$	Subderivation of $\mathcal{D}_1$
	$\Psi \vdash \tau$	Premise of $\mathcal{D}_2$
	$\Psi \vdash [\tau/\beta]A \leq [\tau/\beta]B_0$	By Lemma 5 (Substitution)
	$[\tau/\beta]A = A$	β cannot appear in A
	$\Psi \vdash A \leq [\tau/\beta]B_0$	By above equality
	$\Psi \vdash [\tau/\beta]B_0 \leq C$	Subderivation of $\mathcal{D}_2$
☞	$\Psi \vdash A \leq C$	By i.h. (one less $\forall$ quantifier in B)

□

A′.6 Invertibility of $\leq\forall$R

Lemma 7 (Invertibility).
If $\mathcal{D}$ derives $\Psi \vdash A \leq \forall\beta.\, B$ then $\mathcal{D}'$ derives $\Psi, \beta \vdash A \leq B$ where $\mathcal{D}' < \mathcal{D}$.

Proof. By induction on the given derivation $\mathcal{D}$.

- **Cases** $\leq$Var, $\leq$Unit, $\leq\to$: Impossible: the supertype cannot have the form $\forall\beta.\, B$.

- **Case** $\dfrac{\Psi, \beta \vdash A \leq B}{\Psi \vdash A \leq \forall\beta.\, B} \leq\forall\mathsf{R}$

  The subderivation is exactly what we need, and is strictly smaller than $\mathcal{D}$.

- **Case** $\dfrac{\Psi \vdash \tau \qquad \overset{\mathcal{D}_0}{\Psi \vdash [\tau/\alpha]A_0 \leq \forall\beta.\, B}}{\Psi \vdash \forall\alpha.\, A_0 \leq \forall\beta.\, B} \leq\forall\mathsf{L}$

  By i.h., $\mathcal{D}_0'$ derives $\Psi, \beta \vdash [\tau/\alpha]A_0 \leq B$ where $\mathcal{D}_0' < \mathcal{D}_0$.
  By $\leq\forall\mathsf{L}$, $\mathcal{D}'$ derives $\Psi, \beta \vdash \forall\alpha.\, A_0 \leq B$; since $\mathcal{D}_0' < \mathcal{D}_0$, we have $\mathcal{D}' < \mathcal{D}$. □

A′.7 Non-Circularity and Equality

Lemma 8 (Occurrence)**.**

(i) If $\Psi \vdash A \leq \tau$ *then* $\tau \not\sqsupsetneq A$.

(ii) If $\Psi \vdash \tau \leq B$ *then* $\tau \not\sqsupsetneq B$.

Proof. By induction on the given derivation.

- **Cases** $\leq\mathsf{Var}$, $\leq\mathsf{Unit}$: (i), (ii): Here A and B have no subterms at all, so the result is immediate.

- **Case** $\dfrac{\Psi \vdash B_1 \leq A_1 \qquad \Psi \vdash A_2 \leq B_2}{\Psi \vdash A_1 \to A_2 \leq B_1 \to B_2} \leq\to$

  (i) Here, $A = A_1 \to A_2$ and $\tau = B_1 \to B_2$.

$B_1 \not\sqsupsetneq A_1$	By i.h. (ii)
$B_1 \to B_2 \not\preceq A_1$	Suppose $B_1 \to B_2 \preceq A_1$. Then $B_1 \sqsupsetneq A_1$: contradiction.
$B_2 \not\sqsupsetneq A_2$	By i.h. (i)
$B_1 \to B_2 \not\preceq A_2$	Similar

  Suppose (for a contradiction) that $B_1 \to B_2 \sqsupsetneq A_1 \to A_2$.
  Now $B_1 \to B_2 \preceq A_1$ or $B_1 \to B_2 \preceq A_2$.
  But above, we showed that both were false: contradiction.
  Therefore, $B_1 \to B_2 \not\prec A_1 \to A_2$.
  Therefore, $B_1 \to B_2 \not\sqsupsetneq A_1 \to A_2$.

  (ii) Here, $A = \tau$ and $B = B_1 \to B_2$.
  Symmetric to the previous case.

- **Case** $\dfrac{\Psi \vdash \tau' \qquad \Psi \vdash [\tau'/\alpha]A_0 \leq \tau}{\Psi \vdash \forall\alpha.\, A_0 \leq \tau} \leq\forall\mathsf{L}$

  In part (ii), this case cannot arise, so we prove part (i).

  By i.h. (i), $\tau \not\sqsupsetneq [\tau'/\alpha]A_0$.
  It follows from the definition of $\sqsupsetneq$ that $\tau \not\sqsupsetneq \forall\alpha.\, A_0$.

- **Case** $\dfrac{\Psi, \beta \vdash \tau \leq B_0}{\Psi \vdash \tau \leq \forall\beta.\, B_0} \leq\forall\mathsf{R}$

  In part (i), this case cannot arise, so we prove part (ii).

  Similar to the $\leq\forall\mathsf{L}$ case. □

Lemma 9 (Monotype Equality)**.** *If* $\Psi \vdash \sigma \leq \tau$ *then* $\sigma = \tau$.

Proof. By induction on the given derivation.

- **Case** $\leq$Var: Immediate.
- **Case** $\leq$Unit: Immediate.
- **Case** $\dfrac{\Psi \vdash B_1 \leq A_1 \qquad \Psi \vdash A_2 \leq B_2}{\Psi \vdash A_1 \to A_2 \leq B_1 \to B_2} \leq\to$

  By i.h. on each subderivation, $B_1 = A_1$ and $A_2 = B_2$. Therefore $A_1 \to A_2 = B_1 \to B_2$.
- **Case** $\leq\forall$L: Here $\sigma = \forall\alpha.\, A_0$, which is not a monotype, so this case is impossible.
- **Case** $\leq\forall$R: Here $\tau = \forall\beta.\, B_0$, which is not a monotype, so this case is impossible. ☐

B′ Type Assignment

Theorem 1 (Completeness of Bidirectional Typing). *If* $\Psi \vdash e : A$ *then there exists* e' *such that* $\Psi \vdash e' \Rightarrow A$ *and* $|e'| = e$.

Proof. By induction on the derivation of $\Psi \vdash e : A$.

- **Case** $\dfrac{x : A \in \Psi}{\Psi \vdash x : A}$ AVar

  Immediate, by rule DeclVar.

- **Case** $\dfrac{\Psi, x : A \vdash e : B}{\Psi \vdash \lambda x.\, e : A \to B}$ A$\to$I

  By inversion, we have $\Psi, x : A \vdash e : B$.
  By induction, we have $\Psi, x : A \vdash e' \Rightarrow B$, where $|e'| = e$.
  By Lemma 3 (Reflexivity of Declarative Subtyping), $\Psi \vdash B \leq B$.
  By rule DeclSub, $\Psi, x : A \vdash e' \Leftarrow B$.
  By rule Decl$\to$I, $\Psi \vdash \lambda x.\, e' \Leftarrow A \to B$.
  By rule DeclAnno, $\Psi \vdash ((\lambda x.\, e') : A \to B) \Rightarrow A \to B$.
  By definition, $|((\lambda x.\, e') : A \to B)| = |\lambda x.\, e'| = \lambda x.\, |e'| = \lambda x.\, e$.

- **Case** $\dfrac{\Psi \vdash e_1 : A \to B \qquad \Psi \vdash e_2 : A}{\Psi \vdash e_1\ e_2 : B}$ A$\to$E

  By induction, $\Psi \vdash e_1' \Rightarrow A \to B$ and $|e_1'| = e_1$.
  By induction, $\Psi \vdash e_2' \Rightarrow A$ and $|e_2'| = e_2$.
  By Lemma 3 (Reflexivity of Declarative Subtyping), $\Psi \vdash A \leq A$.
  By rule DeclSub, $\Psi \vdash e_2' \Leftarrow A$.
  By rule Decl$\to$App, $\Psi \vdash A \to B \bullet e_2' \Rightarrow\!\!\!\Rightarrow B$.
  By rule Decl$\to$E, $\Psi \vdash e_1'\ e_2' \Rightarrow B$.
  By definition, $|e_1'\ e_2'| = |e_1'|\ |e_2'| = e_1\ e_2$.

- **Case** $\dfrac{\Psi, \alpha \vdash e : A}{\Psi \vdash e : \forall\alpha.\, A}$ A$\forall$I

  By induction, $\Psi, \alpha \vdash e' \Rightarrow A$ where $|e'| = e$.
  By Lemma 3 (Reflexivity of Declarative Subtyping), $\Psi, \alpha \vdash A \leq A$.
  By rule DeclSub, $\Psi, \alpha \vdash e' \Leftarrow A$.
  By rule Decl$\forall$I, $\Psi \vdash e' \Leftarrow \forall\alpha.\, A$.
  By rule DeclAnno, $\Psi \vdash e' : \forall\alpha.\, A \Rightarrow \forall\alpha.\, A$.
  By definition, $|e' : \forall\alpha.\, A| = |e'| = e$.

- **Case** $\dfrac{\Psi \vdash e : \forall\alpha.\, A \qquad \Psi \vdash \tau}{\Psi \vdash e : [\tau/\alpha]A}\ \mathsf{A\forall E}$

  By induction, $\Psi \vdash e' \Rightarrow \forall\alpha.\, A$ where $|e'| = e$.
  By Lemma 3 (Reflexivity of Declarative Subtyping), $\Psi \vdash [\tau/\alpha]A \leq [\tau/\alpha]A$.
  By $\leq\forall\mathsf{L}$, $\Psi \vdash \forall\alpha.\, A \leq [\tau/\alpha]A$.
  By rule DeclSub, $\Psi \vdash e' \Leftarrow [\tau/\alpha]A$.
  By rule DeclAnno, $\Psi \vdash (e' : [\tau/\alpha]A) \Leftarrow [\tau/\alpha]A$.
  By definition, $|e' : [\tau/\alpha]A| = |e'| = e$. □

Lemma 10 (Subtyping Coercion). *If* $\Psi \vdash A \leq B$ *then there exists* f *which is* βη*-equal to the identity such that* $\Psi \vdash f : A \to B$.

Proof. By induction on the derivation of $\Psi \vdash A \leq B$.

- **Case** $\dfrac{\alpha \in \Psi}{\Psi \vdash \alpha \leq \alpha}\ \leq\mathsf{Var}$

  Choose $f = \lambda x.\, x$.
  Clearly $\Psi \vdash \lambda x.\, x : \alpha \to \alpha$.

- **Case**

  $\dfrac{}{\Psi \vdash 1 \leq 1}\ \leq\mathsf{Unit}$

  Choose $f = \lambda x.\, x$.
  Clearly $\Psi \vdash \lambda x.\, x : 1 \to 1$.

- **Case** $\dfrac{\Psi \vdash B_1 \leq A_1 \qquad \Psi \vdash A_2 \leq B_2}{\Psi \vdash A_1 \to A_2 \leq B_1 \to B_2}\ \leq\!\to$

  By induction, we have $g : B_1 \to A_1$, which is βη-equal to the identity.
  By induction, we have $k : A_2 \to B_2$, which is βη-equal to the identity.
  Let f be $\lambda h.\, k \circ h \circ g$.
  It is easy to verify that $\Psi \vdash f : (A_1 \to A_2) \to (B_1 \to B_2)$.
  Since k and g are identities, $f =_{\beta\eta} \lambda h.\, h$.

- **Case** $\dfrac{\Psi \vdash \tau \qquad \Psi \vdash [\tau/\alpha]A \leq B}{\Psi \vdash \forall\alpha.\, A \leq B}\ \leq\forall\mathsf{L}$

  By induction, $g : [\tau/\alpha]A \to B$.
  Let $f \triangleq \lambda x.\, g\ x$.
  f is an eta-expansion of g, which is βη-equal to the identity. Hence f is too.
  Also, $\lambda x.\, g\ x : (\forall\alpha.\, A) \to B$, using the Decl∀E rule on x.

- **Case** $\dfrac{\Psi, \beta \vdash A \leq B}{\Psi \vdash A \leq \forall\beta.\, B}\ \leq\forall\mathsf{R}$

  By induction, we have g such that $\Psi, \beta \vdash g : A \to B$.
  Let $f \triangleq \lambda x.\, g\ x$.
  Use the following derivation:

$$\dfrac{\dfrac{\dfrac{\mathsf{Weaken}\ \dfrac{\vdots}{\Psi, \beta \vdash g : A \to B}}{\Psi, x : A, \beta \vdash g : A \to B} \qquad \Psi, x : A, \beta \vdash x : A}{\dfrac{\Psi, x : A, \beta \vdash g\ x : B}{\Psi, x : A \vdash g\ x : \forall\beta.\, B}}}{\Psi \vdash \lambda x.\, g\ x : A \to \forall\beta.\, B}$$

□

Lemma 11 (Application Subtyping). *If* $\Psi \vdash A \bullet e \Rrightarrow C$ *then there exists* B *such that* $\Psi \vdash A \leq B \to C$ *and* $\Psi \vdash e \Leftarrow B$ *by a smaller derivation.*

Proof. By induction on the given derivation $\mathcal{D}$.

- **Case** $\dfrac{\Psi \vdash e \Leftarrow B}{\Psi \vdash B \to C \bullet e \Rrightarrow C}$ Decl→App

  ☞ $\mathcal{D}' :: \Psi \vdash e \Leftarrow B$ — Subderivation
  ☞ $\mathcal{D}' < \mathcal{D}$ — $\mathcal{D}'$ is a subderivation of $\mathcal{D}$
  ☞ $\Psi \vdash \underbrace{B \to C}_{A} \leq B \to C$ — By Lemma 3 (Reflexivity of Declarative Subtyping)

- **Case** $\dfrac{\Psi \vdash \tau \qquad \Psi \vdash [\tau/\alpha]A_0 \bullet e \Rrightarrow C}{\Psi \vdash \forall\alpha.\, A_0 \bullet e \Rrightarrow C}$ Decl∀App

  $\Psi \vdash \tau$ — Subderivation
  $\Psi \vdash [\tau/\alpha]A_0 \bullet e \Rrightarrow C$ — Subderivation
  $\Psi \vdash [\tau/\alpha]A_0 \leq B \to C$ — By i.h.
  ☞ $\mathcal{D}' :: \Psi \vdash e \Leftarrow B$ — "
  ☞ $\mathcal{D}' < \mathcal{D}$ — "
  ☞ $\Psi \vdash \forall\alpha.\, A_0 \leq B \to C$ — By $\leq\forall$L ∎

Theorem 2 (Soundness of Bidirectional Typing). *We have that:*

- *If* $\Psi \vdash e \Leftarrow A$*, then there is an* e' *such that* $\Psi \vdash e' : A$ *and* $e' =_{\beta\eta} |e|$.
- *If* $\Psi \vdash e \Rightarrow A$*, then there is an* e' *such that* $\Psi \vdash e' : A$ *and* $e' =_{\beta\eta} |e|$.

Proof.

- **Case** $\dfrac{(x : A) \in \Psi}{\Psi \vdash x \Rightarrow A}$ DeclVar

  By rule AVar, $\Psi \vdash x : A$.
  Note $x =_{\beta\eta} x$.

- **Case** $\dfrac{\Psi \vdash e \Rightarrow A \qquad \Psi \vdash A \leq B}{\Psi \vdash e \Leftarrow B}$ DeclSub

  By induction, $\Psi \vdash e' : A$ and $e' =_{\beta\eta} |e|$.
  By Lemma 10 (Subtyping Coercion), $f : A \to B$ such that $f =_{\beta\eta} \mathrm{id}$.
  By $A{\to}E$, $\Psi \vdash f\ e' : B$.
  Note $f\ e' =_{\beta\eta} \mathrm{id}\ e' =_{\beta\eta} e' =_{\beta\eta} |e|$.

- **Case** $\dfrac{\Psi \vdash e \Leftarrow A}{\Psi \vdash (e : A) \Rightarrow A}$ DeclAnno

  By induction, $\Psi \vdash e' : A$ such that $e' =_{\beta\eta} |e|$.
  Note $e' =_{\beta\eta} |e| = |e : A|$.

- **Case** $\dfrac{}{\Psi \vdash () \Leftarrow 1}$ Decl1I

  By AUnit, $\Psi \vdash () : 1$.
  Note $() =_{\beta\eta} ()$.

- **Case**
$$\frac{}{\Psi \vdash () \Rightarrow 1}\ \text{Decl1I}{\Rightarrow}$$
By AUnit, $\Psi \vdash () : 1$.
Note $() =_{\beta\eta} ()$.

- **Case**
$$\frac{\Psi, \alpha \vdash e \Leftarrow A}{\Psi \vdash e \Leftarrow \forall\alpha.\, A}\ \text{Decl}\forall\text{I}$$
By induction, $\Psi, \alpha \vdash e' : A$ such that $e' =_{\beta\eta} |e|$.
By rule A$\forall$I, $\Psi \vdash e' : \forall\alpha.\, A$.

- **Case**
$$\frac{\Psi, x : A \vdash e \Leftarrow B}{\Psi \vdash \lambda x.\, e \Leftarrow A \to B}\ \text{Decl}{\to}\text{I}$$
By induction, $\Psi, x : A \vdash e' : B$ such that $e' =_{\beta\eta} |e|$.
By A$\to$I, $\Psi \vdash \lambda x.\, e' : A \to B$.
Note $\lambda x.\, e' =_{\beta\eta} \lambda x.\, |e| = |\lambda x.\, e|$.

- **Case**
$$\frac{\Psi \vdash \sigma \to \tau \qquad \Psi, x : \sigma \vdash e \Leftarrow \tau}{\Psi \vdash \lambda x.\, e \Rightarrow \sigma \to \tau}\ \text{Decl}{\to}\text{I}{\Rightarrow}$$
By induction, $\Psi, x : \sigma \vdash e' : \tau$ such that $e' =_{\beta\eta} |e|$.
By A$\to$I, $\Psi \vdash \lambda x.\, e' : \sigma \to \tau$.
Note $\lambda x.\, e' =_{\beta\eta} \lambda x.\, |e| = |\lambda x.\, e|$.

- **Case**
$$\frac{\Psi \vdash e_1 \Rightarrow A \qquad \Psi \vdash A \bullet e_2 \Rrightarrow C}{\Psi \vdash e_1\, e_2 \Rightarrow C}\ \text{Decl}{\to}\text{E}$$
By induction, $\Psi \vdash e_1' : A$ such that $e_1' =_{\beta\eta} |e_1|$.
By Lemma 11 (Application Subtyping), there is a B such that
1. $\Psi \vdash A \leq B \to C$, and
2. $\Psi \vdash e_2 \Leftarrow B$, which is no bigger than $\Psi \vdash A \bullet e_2 \Rrightarrow C$.
By Lemma 10 (Subtyping Coercion), we have f such that $\Psi \vdash f : A \to B \to C$ and $f =_{\beta\eta} \mathsf{id}$.
By induction, we get $\Psi \vdash e_2' : B$ and $e_2' =_{\beta\eta} |e_2|$.
By A$\to$E twice, $\Psi \vdash f\ e_1'\ e_2' : C$.
Note $f\ e_1'\ e_2' =_{\beta\eta} \mathsf{id}\ e_1'\ e_2' =_{\beta\eta} e_1'\ e_2' =_{\beta\eta} |e_1|\ e_2' =_{\beta\eta} |e_1|\ |e_2| = |e_1\ e_2|$. □

C′ Robustness of Typing

Theorem 3 (Substitution).
Assume $\Psi \vdash e \Rightarrow A$.

- *If* $\Psi, x : A \vdash e' \Leftarrow C$ *then* $\Psi \vdash [e/x]e' \Leftarrow C$.
- *If* $\Psi, x : A \vdash e' \Rightarrow C$ *then* $\Psi \vdash [e/x]e' \Rightarrow C$.
- *If* $\Psi, x : A \vdash B \bullet e' \Rrightarrow C$ *then* $\Psi \vdash B \bullet [e/x]e' \Rrightarrow C$.

Proof. By a straightforward induction on the given derivation. □

Lemma 12 (Type Substitution).
Assume $\Psi \vdash \tau$.

- *If* $\Psi, \alpha, \Psi' \vdash e' \Leftarrow C$ *then* $\Psi, [\tau/\alpha]\Psi' \vdash [\tau/\alpha]e' \Leftarrow [\tau/\alpha]C$.
- *If* $\Psi, \alpha, \Psi' \vdash e' \Rightarrow C$ *then* $\Psi, [\tau/\alpha]\Psi' \vdash [\tau/\alpha]e' \Rightarrow [\tau/\alpha]C$.

- *If* $\Psi, \alpha, \Psi' \vdash B \bullet e' \Rrightarrow C$ *then* $\Psi, [\tau/\alpha]\Psi' \vdash [\tau/\alpha]B \bullet [\tau/\alpha]e' \Rrightarrow [A/\alpha]C$.

Moreover, the resulting derivation contains no more applications of typing rules than the given one. (Internal subtyping derivations, however, may grow.)

Proof. By induction on the given derivation.

In the DeclVar case, split on whether the variable being typed is in Ψ or Ψ'; the former is immediate, and in the latter, use the fact that $(x : C) \in \Psi'$ implies $(x : [\tau/\alpha]C) \in [\tau/\alpha]\Psi'$.

In the DeclSub case, use the i.h. and Lemma 5 (Substitution).

In the DeclAnno case, we are substituting in the annotation in the term, as well as in the type.

In Decl$\to$I, Decl$\to$I$\Rightarrow$ and Decl$\forall$I, we add to the context in the premise, which is why the statement is generalized for nonempty Ψ'. $\square$

Lemma 13 (Subsumption). *Suppose* $\Psi' \leq \Psi$. *Then:*

(i) If $\Psi \vdash e \Leftarrow A$ *and* $\Psi \vdash A \leq A'$ *then* $\Psi' \vdash e \Leftarrow A'$.

(ii) If $\Psi \vdash e \Rightarrow A$ *then there exists* A' *such that* $\Psi \vdash A' \leq A$ *and* $\Psi' \vdash e \Rightarrow A'$.

(iii) If $\Psi \vdash C \bullet e \Rrightarrow A$ *and* $\Psi \vdash C' \leq C$
then there exists A' *such that* $\Psi \vdash A' \leq A$ *and* $\Psi' \vdash C' \bullet e \Rrightarrow A'$.

Proof. By mutual induction: in (i), by lexicographic induction on the derivation of the checking judgment, then of the subtyping judgment; in (ii), by induction on the derivation of the synthesis judgment; in (iii), by lexicographic induction on the derivation of the application judgment, then of the subtyping judgment.

For part (i), checking:

- **Case** $$\dfrac{\Psi \vdash e \Rightarrow B \qquad \Psi \vdash B \leq A}{\Psi \vdash e \Leftarrow A}\ \text{DeclSub}$$

	$\Psi \vdash e \Rightarrow B$	Subderivation
	$\Psi' \vdash e \Rightarrow B'$	By i.h.
	$\Psi \vdash B' \leq B$	"
	$\Psi \vdash B \leq A$	Subderivation
	$\Psi \vdash A \leq A'$	Given
	$\Psi \vdash B' \leq A'$	By Lemma 6 (Transitivity of Declarative Subtyping) (twice)
	$\Psi' \vdash B' \leq A'$	By weakening
☞	$\Psi' \vdash e \Leftarrow A'$	By DeclSub

- **Case** $$\dfrac{}{\Psi \vdash () \Leftarrow 1}\ \text{Decl1I}$$

	$\Psi' \vdash () \Rightarrow 1$	By Decl1I$\Rightarrow$
	$\Psi \vdash 1 \leq A'$	Given
	$\Psi' \vdash 1 \leq A'$	By weakening
☞	$\Psi' \vdash () \Leftarrow A'$	By DeclSub

- **Case** $$\dfrac{\Psi, \alpha \vdash e \Leftarrow A_0}{\Psi \vdash e \Leftarrow \forall\alpha.\, A_0}\ \text{Decl}\forall\text{I}$$

We consider cases of $\Psi \vdash \forall\alpha.\, A_0 \leq A'$:

- **Case** $\dfrac{\Psi, \beta \vdash \forall\alpha.\, A_0 \leq B}{\Psi \vdash \forall\alpha.\, A_0 \leq \forall\beta.\, B} \leq\forall R$

$\Psi, \beta \vdash \forall\alpha.\, A_0 \leq B$	Subderivation
$\Psi \vdash e \Leftarrow \forall\alpha.\, A_0$	Given
$\Psi' \vdash e \Leftarrow B$	By i.h. (i)
☞ $\Psi' \vdash e \Leftarrow \underbrace{\forall\beta.\, B}_{A'}$	By Decl∀I

- **Case** $\dfrac{\Psi \vdash \tau \qquad \Psi \vdash [\tau/\alpha]A_0 \leq A'}{\Psi \vdash \forall\alpha.\, A_0 \leq A'} \leq\forall L$

$\Psi, \alpha \vdash e \Leftarrow A_0$	Subderivation
$\Psi \vdash e \Leftarrow [\tau/\alpha]A_0$	By Lemma 12 (Type Substitution)
$\Psi \vdash [\tau/\alpha]A_0 \leq A'$	Subderivation
☞ $\Psi' \vdash e \Leftarrow A'$	By i.h. (i)

- **Case** $\dfrac{\Psi, x : A_1 \vdash e_0 \Leftarrow A_2}{\Psi \vdash \lambda x.\, e_0 \Leftarrow A_1 \to A_2}$ Decl→I

  We consider cases of $\Psi \vdash A_1 \to A_2 \leq A'$:

  - **Case** $\dfrac{\Psi \vdash B_1 \leq A_1 \qquad \Psi \vdash A_2 \leq B_2}{\Psi \vdash A_1 \to A_2 \leq B_1 \to B_2} \leq\to$

$\Psi \leq \Psi'$	Given
$\Psi \vdash B_1 \leq A_1$	Subderivation
$\Psi', x : B_1 \leq \Psi, x : A_1$	By CtxSubVar
$\Psi', x : B_1 \vdash e_0 \Leftarrow B_2$	By i.h. (i)
☞ $\Psi' \vdash \lambda x.\, e_0 \Leftarrow B_1 \to B_2$	By Decl→I

  - **Case** $\dfrac{\Psi, \beta \vdash A_1 \to A_2 \leq B'}{\Psi \vdash A_1 \to A_2 \leq \forall\beta.\, B'} \leq\forall R$

$\Psi, \beta \vdash A_1 \to A_2 \leq B'$	Subderivation
$\Psi, \beta \vdash \lambda x.\, e_0 \Leftarrow A_1 \to A_2$	By weakening
$\Psi', \beta \vdash \lambda x.\, e_0 \Leftarrow B'$	By i.h. (i)
☞ $\Psi' \vdash \lambda x.\, e_0 \Leftarrow \forall\beta.\, B'$	By Decl∀I

For part (ii), synthesis:

- **Case** $\dfrac{(x : A) \in \Psi}{\Psi \vdash x \Rightarrow A}$ DeclVar

  By inversion on $\Psi' \leq \Psi$, we have $(x : A') \in \Psi'$ where $\Psi \vdash A' \leq A$.
  By DeclVar, $\Psi' \vdash x \Rightarrow A'$.

- **Case** $\dfrac{\Psi \vdash e_0 \Leftarrow A}{\Psi \vdash (e_0 : A) \Rightarrow A}$ DeclAnno

  Let $A' = A$.

$\Psi \vdash e_0 \Leftarrow A$	Subderivation
$\Psi' \vdash e_0 \Leftarrow A$	By i.h.
☞ $\Psi' \vdash (e_0 : A) \Rightarrow A'$	By DeclAnno and $A' = A$
☞ $\Psi \vdash A' \leq A$	By Lemma 3 (Reflexivity of Declarative Subtyping)

- **Case**

$$\frac{}{\Psi \vdash () \Rightarrow 1}\ \mathsf{Decl1I}\Rightarrow$$

Let $A' = 1$.

☞ $\Psi' \vdash () \Rightarrow 1$ By $\mathsf{Decl1I}\Rightarrow$

☞ $\Psi \vdash 1 \leq 1$ By $\leq$Unit

- **Case**

$$\frac{\Psi \vdash \sigma \to \tau \qquad \Psi, x : \sigma \vdash e_0 \Leftarrow \tau}{\Psi \vdash \lambda x.\, e_0 \Rightarrow \sigma \to \tau}\ \mathsf{Decl}{\to}\mathsf{I}\Rightarrow$$

Let $A' = \sigma \to \tau$.

	$\Psi' \leq \Psi$	Given
	$\Psi \vdash \sigma \leq \sigma$	By Lemma 3 (Reflexivity of Declarative Subtyping)
	$\Psi', x : \sigma \leq \Psi, x : \sigma$	By CtxSubVar
	$\Psi, x : \sigma \vdash e_0 \Leftarrow \tau$	Subderivation
	$\Psi \vdash \tau \leq \tau$	By Lemma 3 (Reflexivity of Declarative Subtyping)
	$\Psi', x : \sigma \vdash e_0 \Leftarrow \tau$	By i.h. (i) with τ
☞	$\Psi \vdash A' \leq \sigma \to \tau$	By Lemma 3 (Reflexivity of Declarative Subtyping)
☞	$\Psi' \vdash \lambda x.\, e_0 \Rightarrow A'$	By $\mathsf{Decl}{\to}\mathsf{I}\Rightarrow$

- **Case**

$$\frac{\Psi \vdash e_1 \Rightarrow C \qquad \Psi \vdash C \bullet e_2 \Rightarrow\!\!\!\Rightarrow A}{\Psi \vdash e_1\, e_2 \Rightarrow A}\ \mathsf{Decl}{\to}\mathsf{E}$$

	$\Psi \vdash e_1 \Rightarrow C$	Subderivation
	$\Psi' \vdash e_1 \Rightarrow C'$	By i.h. (ii)
	$\Psi \vdash C' \leq C$	"
	$\Psi \vdash C \bullet e_2 \Rightarrow\!\!\!\Rightarrow A$	Subderivation
☞	$\Psi \vdash A' \leq A$	By i.h. (iii)
	$\Psi' \vdash C' \bullet e_2 \Rightarrow\!\!\!\Rightarrow A'$	"
☞	$\Psi' \vdash e_1\, e_2 \Rightarrow A'$	By $\mathsf{Decl}{\to}\mathsf{E}$

For part (iii), application:

- **Case**

$$\frac{\Psi \vdash \tau \qquad \Psi \vdash [\tau/\alpha]C_0 \bullet e \Rightarrow\!\!\!\Rightarrow A}{\Psi \vdash \forall\alpha.\, C_0 \bullet e \Rightarrow\!\!\!\Rightarrow A}\ \mathsf{Decl}\forall\mathsf{App}$$

	$\Psi \vdash C' \leq \forall\alpha.\, C_0$	Given
	$\Psi, \alpha \vdash C' \leq C_0$	By Lemma 7 (Invertibility)
	$\Psi \vdash [\tau/\alpha]C' \leq [\tau/\alpha]C_0$	By Lemma 5 (Substitution)
	$\Psi \vdash C' \leq [\tau/\alpha]C_0$	α cannot appear in C'
	$\Psi \vdash [\tau/\alpha]C_0 \bullet e \Rightarrow\!\!\!\Rightarrow A$	Subderivation
☞	$\Psi' \vdash C' \bullet e \Rightarrow\!\!\!\Rightarrow A'$	By i.h. (iii)
☞	$\Psi' \vdash A' \leq A$	"

- **Case**

$$\frac{\Psi \vdash e \Leftarrow C_0}{\Psi \vdash C_0 \to A \bullet e \Rightarrow\!\!\!\Rightarrow A}\ \mathsf{Decl}{\to}\mathsf{App}$$

$\Psi \vdash C' \leq C_0 \to A$ Given

  - **Case**

$$\frac{\Psi \vdash C_0 \leq C_1' \qquad \Psi \vdash C_2' \leq A}{\Psi \vdash C_1' \to C_2' \leq C_0 \to A}\ {\leq}{\to}$$

Let $A' = C_2'$.

	$\Psi \vdash e \Leftarrow C_0$	Subderivation
	$\Psi \vdash C_0 \le C_1'$	Subderivation
	$\Psi' \vdash e \Leftarrow C_1'$	By i.h.
	$\Psi' \vdash C_1' \to C_2' \bullet e \Rightarrow\!\!\!\Rightarrow C_2'$	By Decl$\to$App
☞	$\Psi' \vdash C_1' \to A' \bullet e \Rightarrow\!\!\!\Rightarrow A'$	$A' = C_2'$
	$\Psi \vdash C_2' \le A$	Subderivation
☞	$\Psi \vdash A' \le A$	$A' = C_2'$

– **Case** $$\frac{\Psi \vdash \tau \qquad \Psi \vdash [\tau/\beta]B \le C_0 \to A}{\Psi \vdash \forall\beta.\, B \le C_0 \to A}\ \le\forall L$$

	$\Psi \vdash [\tau/\beta]B \le C_0 \to A$	Subderivation
	$\Psi' \vdash [\tau/\beta]B \bullet e \Rightarrow\!\!\!\Rightarrow A'$	By i.h. (iii)
☞	$\Psi \vdash A' \le A$	"
	$\Psi \vdash \tau$	Subderivation
	$\Psi' \vdash \tau$	By weakening
☞	$\Psi' \vdash \forall\beta.\, B \bullet e \Rightarrow\!\!\!\Rightarrow A'$	By Decl$\forall$App

□

Theorem 4 (Inverse Substitution). *Assume* $\Psi \vdash e \Leftarrow A$. *Then:*

(i) If $\Psi \vdash [(e : A)/x]e' \Leftarrow C$ *then* $\Psi, x : A \vdash e' \Leftarrow C$.

(ii) If $\Psi \vdash [(e : A)/x]e' \Rightarrow C$ *then* $\Psi, x : A \vdash e' \Rightarrow C$.

(iii) If $\Psi \vdash B \bullet [(e : A)/x]e' \Rightarrow\!\!\!\Rightarrow C$ *then* $\Psi, x : A \vdash B \bullet e' \Rightarrow\!\!\!\Rightarrow C$.

Proof. By mutual induction on the typing derivation.

(i) $\Psi \vdash [(e : A)/x]e' \le C$.

Now, we consider whether or not $e' = x$:

- **Case** $e' = x$:
  Note $[(e' : A)/x]x = e' : A$. Hence $\Psi \vdash (e : A) \Leftarrow C$.
  By inversion, $\Psi \vdash A \le C$.
  By DeclVar, $\Psi, x : A \vdash x \Rightarrow A$.
  By DeclSub, $\Psi, x : A \vdash x \Leftarrow C$.
- **Case** $e' \ne x$:
  We now proceed by cases on the derivation of $\Psi \vdash [(e : A)/x]e' \le C$.

  – **Case** $$\frac{\Psi \vdash [(e : A)/x]e' \Rightarrow A \qquad \Psi \vdash A \le C}{\Psi \vdash [(e : A)/x]e' \Leftarrow C}\ \text{DeclSub}$$
  By induction, $\Psi, x : A \vdash e' \Rightarrow A$.
  By DeclSub, $\Psi, x : A \vdash e' \Rightarrow C$.

  – **Case** $$\frac{}{\Psi \vdash () \Leftarrow 1}\ \text{Decl1I}$$
  Since $[(e : A)/x]e' = ()$, it follows that $e' = ()$.
  By Decl1I, $\Psi, x : A \vdash () \Leftarrow 1$.

  – **Case** $$\frac{\Psi, \alpha \vdash [(e : A)/x]e' \Leftarrow C'}{\Psi \vdash [(e : A)/x]e' \Leftarrow \forall\alpha.\, C'}\ \text{Decl}\forall\text{I}$$
  By induction, $\Psi, \alpha, x : A \vdash e' \Leftarrow C'$.
  By exchange, $\Psi, x : A, \alpha \vdash e' \Leftarrow C'$.
  By Decl∀I, $\Psi, x : A \vdash e' \Leftarrow \forall\alpha.\, C'$.

– **Case** $$\dfrac{\Psi, y : B \vdash e'' \Leftarrow C}{\Psi \vdash \underbrace{\lambda y.\, e''}_{[(e:A)/x]e'} \Leftarrow B \to C}\ \text{Decl}\to\text{I}$$
We assume $[(e : A)/x]e' = \lambda y.\, e''$.
By definition there is e_2 such that $e' = \lambda y.\, e_2$ and $e'' = [(e : A)/x]e_2$.
So $\Psi, y : B \vdash [(e : A)/x]e_2 \Leftarrow C$.
By induction, $\Psi, y : B, x : A \vdash e_2 \Leftarrow C$.
By exchange and Decl→I, $\Psi, x : A \vdash \lambda y.\, e_2 \Leftarrow B \to C$.
Hence Decl→I, $\Psi, x : A \vdash e' \Leftarrow B \to C$.

(ii) $\Psi \vdash [(e : A)/x]e' \Rightarrow C$.

- **Case** $e' = x$:
Note $[(e' : A)/x]x = e' : A$.
Hence $\Psi \vdash e : A \Rightarrow C$.
By DeclAnno, $\Psi \vdash e : A \Rightarrow A$.
Therefore $C = A$.
By DeclVar, $\Psi, x : A \vdash e : A \Rightarrow A$.

- **Case** $e' \neq x$:
We now proceed by cases on the derivation of $\Psi \vdash [(e : A)/x]e' \leq C$.

– **Case** $$\dfrac{(y : C) \in \Psi}{\Psi \vdash y \Rightarrow C}\ \text{DeclVar}$$
Since $[(e : A)/x]e' = y$, we know that $e' = y$.
By DeclVar, $\Psi, x : A \vdash y \Rightarrow C$.

– **Case** $$\dfrac{\Psi \vdash e'' \Leftarrow C}{\Psi \vdash \underbrace{(e'' : C)}_{[(e:A)/x]e'} \Rightarrow C}\ \text{DeclAnno}$$
We know $[(e : A)/x]e' = e'' : C$ and $e' \neq x$.
Hence there is e_2 such that $e' = e_2 : C$ and $[(e : A)/x]e_2 = e''$.
So $\Psi \vdash [(e : A)/x]e_2 \Leftarrow C$.
By induction, $\Psi, x : A \vdash e_2 \Leftarrow C$.
By DeclAnno, $\Psi, x : A \vdash (e_2 : C) \Rightarrow C$.
By equality, $\Psi, x : A \vdash (e') \Rightarrow C$.

– **Case** $$\dfrac{}{\Psi \vdash () \Rightarrow 1}\ \text{Decl1I}{\Rightarrow}$$
Since $[(e : A)/x]e' = ()$, it follows that $e' = ()$.
By Decl1I⇒, $\Psi, x : A \vdash () \Rightarrow 1$.

– **Case** $$\dfrac{\Psi \vdash \sigma \to \tau \qquad \Psi, x : \sigma \vdash e \Leftarrow \tau}{\Psi \vdash \lambda x.\, e \Rightarrow \sigma \to \tau}\ \text{Decl}\to\text{I}{\Rightarrow}$$
We assume $[(e : A)/x]e' = \lambda y.\, e''$.
By definition there is e_2 such that $e' = \lambda y.\, e_2$ and $e'' = [(e : A)/x]e_2$.
So $\Psi, y : \sigma \vdash [(e : A)/x]e_2 \Leftarrow \tau$.
By induction, $\Psi, y : \sigma, x : A \vdash e_2 \Leftarrow \tau$.
By exchange and Decl→I, $\Psi, x : A \vdash \lambda y.\, e_2 \Leftarrow \sigma \to \tau$.
Hence Decl→I⇒, $\Psi, x : A \vdash e' \Rightarrow \sigma \to \tau$.

– **Case** $$\dfrac{\Psi \vdash e_1 \Rightarrow B \qquad \Psi \vdash B \bullet e_2 \Rrightarrow C}{\Psi \vdash \underbrace{e_1\, e_2}_{[(e:A)/x]e'} \Rightarrow C}\ \text{Decl}\to\text{E}$$

Note that $[(e : A)/x]e' = e_1\ e_2$.
So there are e'_1 and e'_2 such that $e' = e'_1\ e'_2$ and $[(e : A)/x]e'_1 = e_1$ and $[(e : A)/x]e'_2 = e_2$.
So $\Psi \vdash [(e : A)/x]e'_1 \Rightarrow B$ and $\Psi \vdash B \bullet [(e : A)/x]e'_2 \Rrightarrow C$.
By induction, $\Psi, x : A \vdash e'_1 \Rightarrow B$.
By induction $\Psi, x : A \vdash B \bullet e'_2 \Rrightarrow C$.
By Decl$\to$E, $\Psi, x : A \vdash e'_1\ e'_2 \Rightarrow C$.
By equality, $\Psi, x : A \vdash e' \Rightarrow C$.

(iii) $\Psi \vdash [(e : A)/x]e' \bullet A \Rrightarrow C$.

We proceed by cases on the derivation of $\Psi \vdash [(e : A)/x]e' \bullet A \Rrightarrow C$.

- **Case** $$\frac{\Psi \vdash \tau \qquad \Psi \vdash [\tau/\alpha]B \bullet [(e : A)/x]e' \Rrightarrow C}{\Psi \vdash \forall\alpha.\, B \bullet [(e : A)/x]e' \Rrightarrow C}\ \text{Decl}\forall\text{App}$$
  By inversion, $\Psi \vdash [\tau/\alpha]B \bullet [(e : A)/x]e' \Rrightarrow C$.
  By induction, $\Psi, x : A \vdash [\tau/\alpha]B \bullet e' \Rrightarrow C$.
  By Decl$\forall$App, $\Psi, x : A \vdash \forall\alpha.\, B \bullet e' \Rrightarrow C$.

- **Case** $$\frac{\Psi \vdash [(e : A)/x]e' \Leftarrow B}{\Psi \vdash B \to C \bullet [(e : A)/x]e' \Rrightarrow C}\ \text{Decl}\to\text{App}$$
  By inversion, $\Psi \vdash [(e : A)/x]e' \Leftarrow B$.
  By induction, $\Psi, x : A \vdash e' \Leftarrow B$.
  By Decl$\to$App, $\Psi \vdash B \to C \bullet e' \Rrightarrow C$. □

Theorem 5 (Annotation Removal). *We have that:*

- *If* $\Psi \vdash \big((\lambda x.\, e) : A\big) \Leftarrow C$ *then* $\Psi \vdash \lambda x.\, e \Leftarrow C$.
- *If* $\Psi \vdash (() : A) \Leftarrow C$ *then* $\Psi \vdash () \Leftarrow C$.
- *If* $\Psi \vdash e_1\ (e_2 : A) \Rightarrow C$ *then* $\Psi \vdash e_1\ e_2 \Rightarrow C$.
- *If* $\Psi \vdash (x : A) \Rightarrow A$ *then* $\Psi \vdash x \Rightarrow B$ *and* $\Psi \vdash B \leq A$.
- *If* $\Psi \vdash \big((e_1\ e_2) : A\big) \Rightarrow A$
  then $\Psi \vdash e_1\ e_2 \Rightarrow B$ *and* $\Psi \vdash B \leq A$.
- *If* $\Psi \vdash \big((e : B) : A\big) \Rightarrow A$
  then $\Psi \vdash (e : B) \Rightarrow B$ *and* $\Psi \vdash B \leq A$.
- *If* $\Psi \vdash ((\lambda x.\, e) : \sigma \to \tau) \Rightarrow \sigma \to \tau$ *then* $\Psi \vdash \lambda x.\, e \Rightarrow \sigma \to \tau$.

Proof. All of these follow directly from inversion and Lemma 13 (Subsumption). The one exception is the third, which additionally requires a small induction on the application judgment. □

Theorem 6 (Soundness of Eta).
If $\Psi \vdash \lambda x.\, e\ x \Leftarrow A$ *and* $x \notin FV(e)$, *then* $\Psi \vdash e \Leftarrow A$.

Proof. By induction on the derivation of $\Psi \vdash \lambda x.\, e\ x \Leftarrow A$. There are three non-impossible cases:

- **Case** $$\frac{\Psi, x : B \vdash e\ x \Leftarrow C}{\Psi \vdash \lambda x.\, e\ x \Leftarrow B \to C}\ \text{Decl}\to\text{I}$$
  We have $\Psi, x : B \vdash e\ x \Leftarrow C$.
  By inversion on DeclSub, we get $\Psi, x : B \vdash e\ x \Rightarrow C'$ and $\Psi \vdash C' \leq C$.
  By inversion on Decl$\to$E, we get $\Psi, x : B \vdash e \Rightarrow A'$ and $\Psi, x : B \vdash A' \bullet x \Rrightarrow C'$.
  By thinning, we know that $\Psi \vdash e \Rightarrow A'$.
  By Lemma 11 (Application Subtyping), we get B' so $\Psi, x : B \vdash A' \leq B' \to C'$ and $\Psi, x : B \vdash x \Leftarrow B'$.

By inversion, we know that $\Psi, x : B \vdash x \Rightarrow B$ and $\Psi \vdash B \leq B'$.
By $\leq\!\rightarrow$, $\Psi, x : B \vdash B' \rightarrow C' \leq B \rightarrow C$.
Hence by Lemma 6 (Transitivity of Declarative Subtyping), $\Psi, x : B \vdash A' \leq B \rightarrow C$.
Hence $\Psi \vdash A' \leq B \rightarrow C$.
By DeclSub, $\Psi \vdash e \Leftarrow B \rightarrow C$.

- **Case** $\dfrac{\Psi, \alpha \vdash \lambda x.\, e\, x \Leftarrow B}{\Psi \vdash \lambda x.\, e\, x \Leftarrow \forall \alpha.\, B}$ Decl∀I

  By induction, $\Psi, \alpha \vdash \lambda x.\, e\, x \Leftarrow B$.
  By Decl∀I, $\Psi \vdash \lambda x.\, e\, x \Leftarrow \forall \alpha.\, B$.

- **Case** $\dfrac{\Psi \vdash \lambda x.\, e\, x \Rightarrow B \qquad \Psi \vdash B \leq A}{\Psi \vdash \lambda x.\, e\, x \Leftarrow A}$ DeclSub

  We have $\Psi \vdash \lambda x.\, e\, x \Rightarrow B$ and $\Psi \vdash B \leq A$.
  By inversion on Decl$\rightarrow$I$\Rightarrow$, $\Psi, x : \sigma \vdash e\, x \Leftarrow \tau$ and $B = \sigma \rightarrow \tau$.
  By inversion on DeclSub, we get $\Psi, x : \sigma \vdash e\, x \Rightarrow C_2$ and $\Psi \vdash C_2 \leq \tau$.
  By inversion on Decl$\rightarrow$E, we get $\Psi, x : \sigma \vdash e \Rightarrow C$ and $\Psi, x : \sigma \vdash C \bullet x \Rightarrow\!\!\Rightarrow C_2$.
  By thinning, we know that $\Psi \vdash e \Rightarrow C$.
  By Lemma 11 (Application Subtyping), we get C_1 such that $\Psi, x : \sigma \vdash C \leq C_1 \rightarrow C_2$ and $\Psi, x : \sigma \vdash x \Leftarrow C_1$.
  By inversion on DeclSub, $\Psi, x : \sigma \vdash x \Rightarrow \sigma$ and $\Psi \vdash \sigma \leq C_1$.
  By $\leq\!\rightarrow$, $\Psi, x : \sigma \vdash C_1 \rightarrow C_2 \leq \sigma \rightarrow \tau$.
  Hence by Lemma 6 (Transitivity of Declarative Subtyping), $\Psi, x : \sigma \vdash C \leq \sigma \rightarrow \tau$.
  Hence $\Psi \vdash C \leq \sigma \rightarrow \tau$.
  Hence by Lemma 6 (Transitivity of Declarative Subtyping), $\Psi \vdash C \leq A$.
  By DeclSub, $\Psi \vdash e \Leftarrow A$. □

D′ Properties of Context Extension

D′.1 Syntactic Properties

Lemma 14 (Declaration Preservation). *If $\Gamma \longrightarrow \Delta$, and u is a variable or marker $\blacktriangleright_{\hat{\alpha}}$ declared in Γ, then u is declared in Δ.*

Proof. By a routine induction on $\Gamma \longrightarrow \Delta$. □

Lemma 15 (Declaration Order Preservation). *If $\Gamma \longrightarrow \Delta$ and u is declared to the left of v in Γ, then u is declared to the left of v in Δ.*

Proof. By induction on the derivation of $\Gamma \longrightarrow \Delta$.

- **Case** $\dfrac{}{\cdot \longrightarrow \cdot}$ $\longrightarrow$ID

  This case is impossible.

- **Case** $\dfrac{\Gamma \longrightarrow \Delta}{\Gamma, x : A \longrightarrow \Delta, x : A}$ $\longrightarrow$Var

  There are two cases, depending on whether or not $v = x$.

  - Case $v = x$:
    Since u is declared to the left of v, u is declared in Γ.
    By Lemma 14 (Declaration Preservation), u is declared in Δ.
    Hence u is declared to the left of x in $\Delta, x : A$.

– Case $v \neq x$:
Then v is declared in Γ, and u is declared to the left of v in Γ.
By induction, u is declared to the left of v in Δ.
Hence u is declared to the left of v in $\Delta, x : A$.

- **Case** $$\frac{\Gamma \longrightarrow \Delta}{\Gamma, \alpha \longrightarrow \Delta, \alpha} \longrightarrow\mathsf{Uvar}$$

  This case is similar to the $\longrightarrow$Var case.

- **Case** $$\frac{\Gamma \longrightarrow \Delta}{\Gamma, \hat{\alpha} \longrightarrow \Delta, \hat{\alpha}} \longrightarrow\mathsf{Unsolved}$$

  This case is similar to the $\longrightarrow$Var case.

- **Case** $$\frac{\Gamma \longrightarrow \Delta \qquad [\Delta]\tau = [\Delta]\tau'}{\Gamma, \hat{\alpha} = \tau \longrightarrow \Delta, \hat{\alpha} = \tau'} \longrightarrow\mathsf{Solved}$$

  This case is similar to the $\longrightarrow$Var case.

- **Case** $$\frac{\Gamma \longrightarrow \Delta}{\Gamma, \blacktriangleright_{\hat{\alpha}} \longrightarrow \Delta, \blacktriangleright_{\hat{\alpha}}} \longrightarrow\mathsf{Marker}$$

  This case is similar to the $\longrightarrow$Var case.

- **Case** $$\frac{\Gamma \longrightarrow \Delta}{\Gamma, \hat{\alpha} \longrightarrow \Delta, \hat{\alpha} = \tau} \longrightarrow\mathsf{Solve}$$

  This case is similar to the $\longrightarrow$Var case.

- **Case** $$\frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \hat{\alpha}} \longrightarrow\mathsf{Add}$$

  By induction, u is declared to the left of v in Δ.
  Therefore u is declared to the left of v in $\Delta, \hat{\alpha}$.

- **Case** $$\frac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \hat{\alpha} = \tau} \longrightarrow\mathsf{AddSolved}$$

  By induction, u is declared to the left of v in Δ.
  Therefore u is declared to the left of v in $\Delta, \hat{\alpha} = \tau$. □

Lemma 16 (Reverse Declaration Order Preservation)**.** *If* $\Gamma \longrightarrow \Delta$ *and* u *and* v *are both declared in* Γ *and* u *is declared to the left of* v *in* Δ*, then* u *is declared to the left of* v *in* Γ*.*

Proof. It is given that u and v are declared in Γ. Either u is declared to the left of v in Γ, or v is declared to the left of u. Suppose the latter (for a contradiction). By Lemma 15 (Declaration Order Preservation), v is declared to the left of u in Δ. But we know that u is declared to the left of v in Δ: contradiction. Therefore u is declared to the left of v in Γ. □

Lemma 17 (Substitution Extension Invariance)**.** *If* $\Theta \vdash A$ *and* $\Theta \longrightarrow \Gamma$ *then* $[\Gamma]A = [\Gamma]([\Theta]A)$ *and* $[\Gamma]A = [\Theta]([\Gamma]A)$.

Proof. To show that $[\Gamma]A = [\Theta][\Gamma]A$, observe that $\Theta \vdash A$, and that by definition of $\Theta \longrightarrow \Gamma$, every solved variable in Θ is solved in Γ. Therefore $[\Theta]([\Gamma]A) = [\Gamma]A$, since $\mathsf{unsolved}([\Gamma]A)$ contains no variables that Θ solves.

To show that $[\Gamma]A = [\Gamma][\Theta]A$, we proceed by induction on $|\Gamma \vdash A|$.

- **Case** $\dfrac{\alpha \in \Theta}{\Theta \vdash \alpha}$

  Note that $[\Gamma]\alpha = \alpha = [\Theta]\alpha$, so $[\Gamma]\alpha = [\Gamma][\Theta]\alpha$.

- **Case** $\dfrac{\Theta \vdash A \qquad \Theta \vdash B}{\Theta \vdash A \to B}$

  By induction, $[\Gamma]A = [\Gamma][\Theta]A$.
  By induction, $[\Gamma]B = [\Gamma][\Theta]B$.
  Then

$$\begin{array}{rcll} [\Gamma](A \to B) & = & [\Gamma]A \to [\Gamma]B & \text{By definition of substitution} \\ & = & [\Gamma][\Theta]A \to [\Gamma][\Theta]B & \text{By induction hypothesis (twice)} \\ & = & [\Gamma]([\Theta]A \to [\Theta]B) & \text{By definition of substitution} \\ & = & [\Gamma][\Theta](A \to B) & \text{By definition of substitution} \end{array}$$

- **Case** $\dfrac{\Theta, \alpha \vdash A}{\Theta \vdash \forall\alpha.\, A}$

  By inversion, we have $\Theta, \alpha \vdash A$.
  By rule $\longrightarrow$Uvar, $\Theta, \alpha \longrightarrow \Gamma, \alpha$.
  By induction, $[\Gamma, \alpha]A = [\Gamma, \alpha][\Theta, \alpha]A$.
  By definition, $[\Gamma]A = [\Gamma][\Theta]A$.
  Then

$$\begin{array}{rcll} [\Gamma]\forall\alpha.\, A & = & \forall\alpha.\, [\Gamma]A & \text{By definition} \\ & = & \forall\alpha.\, [\Gamma][\Theta]A & \text{By conclusion above} \\ & = & [\Gamma](\forall\alpha.\, [\Theta]A) & \text{By definition} \\ & = & [\Gamma][\Theta](\forall\alpha.\, A) & \text{By definition} \\ & = & [\Gamma, \alpha][\Theta, \alpha](\forall\alpha.\, A) & \text{By definition} \end{array}$$

- **Case**

$$\dfrac{}{\underbrace{\Theta_0, \hat{\alpha}, \Theta_1}_{\Theta} \vdash \hat{\alpha}}$$

  Note that $[\Theta]\hat{\alpha} = \hat{\alpha}$.
  Hence $[\Gamma][\Theta]\hat{\alpha} = [\Gamma]\hat{\alpha}$.

- **Case**

$$\dfrac{}{\Theta_0, \hat{\alpha} = \tau, \Theta_1 \vdash \hat{\alpha}}$$

  From $\Theta \longrightarrow \Gamma$, By a nested induction we get $\Gamma = \Gamma_0, \hat{\alpha} = \tau', \Gamma_1$, and $[\Gamma]\tau' = [\Gamma]\tau$.
  Note that $|\Theta \vdash \tau| < |\Theta \vdash \hat{\alpha}|$.
  By induction, $[\Gamma]\tau = [\Gamma][\Theta]\tau$.
  Hence

$$\begin{array}{rcll} [\Gamma]\hat{\alpha} & = & [\Gamma]\tau' & \text{By definition} \\ & = & [\Gamma]\tau & \text{From the extension judgment} \\ & = & [\Gamma][\Theta]\tau & \text{From the induction hypothesis} \\ & = & [\Gamma][\Theta]\hat{\alpha} & \text{By definition} \end{array}$$

  □

Lemma 18 (Extension Equality Preservation).
If $\Gamma \vdash A$ *and* $\Gamma \vdash B$ *and* $[\Gamma]A = [\Gamma]B$ *and* $\Gamma \longrightarrow \Delta$*, then* $[\Delta]A = [\Delta]B$.

Proof. By induction on the derivation of $\Gamma \longrightarrow \Delta$.

- **Case**
$$\dfrac{}{\underbrace{\cdot}_{\Gamma} \longrightarrow \underbrace{\cdot}_{\Delta}} \longrightarrow\mathsf{ID}$$
We have $[\Gamma]A = [\Gamma]B$, but $\Gamma = \Delta$, so $[\Delta]A = [\Delta]B$.

- **Case** $$\dfrac{\Gamma' \longrightarrow \Delta'}{\Gamma', x : C \longrightarrow \Delta', x : C} \longrightarrow\mathsf{Var}$$
We have $[\Gamma', x : C]A = [\Gamma', x : C]B$.
By definition of substitution, $[\Gamma']A = [\Gamma']B$.
By i.h., $[\Delta']A = [\Delta']B$.
By definition of substitution, $[\Delta', x : C]A = [\Delta', x : C]B$.

- **Case** $$\dfrac{\Gamma' \longrightarrow \Delta'}{\Gamma', \alpha \longrightarrow \Delta', \alpha} \longrightarrow\mathsf{Uvar}$$
We have $[\Gamma', \alpha]A = [\Gamma', \alpha]B$.
By definition of substitution, $[\Gamma']A = [\Gamma']B$.
By i.h., $[\Delta']A = [\Delta']B$.
By definition of substitution, $[\Delta', \alpha]A = [\Delta', \alpha]B$.

- **Case** $$\dfrac{\Gamma' \longrightarrow \Delta'}{\Gamma', \hat{\alpha} \longrightarrow \Delta', \hat{\alpha}} \longrightarrow\mathsf{Unsolved}$$
Similar to the $\longrightarrow$Uvar case.

- **Case** $$\dfrac{\Gamma' \longrightarrow \Delta'}{\Gamma', \blacktriangleright_{\hat{\alpha}} \longrightarrow \Delta', \blacktriangleright_{\hat{\alpha}}} \longrightarrow\mathsf{Marker}$$
Similar to the $\longrightarrow$Uvar case.

- **Case** $$\dfrac{\Gamma \longrightarrow \Delta'}{\Gamma \longrightarrow \Delta', \hat{\alpha}} \longrightarrow\mathsf{Add}$$
We have $[\Gamma]A = [\Gamma]B$.
By i.h., $[\Delta']A = [\Delta']B$.
By definition of substitution, $[\Delta', \hat{\alpha}]A = [\Delta', \hat{\alpha}]B$.

- **Case** $$\dfrac{\Gamma \longrightarrow \Delta'}{\Gamma \longrightarrow \Delta', \hat{\alpha} = \tau} \longrightarrow\mathsf{AddSolved}$$
We have $[\Gamma]A = [\Gamma]B$.
By i.h., $[\Delta']A = [\Delta']B$.
We implicitly assume that Δ is well-formed, so $\hat{\alpha} \notin \mathsf{dom}(\Delta')$.
Since $\Gamma \longrightarrow \Delta'$ and $\hat{\alpha} \notin \mathsf{dom}(\Delta')$, it follows that $\hat{\alpha} \notin \mathsf{dom}(\Gamma)$.
We have $\Gamma \vdash A$ and $\Gamma \vdash B$, so $\hat{\alpha} \notin (FV(A) \cup FV(B))$.
Therefore, by definition of substitution, $[\Delta', \hat{\alpha} = \tau]A = [\Delta', \hat{\alpha} = \tau]B$.

- **Case** $$\dfrac{\Gamma' \longrightarrow \Delta' \qquad [\Delta']\tau = [\Delta']\tau'}{\Gamma', \hat{\alpha} = \tau \longrightarrow \Delta', \hat{\alpha} = \tau'} \longrightarrow\mathsf{Solved}$$
We have $[\Gamma', \hat{\alpha} = \tau]A = [\Gamma', \hat{\alpha} = \tau]B$.

By definition, $[\Gamma', \hat{\alpha} = \tau]A = [\Gamma', \hat{\alpha} = \tau]\tau$, but we implicitly assume that Γ is well-formed, so $\hat{\alpha} \notin FV(\tau)$, so actually $[\Gamma', \hat{\alpha} = \tau]A = [\Gamma']\tau$.
Combined with similar reasoning for B, we get

$$[\Gamma'][\tau/\hat{\alpha}]A \;=\; [\Gamma'][\tau/\hat{\alpha}]B$$

By i.h., $[\Delta'][\tau/\hat{\alpha}]A = [\Delta'][\tau/\hat{\alpha}]B$.
By distributivity of substitution, $\left[[\Delta']\tau/\hat{\alpha}\right][\Delta']A = \left[[\Delta']\tau/\hat{\alpha}\right][\Delta']B$.
Using the premise $[\Delta']\tau = [\Delta']\tau'$, we get $[[\Delta']\tau'/\hat{\alpha}][\Delta']A = [[\Delta']\tau'/\hat{\alpha}][\Delta']B$.
By distributivity of substitution (in the other direction), $[\Delta'][\tau'/\hat{\alpha}]A = [\Delta'][\tau'/\hat{\alpha}]B$.
It follows from the definition of substitution that $[\Delta', \hat{\alpha} = \tau']A = [\Delta', \hat{\alpha} = \tau']B$.

- **Case** $\dfrac{\Gamma' \longrightarrow \Delta'}{\Gamma', \hat{\alpha} \longrightarrow \Delta', \hat{\alpha} = \tau} \longrightarrow\mathsf{Solve}$

  We have $[\Gamma', \hat{\alpha}]A = [\Gamma', \hat{\alpha}]B$.
  By definition of substitution, $[\Gamma']A = [\Gamma']B$.
  By i.h., $[\Delta'][\tau/\hat{\alpha}]A = [\Delta'][\tau/\hat{\alpha}]B$.
  It follows from the definition of substitution that $[\Delta', \hat{\alpha} = \tau]A = [\Delta', \hat{\alpha} = \tau]B$. □

Lemma 19 (Reflexivity). *If Γ is well-formed, then $\Gamma \longrightarrow \Gamma$.*

Proof. By induction on the structure of Γ.

- **Case** $\Gamma = \cdot$: Apply rule $\longrightarrow\mathsf{ID}$.
- **Case** $\Gamma = (\Gamma', \alpha)$: By i.h., $\Gamma' \longrightarrow \Gamma'$. By rule $\longrightarrow\mathsf{Uvar}$, we get $\Gamma', \alpha \longrightarrow \Gamma', \alpha$.
- **Case** $\Gamma = (\Gamma', \hat{\alpha})$: By i.h., $\Gamma' \longrightarrow \Gamma'$. By rule $\longrightarrow\mathsf{Unsolved}$, we get $\Gamma', \hat{\alpha} \longrightarrow \Gamma', \hat{\alpha}$.
- **Case** $\Gamma = (\Gamma', \hat{\alpha} = \tau)$:
  By i.h., $\Gamma' \longrightarrow \Gamma'$.
  Clearly, $[\Gamma']\tau = [\Gamma']\tau$, so we can apply $\longrightarrow\mathsf{Solved}$ to get $\Gamma', \hat{\alpha} = \tau \longrightarrow \Gamma', \hat{\alpha} = \tau$.
- **Case** $\Gamma = (\Gamma', \blacktriangleright_{\hat{\alpha}})$: By i.h., $\Gamma' \longrightarrow \Gamma'$. By rule $\longrightarrow\mathsf{Marker}$, we get $\Gamma', \blacktriangleright_{\hat{\alpha}} \longrightarrow \Gamma', \blacktriangleright_{\hat{\alpha}}$. □

Lemma 20 (Transitivity). *If $\Gamma \longrightarrow \Delta$ and $\Delta \longrightarrow \Theta$, then $\Gamma \longrightarrow \Theta$.*

Proof. By induction on the derivation of $\Delta \longrightarrow \Theta$.

- **Case** $\longrightarrow\mathsf{ID}$:
  In this case $\Theta = \Delta$.
  Hence $\Gamma \longrightarrow \Delta$ suffices.

- **Case** $\dfrac{\Delta' \longrightarrow \Theta'}{\Delta', \alpha \longrightarrow \Theta', \alpha} \longrightarrow\mathsf{Uvar}$

  We have $\Delta = (\Delta', \alpha)$ and $\Theta = (\Theta', \alpha)$.
  By inversion on $\Gamma \longrightarrow \Delta$, we have $\Gamma = (\Gamma', \alpha)$ and $\Gamma' \longrightarrow \Delta'$.
  By i.h., $\Gamma' \longrightarrow \Theta'$.
  Applying rule $\longrightarrow\mathsf{Uvar}$ gives $\Gamma', \alpha \longrightarrow \Theta', \alpha$.

- **Case** $\dfrac{\Delta' \longrightarrow \Theta'}{\Delta', \hat{\alpha} \longrightarrow \Theta', \hat{\alpha}} \longrightarrow\mathsf{Uvar}$

  We have $\Delta = (\Delta', \hat{\alpha})$ and $\Theta = (\Theta', \hat{\alpha})$.
  Either of two rules could have derived $\Gamma \longrightarrow \Delta$:

- **Case** $\dfrac{\Gamma' \longrightarrow \Delta'}{\Gamma', \hat{\alpha} \longrightarrow \Delta', \hat{\alpha}} \longrightarrow\!\mathsf{Unsolved}$

  Here we have $\Gamma = (\Gamma', \hat{\alpha})$ and $\Gamma' \longrightarrow \Delta'$.
  By i.h., $\Gamma' \longrightarrow \Theta'$.
  Applying rule $\longrightarrow$Unsolved gives $\Gamma', \hat{\alpha} \longrightarrow \Theta', \hat{\alpha}$.

- **Case** $\dfrac{\Gamma \longrightarrow \Delta'}{\Gamma \longrightarrow \Delta', \hat{\alpha}} \longrightarrow\!\mathsf{Add}$

  By i.h., $\Gamma \longrightarrow \Theta'$.
  By rule $\longrightarrow$Add, we get $\Gamma \longrightarrow \Theta', \hat{\alpha}$.

- **Case** $\dfrac{\Delta' \longrightarrow \Theta' \qquad [\Theta']\tau_1 = [\Theta']\tau_2}{\Delta', \hat{\alpha} = \tau_1 \longrightarrow \Theta', \hat{\alpha} = \tau_2} \longrightarrow\!\mathsf{Solved}$

  In this case $\Delta = (\Delta', \hat{\alpha} = \tau_1)$ and $\Theta = (\Theta', \hat{\alpha} = \tau_2)$.
  One of three rules must have derived $\Gamma \longrightarrow \Delta', \hat{\alpha} = \tau$:

  - **Case** $\dfrac{\Gamma' \longrightarrow \Delta' \qquad [\Delta']\tau_0 = [\Delta']\tau_1}{\Gamma', \hat{\alpha} = \tau_0 \longrightarrow \Delta', \hat{\alpha} = \tau_1} \longrightarrow\!\mathsf{Solved}$

    Here, $\Gamma = (\Gamma', \hat{\alpha} = \tau_0)$ and $\Delta = (\Delta', \hat{\alpha} = \tau_1)$.
    By i.h., we have $\Gamma' \longrightarrow \Theta'$.
    The premises of the respective $\longrightarrow$ derivations give us $[\Delta']\tau_0 = [\Delta']\tau_1$ and $[\Theta']\tau_1 = [\Theta']\tau_2$.
    We know that $\Gamma' \vdash \tau_0$ and $\Delta' \vdash \tau_1$ and $\Theta' \vdash \tau_2$.
    By extension weakening (Lemma 24 (Extension Weakening)), $\Theta' \vdash \tau_0$.
    By extension weakening (Lemma 24 (Extension Weakening)), $\Theta' \vdash \tau_1$.
    Since $[\Delta']\tau_0 = [\Delta']\tau_1$, we know that $[\Theta'][\Delta']\tau_0 = [\Theta'][\Delta']\tau_1$.
    By Lemma 17 (Substitution Extension Invariance), $[\Theta'][\Delta']\tau_0 = [\Theta']\tau_0$.
    By Lemma 17 (Substitution Extension Invariance), $[\Theta'][\Delta']\tau_1 = [\Theta']\tau_1$.
    So $[\Theta']\tau_0 = [\Theta']\tau_1$.

    Hence by transitivity of equality, $[\Theta']\tau_0 = [\Theta']\tau_1 = [\Theta']\tau_2$.
    By rule $\longrightarrow$Solved, $\Gamma', \hat{\alpha} = \tau \longrightarrow \Theta', \hat{\alpha} = \tau_2$.

  - **Case** $\dfrac{\Gamma \longrightarrow \Delta'}{\Gamma \longrightarrow \Delta', \hat{\alpha} = \tau_1} \longrightarrow\!\mathsf{AddSolved}$

    By induction, we have $\Gamma \longrightarrow \Theta'$.
    By rule $\longrightarrow$AddSolved, we get $\Gamma \longrightarrow \Theta', \hat{\alpha} = \tau_2$.

  - **Case** $\dfrac{\Gamma' \longrightarrow \Delta'}{\Gamma', \hat{\alpha} \longrightarrow \Delta', \hat{\alpha} = \tau_1} \longrightarrow\!\mathsf{Solve}$

    We have $\Gamma = (\Gamma', \hat{\alpha})$.
    By induction, $\Gamma' \longrightarrow \Theta'$.
    By rule $\longrightarrow$Solve, we get $\Gamma', \hat{\alpha} \longrightarrow \Theta', \hat{\alpha} = \tau_2$.

- **Case** $\dfrac{\Delta' \longrightarrow \Theta'}{\Delta', \blacktriangleright_{\hat{\alpha}} \longrightarrow \Theta', \blacktriangleright_{\hat{\alpha}}} \longrightarrow\!\mathsf{Marker}$

  In this case we know $\Delta = (\Delta', \blacktriangleright_{\hat{\alpha}})$ and $\Theta = (\Theta', \blacktriangleright_{\hat{\alpha}})$.
  Since $\Delta = (\Delta', \blacktriangleright_{\hat{\alpha}})$, only $\longrightarrow$Marker could derive $\Gamma \longrightarrow \Delta$, so by inversion, $\Gamma = (\Gamma', \blacktriangleright_{\hat{\alpha}})$ and $\Gamma' \longrightarrow \Delta'$.
  By induction, we have $\Gamma' \longrightarrow \Theta'$.
  Applying rule $\longrightarrow$Marker gives $\Gamma', \blacktriangleright_{\hat{\alpha}} \longrightarrow \Theta', \blacktriangleright_{\hat{\alpha}}$.

- **Case** $\dfrac{\Delta \longrightarrow \Theta'}{\Delta \longrightarrow \Theta', \hat{\alpha}} \longrightarrow\mathsf{Add}$

  In this case, we have $\Theta = (\Theta', \hat{\alpha})$.
  By induction, we get $\Gamma \longrightarrow \Theta'$.
  By rule $\longrightarrow$Add, we get $\Gamma \longrightarrow \Theta', \hat{\alpha}$.

- **Case** $\dfrac{\Delta \longrightarrow \Theta'}{\Delta \longrightarrow \Theta', \hat{\alpha} = \tau} \longrightarrow\mathsf{AddSolved}$

  In this case, we have $\Theta = (\Theta', \hat{\alpha} = \tau)$.
  By induction, we get $\Gamma \longrightarrow \Theta'$.
  By rule $\longrightarrow$AddSolved, we get $\Gamma \longrightarrow \Theta', \hat{\alpha} = \tau$.

- **Case** $\dfrac{\Delta' \longrightarrow \Theta'}{\Delta', \hat{\alpha} \longrightarrow \Theta', \hat{\alpha} = \tau} \longrightarrow\mathsf{Solve}$

  In this case, we have $\Delta = (\Delta', \hat{\alpha})$ and $\Theta = (\Theta', \hat{\alpha} = \tau)$.
  One of two rules could have derived $\Gamma \longrightarrow \Delta', \hat{\alpha}$:

  - **Case** $\dfrac{\Gamma' \longrightarrow \Delta'}{\Gamma', \hat{\alpha} \longrightarrow \Delta', \hat{\alpha}} \longrightarrow\mathsf{Unsolved}$

    In this case, we have $\Gamma = (\Gamma', \hat{\alpha})$ and $\Gamma' \longrightarrow \Delta'$ and $\Delta' \longrightarrow \Theta'$.
    By induction, we have $\Gamma' \longrightarrow \Theta'$.
    By rule $\longrightarrow$Solve, we get $\Gamma', \hat{\alpha} \longrightarrow \Theta', \hat{\alpha} = \tau$.

  - **Case** $\dfrac{\Gamma \longrightarrow \Delta'}{\Gamma \longrightarrow \Delta', \hat{\alpha}} \longrightarrow\mathsf{Add}$

    In this case, we have $\Gamma \longrightarrow \Delta'$ and $\Delta' \longrightarrow \Theta'$.
    By induction, we have $\Gamma \longrightarrow \Theta'$.
    By rule $\longrightarrow$Solve, we get $\Gamma \longrightarrow \Theta', \hat{\alpha} = \tau$. □

Lemma 21 (Right Softness). *If* $\Gamma \longrightarrow \Delta$ *and* Θ *is soft (and* (Δ, Θ) *is well-formed) then* $\Gamma \longrightarrow \Delta, \Theta$.

Proof. By induction on Θ, applying rules $\longrightarrow$Add and $\longrightarrow$AddSolved as needed. □

Lemma 22 (Evar Input).
If $\Gamma, \hat{\alpha} \longrightarrow \Delta$ *then* $\Delta = (\Delta_0, \Delta_{\hat{\alpha}}, \Theta)$ *where* $\Gamma \longrightarrow \Delta_0$*, and* $\Delta_{\hat{\alpha}}$ *is either* $\hat{\alpha}$ *or* $\hat{\alpha} = \tau$*, and* Θ *is soft.*

Proof. By induction on the given derivation.

- **Cases** $\longrightarrow$ID, $\longrightarrow$Var, $\longrightarrow$Uvar, $\longrightarrow$Solved, $\longrightarrow$Marker:
  Impossible: the left-hand context cannot have the form $\Gamma, \hat{\alpha}$.

- **Case** $\dfrac{\Gamma \longrightarrow \Delta_0}{\Gamma, \hat{\alpha} \longrightarrow \underbrace{\Delta_0, \hat{\alpha}}_{\Delta}} \longrightarrow\mathsf{Unsolved}$

  Let $\Theta = \cdot$, which is vacuously soft. Therefore $\Delta = (\Delta_0, \hat{\alpha}) = (\Delta_0, \hat{\alpha}, \Theta)$; the subderivation is the rest of the result.

- **Case** $\dfrac{\Gamma \longrightarrow \Delta_0}{\Gamma, \hat{\alpha} \longrightarrow \underbrace{\Delta_0, \hat{\alpha} = \tau}_{\Delta}} \longrightarrow\mathsf{Solve}$

  Let $\Theta = \cdot$, which is vacuously soft. Therefore $\Delta = (\Delta_0, \hat{\alpha}) = (\Delta_0, \hat{\alpha} = \tau, \Theta)$; the subderivation is the rest of the result.

- **Case**
$$\frac{\Gamma, \hat{\alpha} \longrightarrow \Delta_0}{\Gamma, \hat{\alpha} \longrightarrow \underbrace{\Delta_0, \hat{\beta}}_{\Delta}} \longrightarrow\mathsf{Add}$$
Suppose $\hat{\beta} = \hat{\alpha}$.
We have $\Gamma, \hat{\alpha} \longrightarrow \Delta_0$. By Lemma 14 (Declaration Preservation), $\hat{\alpha}$ is declared in Δ_0.
But then $(\Delta_0, \hat{\beta}) = (\Delta_0, \hat{\alpha})$ with multiple $\hat{\alpha}$ declarations,
which violates the implicit assumption that Δ is well-formed. Contradiction.
Therefore $\hat{\beta} \neq \hat{\alpha}$.
By i.h., $\Delta' = (\Delta_0, \Delta_{\hat{\alpha}}, \Theta')$ where $\Gamma \longrightarrow \Delta_0$ and Θ' is soft.
Let $\Theta = (\Theta', \hat{\beta})$. Therefore $(\Delta', \hat{\beta}) = (\Delta_0, \Delta_{\hat{\alpha}}, \Theta', \hat{\beta})$. As Θ' is soft, $(\Theta', \hat{\beta})$ is soft. Since $\Delta = (\Delta', \hat{\beta})$, this gives $\Delta = (\Delta_0, \Delta_{\hat{\alpha}}, \Theta)$.
- **Case** $\longrightarrow$AddSolved: Similar to the case for $\longrightarrow$Add. □

Lemma 23 (Extension Order)**.**

*(i) If $\Gamma_L, \alpha, \Gamma_R \longrightarrow \Delta$ then $\Delta = (\Delta_L, \alpha, \Delta_R)$ where $\Gamma_L \longrightarrow \Delta_L$.
Moreover, if Γ_R is soft then Δ_R is soft.*

*(ii) If $\Gamma_L, \blacktriangleright_{\hat{\alpha}}, \Gamma_R \longrightarrow \Delta$ then $\Delta = (\Delta_L, \blacktriangleright_{\hat{\alpha}}, \Delta_R)$ where $\Gamma_L \longrightarrow \Delta_L$.
Moreover, if Γ_R is soft then Δ_R is soft.*

(iii) If $\Gamma_L, \hat{\alpha}, \Gamma_R \longrightarrow \Delta$ then $\Delta = \Delta_L, \Theta, \Delta_R$ where $\Gamma_L \longrightarrow \Delta_L$ and Θ is either $\hat{\alpha}$ or $\hat{\alpha} = \tau$ for some τ.

(iv) If $\Gamma_L, \hat{\alpha} = \tau, \Gamma_R \longrightarrow \Delta$ then $\Delta = \Delta_L, \hat{\alpha} = \tau', \Delta_R$ where $\Gamma_L \longrightarrow \Delta_L$ and $[\Delta_L]\tau = [\Delta_L]\tau'$.

*(v) If $\Gamma_L, x : A, \Gamma_R \longrightarrow \Delta$ then $\Delta = (\Delta_L, x : A', \Delta_R)$ where $\Gamma_L \longrightarrow \Delta_L$ and $[\Delta_L]A = [\Delta_L]A'$.
Moreover, Γ_R is soft if and only if Δ_R is soft.*

Proof. (i) By induction on the derivation of $\Gamma_L, \alpha, \Gamma_R \longrightarrow \Delta$.

- **Case**
$$\frac{}{\cdot \longrightarrow \cdot} \longrightarrow\mathsf{ID}$$
This case is impossible since $(\Gamma_L, \alpha, \Gamma_R)$ cannot have the form $\cdot$.
- **Cases** $\longrightarrow$Uvar:
We have two cases, depending on whether or not the rightmost variable is α.
  - **Case**
$$\frac{\Gamma \longrightarrow \Delta'}{\Gamma, \alpha \longrightarrow \Delta', \alpha} \longrightarrow\mathsf{Uvar}$$
Let $\Delta_L = \Delta'$, and let $\Delta_R = \cdot$ (which is soft).
We have $\Gamma \longrightarrow \Delta'$, which is $\Gamma_L \longrightarrow \Delta_L$.
  - **Case**
$$\frac{\Gamma_L, \alpha, \Gamma'_R \longrightarrow \Delta'}{\Gamma_L, \alpha, \underbrace{\Gamma'_R, \beta}_{\Gamma_R} \longrightarrow \underbrace{\Delta', \beta}_{\Delta}} \longrightarrow\mathsf{Uvar}$$
By i.h., $\Delta' = (\Delta_L, \alpha, \Delta'_R)$ where $\Gamma_L \longrightarrow \Delta_L$.
Hence $\Delta = (\Delta_L, \alpha, \Delta'_R, \beta)$.
(Since $\beta \in \Gamma_R$, it cannot be the case that Γ_R is soft.)
- **Case**
$$\frac{\Gamma_L, \alpha, \Gamma'_R \longrightarrow \Delta'}{\Gamma_L, \alpha, \underbrace{\Gamma'_R, x : A}_{\Gamma_R} \longrightarrow \underbrace{\Delta', x : A}_{\Delta}} \longrightarrow\mathsf{Var}$$
By i.h., $\Delta' = (\Delta_L, \alpha, \Delta'_R)$ where $\Gamma_L \longrightarrow \Delta_L$.
Hence $\Delta = (\Delta_L, \alpha, \Delta'_R, x : A)$.
(Since $x : A \in \Gamma_R$, it cannot be the case that Γ_R is soft.)

- **Case** $\dfrac{\Gamma_L, \alpha, \Gamma_R' \longrightarrow \Delta'}{\Gamma_L, \alpha, \underbrace{\Gamma_R', \hat{\alpha}}_{\Gamma_R} \longrightarrow \underbrace{\Delta', \hat{\alpha}}_{\Delta}} \longrightarrow\!\mathsf{Unsolved}$

  By i.h., $\Delta' = (\Delta_L, \alpha, \Delta_R')$ where $\Gamma_L \longrightarrow \Delta_L$.
  Hence $\Delta = (\Delta_L, \alpha, \Delta_R', \hat{\alpha})$.
  (If Γ_R is soft, by i.h. Δ_R' is soft, so $\Delta_R = (\Delta_R', \hat{\alpha})$ is soft.)

- **Case** $\dfrac{\Gamma_L, \alpha, \Gamma_R' \longrightarrow \Delta'}{\Gamma_L, \alpha, \underbrace{\Gamma_R', \blacktriangleright_{\hat{\beta}}}_{\Gamma_R'} \longrightarrow \underbrace{\Delta', \blacktriangleright_{\hat{\beta}}}_{\Delta}} \longrightarrow\!\mathsf{Marker}$

  By i.h., $\Delta' = (\Delta_L, \alpha, \Delta_R')$ where $\Gamma_L \longrightarrow \Delta_L$.
  Hence $\Delta = (\Delta_L, \alpha, \Delta_R', \blacktriangleright_{\hat{\beta}})$.
  (Since $\blacktriangleright_{\hat{\beta}} \in \Gamma_R$, it cannot be the case that Γ_R is soft.)

- **Case** $\dfrac{\Gamma_L, \alpha, \Gamma_R' \longrightarrow \Delta' \qquad [\Delta']\tau = [\Delta']\tau'}{\Gamma_L, \alpha, \underbrace{\Gamma_R', \hat{\alpha} = \tau}_{\Gamma_R} \longrightarrow \underbrace{\Delta', \hat{\alpha} = \tau'}_{\Delta'}} \longrightarrow\!\mathsf{Solved}$

  By i.h., $\Delta' = (\Delta_L, \alpha, \Delta_R')$ where $\Gamma_L \longrightarrow \Delta_L$.
  Hence $\Delta = (\Delta_L, \alpha, \Delta_R', \hat{\alpha} = \tau')$.
  (If Γ_R is soft, by i.h. Δ_R' is soft, so $\Delta_R = (\Delta_R', \hat{\alpha} = \tau)$ is soft.)

- **Case** $\dfrac{\Gamma_L, \alpha, \Gamma_R' \longrightarrow \Delta'}{\Gamma_L, \alpha, \underbrace{\Gamma_R', \hat{\alpha}}_{\Gamma_R} \longrightarrow \underbrace{\Delta', \hat{\alpha} = \tau'}_{\Delta}} \longrightarrow\!\mathsf{Solve}$

  By i.h., $\Delta' = (\Delta_L, \alpha, \Delta_R')$ where $\Gamma_L \longrightarrow \Delta_L$.
  Therefore $\Delta = (\Delta_L, \alpha, \Delta_R, \hat{\alpha} = \tau)$.
  (If Γ_R is soft, by i.h. Δ_R' is soft, so $\Delta_R = (\Delta_R', \hat{\alpha} = \tau)$ is soft.)

- **Case** $\dfrac{\Gamma_L, \alpha, \Gamma_R \longrightarrow \Delta'}{\Gamma_L, \alpha, \Gamma_R \longrightarrow \underbrace{\Delta', \hat{\alpha}}_{\Delta}} \longrightarrow\!\mathsf{Add}$

  By i.h., $\Delta' = (\Delta_L, \alpha, \Delta_R')$ where $\Gamma_L \longrightarrow \Delta_L$.
  Therefore $\Delta = (\Delta_L, \alpha, \Delta_R', \hat{\alpha})$.
  (If Γ_R is soft, by i.h. Δ_R' is soft, so $\Delta_R = (\Delta_R', \hat{\alpha})$ is soft.)

- **Case** $\dfrac{\Gamma_L, \alpha, \Gamma_R \longrightarrow \Delta'}{\Gamma_L, \alpha, \Gamma_R \longrightarrow \Delta', \hat{\alpha} = \tau} \longrightarrow\!\mathsf{AddSolved}$

  In this case, we know that $\Delta = (\Delta', \hat{\alpha} = \tau)$.
  By i.h., $\Delta' = (\Delta_L, \alpha, \Delta_R')$ where $\Gamma_L \longrightarrow \Delta_L$.
  Hence $\Delta = (\Delta_L, \alpha, \Delta_R', \hat{\alpha} = \tau)$.
  (If Γ_R is soft, by i.h. Δ_R' is soft, so $\Delta_R = (\Delta_R', \hat{\alpha} = \tau)$ is soft.)

(ii) Similar to the proof of (i), except that the $\longrightarrow$Marker and $\longrightarrow$Uvar cases are swapped.

(iii) Similar to (i), with $\Theta = \hat{\alpha}$ in the $\longrightarrow$Unsolved case and $\Theta = (\hat{\alpha} = \tau)$ in the $\longrightarrow$Solve case.

(iv) Similar to (iii).

(v) Similar to (i), but using the equality premise of $\longrightarrow$Var. □

Lemma 24 (Extension Weakening). *If* $\Gamma \vdash A$ *and* $\Gamma \longrightarrow \Delta$ *then* $\Delta \vdash A$.

Proof. By a straightforward induction on $\Gamma \vdash A$.

In the UvarWF case, we use Lemma 23 (Extension Order) (i). In the EvarWF case, use Lemma 23 (Extension Order) (iii). In the SolvedEvarWF case, use Lemma 23 (Extension Order) (iv).

In the other cases, apply the i.h. to all subderivations, then apply the rule. □

Lemma 25 (Solution Admissibility for Extension). *If* $\Gamma_L \vdash \tau$ *then* $\Gamma_L, \hat{\alpha}, \Gamma_R \longrightarrow \Gamma_L, \hat{\alpha} = \tau, \Gamma_R$.

Proof. By induction on Γ_R.

- Case $\Gamma_R = \cdot$:
  By Lemma 19 (Reflexivity) (reflexivity), $\Gamma_L \longrightarrow \Gamma_L$.
  Applying rule $\longrightarrow$Solve gives $\Gamma_L, \hat{\alpha} \longrightarrow \Gamma_L, \hat{\alpha} = \tau$.
- Case $\Gamma_R = (\Gamma'_R, x : A)$:
  By i.h., $\Gamma_L, \hat{\alpha}, \Gamma'_R \longrightarrow \Gamma_L, \hat{\alpha} = \tau, \Gamma'_R$.
  Applying rule $\longrightarrow$Var gives $\Gamma_L, \hat{\alpha}, \Gamma'_R, x : A \longrightarrow \Gamma_L, \hat{\alpha} = \tau, \Gamma'_R, x : A$.
- Case $\Gamma_R = (\Gamma'_R, \alpha)$: By i.h. and rule $\longrightarrow$Uvar.
- Case $\Gamma_R = (\Gamma'_R, \hat{\beta})$: By i.h. and rule $\longrightarrow$Add.
- Case $\Gamma_R = (\Gamma'_R, \hat{\beta} = \tau')$: By i.h. and rule $\longrightarrow$AddSolved.
- Case $\Gamma_R = (\Gamma'_R, \blacktriangleright_{\hat{\beta}})$: By i.h. and rule $\longrightarrow$Marker. □

Lemma 26 (Solved Variable Addition for Extension). *If* $\Gamma_L \vdash \tau$ *then* $\Gamma_L, \Gamma_R \longrightarrow \Gamma_L, \hat{\alpha} = \tau, \Gamma_R$.

Proof. By induction on Γ_R. The proof is exactly the same as the proof of Lemma 25 (Solution Admissibility for Extension), except that in the $\Gamma_R = \cdot$, we apply rule $\longrightarrow$AddSolved instead of $\longrightarrow$Solve. □

Lemma 27 (Unsolved Variable Addition for Extension). *We have that* $\Gamma_L, \Gamma_R \longrightarrow \Gamma_L, \hat{\alpha}, \Gamma_R$.

Proof. By induction on Γ_R. The proof is exactly the same as the proof of Lemma 25 (Solution Admissibility for Extension), except that in the $\Gamma_R = \cdot$ case, we apply rule $\longrightarrow$Add instead of $\longrightarrow$Solve. □

Lemma 28 (Parallel Admissibility).
If $\Gamma_L \longrightarrow \Delta_L$ *and* $\Gamma_L, \Gamma_R \longrightarrow \Delta_L, \Delta_R$ *then:*

(i) $\Gamma_L, \hat{\alpha}, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha}, \Delta_R$

(ii) *If* $\Delta_L \vdash \tau'$ *then* $\Gamma_L, \hat{\alpha}, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} = \tau', \Delta_R$.

(iii) *If* $\Gamma_L \vdash \tau$ *and* $\Delta_L \vdash \tau'$ *and* $[\Delta_L]\tau = [\Delta_L]\tau'$, *then* $\Gamma_L, \hat{\alpha} = \tau, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} = \tau', \Delta_R$.

Proof. By induction on Δ_R. As always, we assume that all contexts mentioned in the statement of the lemma are well-formed. Hence, $\hat{\alpha} \notin \mathsf{dom}(\Gamma_L) \cup \mathsf{dom}(\Gamma_R) \cup \mathsf{dom}(\Delta_L) \cup \mathsf{dom}(\Delta_R)$.

(i) We proceed by cases of Δ_R. Observe that in all the extension rules, the right-hand context gets smaller, so as we enter subderivations of $\Gamma_L, \Gamma_R \longrightarrow \Delta_L, \Delta_R$, the context Δ_R becomes smaller.

The only tricky part of the proof is that to apply the i.h., we need $\Gamma_L \longrightarrow \Delta_L$. So we need to make sure that as we drop items from the right of Γ_R and Δ_R, we don't go too far and start decomposing Γ_L or Δ_L! It's easy to avoid decomposing Δ_L: when $\Delta_R = \cdot$, we don't need to apply the i.h. anyway. To avoid decomposing Γ_L, we need to reason by contradiction, using Lemma 14 (Declaration Preservation).

- **Case** $\Delta_R = \cdot$:
  We have $\Gamma_L \longrightarrow \Delta_L$. Applying $\longrightarrow$Unsolved to that derivation gives the result.

- **Case** $\Delta_R = (\Delta'_R, \hat{\beta})$: We have $\hat{\beta} \neq \hat{\alpha}$ by the well-formedness assumption.

  The concluding rule of $\Gamma_L, \Gamma_R \longrightarrow \Delta_L, \Delta'_R, \hat{\beta}$ must have been $\longrightarrow$Unsolved or $\longrightarrow$Add. In both cases, the result follows by i.h. and applying $\longrightarrow$Unsolved or $\longrightarrow$Add.

  Note: In $\longrightarrow$Add, the left-hand context doesn't change, so we clearly maintain $\Gamma_L \longrightarrow \Delta_L$. In $\longrightarrow$Unsolved, we can correctly apply the i.h. because $\Gamma_R \neq \cdot$. Suppose, for a contradiction, that $\Gamma_R = \cdot$. Then $\Gamma_L = (\Gamma'_L, \hat{\beta})$. It was given that $\Gamma_L \longrightarrow \Delta_L$, that is, $\Gamma'_L, \hat{\beta} \longrightarrow \Delta_L$. By Lemma 14 (Declaration Preservation), Δ_L has a declaration of $\hat{\beta}$. But then $\Delta = (\Delta_L, \Delta'_R, \hat{\beta})$ is not well-formed: contradiction. Therefore $\Gamma_R \neq \cdot$.

- **Case** $\Delta_R = (\Delta'_R, \hat{\beta} = \tau)$: We have $\hat{\beta} \neq \hat{\alpha}$ by the well-formedness assumption.

  The concluding rule must have been $\longrightarrow$Solved, $\longrightarrow$Solve or $\longrightarrow$AddSolved. In each case, apply the i.h. and then the corresponding rule. (In $\longrightarrow$Solved and $\longrightarrow$Solve, use Lemma 14 (Declaration Preservation) to show $\Gamma_R \neq \cdot$.)

- **Case** $\Delta_R = (\Delta'_R, \alpha)$: The concluding rule must have been $\longrightarrow$Uvar. The result follows by i.h. and applying $\longrightarrow$Uvar.

- **Case** $\Delta_R = (\Delta'_R, \blacktriangleright_{\hat{\beta}})$: Similar to the previous case, with rule $\longrightarrow$Marker.

- **Case** $\Delta_R = (\Delta'_R, x : A)$: Similar to the previous case, with rule $\longrightarrow$Var.

(ii) Similar to part (i), except that when $\Delta_R = \cdot$, apply rule $\longrightarrow$Solve.

(iii) Similar to part (i), except that when $\Delta_R = \cdot$, apply rule $\longrightarrow$Solved, using the given equality to satisfy the second premise. ☐

Lemma 29 (Parallel Extension Solution).
If $\Gamma_L, \hat{\alpha}, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} = \tau', \Delta_R$ *and* $\Gamma_L \vdash \tau$ *and* $[\Delta_L]\tau = [\Delta_L]\tau'$ *then* $\Gamma_L, \hat{\alpha} = \tau, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} = \tau', \Delta_R$.

Proof. By induction on Δ_R.

In the case where $\Delta_R = (\Delta'_R, \hat{\alpha} = \tau')$, we know that rule $\longrightarrow$Solve must have concluded the derivation (we can use Lemma 14 (Declaration Preservation) to get a contradiction that rules out $\longrightarrow$AddSolved); then we have a subderivation $\Gamma_L \longrightarrow \Delta_L$, to which we can apply $\longrightarrow$Solved. ☐

Lemma 30 (Parallel Variable Update).
If $\Gamma_L, \hat{\alpha}, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} = \tau_0, \Delta_R$ *and* $\Gamma_L \vdash \tau_1$ *and* $\Delta_L \vdash \tau_2$ *and* $[\Delta_L]\tau_0 = [\Delta_L]\tau_1 = [\Delta_L]\tau_2$ *then* $\Gamma_L, \hat{\alpha} = \tau_1, \Gamma_R \longrightarrow \Delta_L, \hat{\alpha} = \tau_2, \Delta_R$.

Proof. By induction on Δ_R. Similar to the proof of Lemma 29 (Parallel Extension Solution), but applying $\longrightarrow$Solved at the end. ☐

D′.2 Instantiation Extends

Lemma 31 (Instantiation Extension).
If $\Gamma \vdash \hat{\alpha} :\leqq \tau \dashv \Delta$ *or* $\Gamma \vdash \tau \leqq: \hat{\alpha} \dashv \Delta$ *then* $\Gamma \longrightarrow \Delta$.

Proof. By induction on the given instantiation derivation.

- **Case**
$$\frac{\Gamma \vdash \tau}{\Gamma, \hat{\alpha}, \Gamma' \vdash \hat{\alpha} :\leqq \tau \dashv \Gamma, \hat{\alpha} = \tau, \Gamma'}\ \textsf{InstLSolve}$$
  By Lemma 25 (Solution Admissibility for Extension), $\Gamma, \hat{\alpha}, \Gamma' \longrightarrow \Gamma, \hat{\alpha} = \tau, \Gamma'$.

- **Case**
$$\frac{}{\Gamma[\hat{\alpha}][\hat{\beta}] \vdash \hat{\alpha} :\leqq \hat{\beta} \dashv \Gamma[\hat{\alpha}][\hat{\beta} = \hat{\alpha}]}\ \textsf{InstLReach}$$
  $\Gamma[\hat{\alpha}][\hat{\beta}] = \Gamma_0, \hat{\alpha}, \Gamma_1, \hat{\beta}, \Gamma_2$ for some $\Gamma_0, \Gamma_1, \Gamma_2$.
  By the definition of well-formedness, $\Gamma_0, \hat{\alpha}, \Gamma_1 \vdash \hat{\alpha}$.
  Therefore, by Lemma 25 (Solution Admissibility for Extension), $\Gamma_0, \hat{\alpha}, \Gamma_1, \hat{\beta}, \Gamma_2 \longrightarrow \Gamma_0, \hat{\alpha}, \Gamma_1, \hat{\beta} = \hat{\alpha}, \Gamma_2$.

- **Case** $$\frac{\Gamma[\hat\alpha_2, \hat\alpha_1, \hat\alpha = \hat\alpha_1 \to \hat\alpha_2] \vdash A_1 \le: \hat\alpha_1 \dashv \Gamma' \qquad \Gamma' \vdash \hat\alpha_2 :\le [\Gamma']A_2 \dashv \Delta}{\Gamma[\hat\alpha] \vdash \hat\alpha :\le A_1 \to A_2 \dashv \Delta}\ \mathsf{InstLArr}$$

  By Lemma 27 (Unsolved Variable Addition for Extension), we can insert an (unsolved) $\hat\alpha_2$, giving $\Gamma[\hat\alpha] \longrightarrow \Gamma[\hat\alpha_2, \hat\alpha]$.
  By Lemma 27 (Unsolved Variable Addition for Extension) again, $\Gamma[\hat\alpha_2, \hat\alpha] \longrightarrow \Gamma[\hat\alpha_2, \hat\alpha_1, \hat\alpha]$.
  By Lemma 25 (Solution Admissibility for Extension), we can solve $\hat\alpha$, giving $\Gamma[\hat\alpha_2, \hat\alpha_1, \hat\alpha] \longrightarrow \Gamma[\hat\alpha_2, \hat\alpha_1, \hat\alpha = \hat\alpha_1 \to \hat\alpha_2]$.
  Then by transitivity (Lemma 20 (Transitivity)), $\Gamma[\hat\alpha] \longrightarrow \Gamma[\hat\alpha_2, \hat\alpha_1, \hat\alpha = \hat\alpha_1 \to \hat\alpha_2]$.
  By i.h. on the first subderivation, $\Gamma[\hat\alpha_2, \hat\alpha_1, \hat\alpha = \hat\alpha_1 \to \hat\alpha_2] \longrightarrow \Gamma'$.
  By i.h. on the second subderivation, $\Gamma' \longrightarrow \Delta$.
  By transitivity (Lemma 20 (Transitivity)), $\Gamma[\hat\alpha_2, \hat\alpha_1, \hat\alpha = \hat\alpha_1 \to \hat\alpha_2] \longrightarrow \Delta$.
  By transitivity (Lemma 20 (Transitivity)), $\Gamma[\hat\alpha] \longrightarrow \Delta$.

- **Case** $$\frac{\Gamma[\hat\alpha], \beta \vdash \hat\alpha :\le B \dashv \Delta, \beta, \Delta'}{\Gamma[\hat\alpha] \vdash \hat\alpha :\le \forall\beta.\, B \dashv \Delta}\ \mathsf{InstLAllR}$$

  By induction, $\Gamma[\hat\alpha], \beta \longrightarrow \Delta, \beta, \Delta'$.
  By Lemma 23 (Extension Order) (i), we have $\Gamma[\hat\alpha] \longrightarrow \Delta$.

- **Case** $$\frac{\Gamma \vdash \tau}{\Gamma, \hat\alpha, \Gamma' \vdash \tau \le: \hat\alpha \dashv \Gamma, \hat\alpha = \tau, \Gamma'}\ \mathsf{InstRSolve}$$

  By Lemma 25 (Solution Admissibility for Extension), we can solve $\hat\alpha$, giving $\Gamma, \hat\alpha, \Gamma' \longrightarrow \Gamma, \hat\alpha = \tau, \Gamma'$.

- **Case** $$\frac{}{\Gamma[\hat\alpha][\hat\beta] \vdash \hat\beta \le: \hat\alpha \dashv \Gamma[\hat\alpha][\hat\beta = \hat\alpha]}\ \mathsf{InstRReach}$$

  $\Gamma[\hat\alpha][\hat\beta] = \Gamma_0, \hat\alpha, \Gamma_1, \hat\beta, \Gamma_2$ for some $\Gamma_0, \Gamma_1, \Gamma_2$.
  By the definition of well-formedness, $\Gamma_0, \hat\alpha, \Gamma_1 \vdash \hat\alpha$.
  Hence by Lemma 25 (Solution Admissibility for Extension), we can solve $\hat\beta$, giving $\Gamma_0, \hat\alpha, \Gamma_1, \hat\beta, \Gamma_2 \longrightarrow \Gamma_0, \hat\alpha, \Gamma_1, \hat\beta = \hat\alpha, \Gamma_2$.

- **Case** $$\frac{\Gamma[\hat\alpha_2, \hat\alpha_1, \hat\alpha = \hat\alpha_1 \to \hat\alpha_2] \vdash \hat\alpha_1 :\le A_1 \dashv \Gamma' \qquad \Gamma' \vdash [\Gamma']A_2 \le: \hat\alpha_2 \dashv \Delta}{\Gamma[\hat\alpha] \vdash A_1 \to A_2 \le: \hat\alpha \dashv \Delta}\ \mathsf{InstRArr}$$

  Because the contexts here are the same as in InstLArr, this is the same as the InstLArr case.

- **Case** $$\frac{\Gamma[\hat\alpha], \blacktriangleright_{\hat\beta}, \hat\beta \vdash [\hat\beta/\beta]B \le: \hat\alpha \dashv \Delta, \blacktriangleright_{\hat\beta}, \Delta'}{\Gamma[\hat\alpha] \vdash \forall\beta.\, B \le: \hat\alpha \dashv \Delta}\ \mathsf{InstRAllL}$$

  By i.h., $\Gamma[\hat\alpha], \blacktriangleright_{\hat\beta}, \hat\beta \longrightarrow \Delta, \blacktriangleright_{\hat\beta}, \Delta'$.
  By Lemma 23 (Extension Order) (ii), $\Gamma[\hat\alpha] \longrightarrow \Delta$. ☐

D′.3 Subtyping Extends

Lemma 32 (Subtyping Extension).
If $\Gamma \vdash A <: B \dashv \Delta$ *then* $\Gamma \longrightarrow \Delta$.

Proof. By induction on the given derivation.
For cases <:Var, <:Unit, <:Exvar, we have $\Delta = \Gamma$, so Lemma 19 (Reflexivity) suffices.

- **Case** $$\frac{\Gamma \vdash B_1 <: A_1 \dashv \Theta \qquad \Theta \vdash [\Omega]A_2 <: [\Omega]B_2 \dashv \Delta}{\Gamma \vdash A_1 \to A_2 <: B_1 \to B_2 \dashv \Delta}\ {<:}{\to}$$

  By IH on each subderivation, $\Gamma \longrightarrow \Theta$ and $\Theta \longrightarrow \Delta$.

  By Lemma 20 (Transitivity) (transitivity), $\Gamma \longrightarrow \Delta$, which was to be shown.

- **Case** $\dfrac{\Gamma, \blacktriangleright_{\hat\alpha}, \hat\alpha \vdash [\hat\alpha/\alpha]A <: B \dashv \Delta, \blacktriangleright_{\hat\alpha}, \Theta}{\Gamma \vdash \forall\alpha.\, A <: B \dashv \Delta}$ <:∀L

  By IH, $\Gamma, \blacktriangleright_{\hat\alpha}, \hat\alpha \longrightarrow \Delta, \blacktriangleright_{\hat\alpha}, \Theta$.

  By Lemma 23 (Extension Order) (ii) with $\Gamma_L = \Gamma$ and $\Gamma'_L = \Delta$ and $\Gamma_R = \hat\alpha$ and $\Gamma'_R = \Theta$, we obtain

$$\Gamma \longrightarrow \Delta$$

- **Case** $\dfrac{\Gamma, \beta \vdash A <: B \dashv \Delta, \beta, \Theta}{\Gamma \vdash A <: \forall\beta.\, B \dashv \Delta}$ <:∀R

  By IH, we have $\Gamma, \beta \longrightarrow \Delta, \beta, \Theta$.

  By Lemma 23 (Extension Order) (i), we obtain $\Gamma \longrightarrow \Delta$, which was to be shown.

- **Cases** <:InstantiateL, <:InstantiateR: In each of these rules, the premise has the same input and output contexts as the conclusion, so Lemma 31 (Instantiation Extension) suffices. □

E′ Decidability of Instantiation

Lemma 33 (Left Unsolvedness Preservation).
If $\underbrace{\Gamma_0, \hat\alpha, \Gamma_1}_{\Gamma} \vdash \hat\alpha :\leqq A \dashv \Delta$ *or* $\underbrace{\Gamma_0, \hat\alpha, \Gamma_1}_{\Gamma} \vdash A \leqq: \hat\alpha \dashv \Delta$*, and* $\hat\beta \in \mathsf{unsolved}(\Gamma_0)$*, then* $\hat\beta \in \mathsf{unsolved}(\Delta)$.

Proof. By induction on the given derivation.

- **Case** $\dfrac{\Gamma_0 \vdash \tau}{\underbrace{\Gamma_0, \hat\alpha, \Gamma_1}_{\Gamma} \vdash \hat\alpha :\leqq \tau \dashv \Gamma_0, \hat\alpha = \tau, \Gamma_1}$ InstLSolve

  Immediate, since to the left of $\hat\alpha$, the contexts Δ and Γ are the same.

- **Case** $\dfrac{}{\Gamma[\hat\alpha][\hat\beta] \vdash \hat\alpha :\leqq \hat\beta \dashv \Gamma[\hat\alpha][\hat\beta = \hat\alpha]}$ InstLReach

  Immediate, since to the left of $\hat\alpha$, the contexts Δ and Γ are the same.

- **Case** $\dfrac{\Gamma[\hat\alpha_2, \hat\alpha_1, \hat\alpha = \hat\alpha_1 \to \hat\alpha_2] \vdash A_1 \leqq: \hat\alpha_1 \dashv \Gamma' \qquad \Gamma' \vdash \hat\alpha_2 :\leqq [\Gamma']A_2 \dashv \Delta}{\Gamma[\hat\alpha] \vdash \hat\alpha :\leqq A_1 \to A_2 \dashv \Delta}$ InstLArr

  We have $\hat\beta \in \mathsf{unsolved}(\Gamma_0)$. Therefore $\hat\beta \in \mathsf{unsolved}(\Gamma_0, \hat\alpha_2)$.
  Clearly, $\hat\alpha_2 \in \mathsf{unsolved}(\Gamma_0, \hat\alpha_2)$.
  We have two subderivations:

$$\Gamma_0, \hat\alpha_2, \hat\alpha_1, \hat\alpha = \hat\alpha_1 \to \hat\alpha_2, \Gamma_1 \vdash A_1 \leqq: \hat\alpha_1 \dashv \Gamma' \quad (1)$$
$$\Gamma' \vdash \hat\alpha_2 :\leqq [\Gamma']A_2 \dashv \Delta \quad (2)$$

  By induction on (1), $\hat\beta \in \mathsf{unsolved}(\Gamma')$.
  Also by induction on (1), with $\hat\alpha_2$ playing the role of $\hat\beta$, we get $\hat\alpha_2 \in \mathsf{unsolved}(\Gamma')$.
  Since $\hat\beta \in \Gamma_0$, it is declared to the left of $\hat\alpha_2$ in $\Gamma_0, \hat\alpha_2, \hat\alpha_1, \hat\alpha = \hat\alpha_1 \to \hat\alpha_2, \Gamma_1$.
  Hence by Lemma 15 (Declaration Order Preservation), $\hat\beta$ is declared to the left of $\hat\alpha_2$ in Γ'. That is, $\Gamma' = (\Gamma'_0, \hat\alpha_2, \Gamma'_1)$, where $\hat\beta \in \mathsf{unsolved}(\Gamma'_0)$.
  By induction on (2), $\hat\beta \in \mathsf{unsolved}(\Delta)$.

- **Case** $\dfrac{\Gamma_0, \hat{\alpha}, \Gamma_1, \beta \vdash \hat{\alpha} :\leqq B \dashv \Delta, \beta, \Delta'}{\Gamma_0, \hat{\alpha}, \Gamma_1 \vdash \hat{\alpha} :\leqq \forall\beta.\, B \dashv \Delta}$ InstLAllR

  We have $\hat{\beta} \in \mathsf{unsolved}(\Gamma_0)$.
  By induction, $\hat{\beta} \in \mathsf{unsolved}(\Delta, \beta, \Delta')$.
  Note that $\hat{\beta}$ is declared to the left of β in $\Gamma_0, \hat{\alpha}, \Gamma_1, \beta$.
  By Lemma 15 (Declaration Order Preservation), $\hat{\beta}$ is declared to the left of β in (Δ, β, Δ'), that is, in Δ. Since $\hat{\beta} \in \mathsf{unsolved}(\Delta, \beta, \Delta')$, we have $\hat{\beta} \in \mathsf{unsolved}(\Delta)$.

- **Cases** InstRSolve, InstRReach: Similar to the InstLSolve and InstLReach cases.

- **Case** $\dfrac{\Gamma[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2] \vdash \hat{\alpha}_1 :\leqq A_1 \dashv \Gamma' \qquad \Gamma' \vdash [\Gamma']A_2 \leqq: \hat{\alpha}_2 \dashv \Delta}{\Gamma[\hat{\alpha}] \vdash A_1 \to A_2 \leqq: \hat{\alpha} \dashv \Delta}$ InstRArr

  Similar to the InstLAllR case.

- **Case** $\dfrac{\Gamma[\hat{\alpha}], \blacktriangleright_{\hat{\gamma}}, \hat{\gamma} \vdash [\hat{\gamma}/\beta]B \leqq: \hat{\alpha} \dashv \Delta, \blacktriangleright_{\hat{\gamma}}, \Delta'}{\Gamma[\hat{\alpha}] \vdash \forall\beta.\, B \leqq: \hat{\alpha} \dashv \Delta}$ InstRAllL

  We have $\hat{\beta} \in \mathsf{unsolved}(\Gamma_0)$.
  By induction, $\hat{\beta} \in \mathsf{unsolved}(\Delta, \blacktriangleright_{\hat{\gamma}}, \Delta')$.
  Note that $\hat{\beta}$ is declared to the left of $\blacktriangleright_{\hat{\gamma}}$ in $\Gamma_0, \hat{\alpha}, \Gamma_1, \blacktriangleright_{\hat{\gamma}}, \hat{\gamma}$.
  By Lemma 15 (Declaration Order Preservation), $\hat{\beta}$ is declared to the left of $\blacktriangleright_{\hat{\gamma}}$ in $\Delta, \blacktriangleright_{\hat{\gamma}}, \Delta'$.
  Hence $\hat{\beta}$ is declared in Δ, and we know it is in $\mathsf{unsolved}(\Delta, \blacktriangleright_{\hat{\gamma}}, \Delta')$, so $\hat{\beta} \in \mathsf{unsolved}(\Delta)$. □

Lemma 34 (Left Free Variable Preservation). *If* $\overbrace{\Gamma_0, \hat{\alpha}, \Gamma_1}^{\Gamma} \vdash \hat{\alpha} :\leqq A \dashv \Delta$ *or* $\overbrace{\Gamma_0, \hat{\alpha}, \Gamma_1}^{\Gamma} \vdash A \leqq: \hat{\alpha} \dashv \Delta$, *and* $\Gamma \vdash B$ *and* $\hat{\alpha} \notin FV([\Gamma]B)$ *and* $\hat{\beta} \in \mathsf{unsolved}(\Gamma_0)$ *and* $\hat{\beta} \notin FV([\Gamma]B)$, *then* $\hat{\beta} \notin FV([\Delta]B)$.

Proof. By induction on the given instantiation derivation.

- **Case** $\dfrac{\Gamma_0 \vdash \tau}{\underbrace{\Gamma_0, \hat{\alpha}, \Gamma_1}_{\Gamma} \vdash \hat{\alpha} :\leqq \tau \dashv \underbrace{\Gamma_0, \hat{\alpha} = \tau, \Gamma_1}_{\Delta}}$ InstLSolve

  We have $\hat{\alpha} \notin FV([\Gamma]B)$. Since Δ differs from Γ only in $\hat{\alpha}$, it must be the case that $[\Gamma]B = [\Delta]B$. It is given that $\hat{\beta} \notin FV([\Gamma]B)$, so $\hat{\beta} \notin FV([\Delta]B)$.

- **Case** $\dfrac{}{\underbrace{\Gamma'[\hat{\alpha}][\hat{\gamma}]}_{\Gamma} \vdash \hat{\alpha} :\leqq \hat{\gamma} \dashv \underbrace{\Gamma'[\hat{\alpha}][\hat{\gamma} = \hat{\alpha}]}_{\Delta}}$ InstLReach

  Since Δ differs from Γ only in solving $\hat{\gamma}$ to $\hat{\alpha}$, applying Δ to a type will not introduce a $\hat{\beta}$. We have $\hat{\beta} \notin FV([\Gamma]B)$, so $\hat{\beta} \notin FV([\Delta]B)$.

- **Case** $\dfrac{\Gamma_0 \vdash \tau}{\Gamma_0, \hat{\alpha}, \Gamma_1 \vdash \tau \leqq: \hat{\alpha} \dashv \Gamma_0, \hat{\alpha} = \tau, \Gamma_1}$ InstRSolve

  Similar to the InstLSolve case.

- **Case** $\dfrac{}{\Gamma'[\hat{\alpha}][\hat{\gamma}] \vdash \hat{\gamma} \leqq: \hat{\alpha} \dashv \Gamma'[\hat{\alpha}][\hat{\gamma} = \hat{\alpha}]}$ InstRReach

  Similar to the InstLReach case.

- **Case**

$$\frac{\overbrace{\Gamma_0, \hat\alpha_2, \hat\alpha_1, \hat\alpha = \hat\alpha_1 \to \hat\alpha_2, \Gamma_1}^{\Gamma'} \vdash A_1 \mathrel{\leqq:} \hat\alpha_1 \dashv \Delta \qquad \Delta \vdash \hat\alpha_2 \mathrel{:\leqq} [\Delta]A_2 \dashv \Delta}{\underbrace{\Gamma_0, \hat\alpha, \Gamma_1}_{\Gamma} \vdash \hat\alpha \mathrel{:\leqq} A_1 \to A_2 \dashv \Delta}\ \mathsf{InstLArr}$$

We have $\Gamma \vdash B$ and $\hat\alpha \notin FV([\Gamma]B)$ and $\hat\beta \notin FV([\Gamma]B)$.
By weakening, we get $\Gamma' \vdash B$; since $\hat\alpha \notin FV([\Gamma]B)$ and Γ' only adds a solution for $\hat\alpha$, it follows that $[\Gamma']B = [\Gamma]B$.
Therefore $\hat\alpha_1 \notin FV([\Gamma']B)$ and $\hat\alpha_2 \notin FV([\Gamma']B)$ and $\hat\beta \notin FV([\Gamma']B)$.
Since we have $\hat\beta \in \Gamma_0$, we also have $\hat\beta \in (\Gamma_0, \hat\alpha_2)$.
By induction on the first premise, $\hat\beta \notin FV([\Delta]B)$.
Also by induction on the first premise, with $\hat\alpha_2$ playing the role of $\hat\beta$, we have $\hat\alpha_2 \notin FV([\Delta]B)$.
Note that $\hat\alpha_2 \in \mathsf{unsolved}(\Gamma_0, \hat\alpha_2)$.
By Lemma 33 (Left Unsolvedness Preservation), $\hat\alpha_2 \in \mathsf{unsolved}(\Delta)$.
Therefore Δ has the form $(\Delta_0, \hat\alpha_2, \Delta_1)$.
Since $\hat\beta \neq \hat\alpha_2$, we know that $\hat\beta$ is declared to the left of $\hat\alpha_2$ in $\Gamma_0, \hat\alpha_2$, so by Lemma 15 (Declaration Order Preservation), $\hat\beta$ is declared to the left of $\hat\alpha_2$ in Δ. Hence $\hat\beta \in \Delta_0$.
Furthermore, by Lemma 31 (Instantiation Extension), we have $\Gamma' \longrightarrow \Delta$.
Then by Lemma 24 (Extension Weakening), we have $\Delta \vdash B$. Using induction on the second premise, $\hat\beta \notin FV([\Delta]B)$.

- **Case**

$$\frac{\Gamma_0, \hat\alpha, \Gamma_1, \gamma \vdash \hat\alpha \mathrel{:\leqq} C \dashv \Delta, \gamma, \Delta'}{\underbrace{\Gamma_0, \hat\alpha, \Gamma_1}_{\Gamma} \vdash \hat\alpha \mathrel{:\leqq} \forall\gamma.\, C \dashv \Delta}\ \mathsf{InstLAllR}$$

We have $\Gamma \vdash B$ and $\hat\alpha \notin FV([\Gamma]B)$ and $\hat\beta \in \Gamma_0$ and $\hat\beta \notin FV([\Gamma]B)$.
By weakening, $\Gamma, \gamma \vdash B$; by the definition of substitution, $[\Gamma, \gamma]B = [\Gamma]B$.
Substituting equals for equals, $\hat\alpha \notin FV([\Gamma, \gamma]B)$ and $\hat\beta \notin FV([\Gamma, \gamma]B)$.
By induction, $\hat\beta \notin FV([\Delta, \gamma, \Delta']B)$.
Since $\hat\beta$ is declared to the left of γ in (Γ, γ), we can use Lemma 15 (Declaration Order Preservation) to show that $\hat\beta$ is declared to the left of γ in $(\Delta, \gamma, \Delta')$, that is, in Δ.
We have $\Gamma \vdash B$, so $\gamma \notin FV(B)$. Thus each free variable u in B is in Γ, to the left of γ in (Γ, γ).
Therefore, by Lemma 15 (Declaration Order Preservation), each free variable u in B is in Δ.
Therefore $[\Delta, \gamma, \Delta']B = [\Delta]B$.
Earlier, we obtained $\hat\beta \notin FV([\Delta, \gamma, \Delta']B)$, so substituting equals for equals, $\hat\beta \notin FV([\Delta]B)$.

- **Case**

$$\frac{\Gamma_0, \hat\alpha_2, \hat\alpha_1, \hat\alpha = \hat\alpha_1 \to \hat\alpha_2, \Gamma_1 \vdash \hat\alpha_1 \mathrel{:\leqq} A_1 \dashv \Delta \qquad \Gamma' \vdash [\Delta]A_2 \mathrel{\leqq:} \hat\alpha_2 \dashv \Delta}{\Gamma_0, \hat\alpha, \Gamma_1 \vdash A_1 \to A_2 \mathrel{\leqq:} \hat\alpha \dashv \Delta}\ \mathsf{InstRArr}$$

Similar to the InstLArr case.

- **Case**

$$\frac{\Gamma[\hat\alpha], \blacktriangleright_{\hat\gamma}, \hat\gamma \vdash [\hat\gamma/\gamma]C \mathrel{\leqq:} \hat\alpha \dashv \Delta, \blacktriangleright_{\hat\gamma}, \Delta'}{\Gamma[\hat\alpha] \vdash \forall\gamma.\, C \mathrel{\leqq:} \hat\alpha \dashv \Delta}\ \mathsf{InstRAllL}$$

We have $\Gamma \vdash B$ and $\hat\alpha \notin FV([\Gamma]B)$ and $\hat\beta \in \Gamma_0$ and $\hat\beta \notin FV([\Gamma]B)$.
By weakening, $\Gamma, \blacktriangleright_{\hat\gamma}, \hat\gamma \vdash B$; by the definition of substitution, $[\Gamma, \blacktriangleright_{\hat\gamma}, \hat\gamma]B = [\Gamma]B$.
Substituting equals for equals, $\hat\alpha \notin FV([\Gamma, \blacktriangleright_{\hat\gamma}, \hat\gamma]B)$ and $\hat\beta \notin FV([\Gamma, \blacktriangleright_{\hat\gamma}, \hat\gamma]B)$.
By induction, $\hat\beta \notin FV([\Delta, \blacktriangleright_{\hat\gamma}, \Delta']B)$.
Note that $\hat\beta$ is declared to the left of $\blacktriangleright_{\hat\gamma}$ in $\Gamma, \blacktriangleright_{\hat\gamma}, \hat\gamma$.
By Lemma 15 (Declaration Order Preservation), $\hat\beta$ is declared to the left of $\blacktriangleright_{\hat\gamma}$ in $\Delta, \blacktriangleright_{\hat\gamma}, \Delta'$.
So $\hat\beta$ is declared in Δ.
Now, note that each free variable u in B is in Γ, which is to the left of $\blacktriangleright_{\hat\gamma}$ in $\Gamma, \blacktriangleright_{\hat\gamma}, \hat\gamma$.
Therefore, by Lemma 15 (Declaration Order Preservation), each free variable u in B is in Δ.
Therefore $[\Delta, \blacktriangleright_{\hat\gamma}, \Delta']B = [\Delta]B$.
Earlier, we obtained $\hat\beta \notin FV([\Delta, \blacktriangleright_{\hat\gamma}, \Delta']B)$, so substituting equals for equals, $\hat\beta \notin FV([\Delta]B)$. □

Lemma 35 (Instantiation Size Preservation). *If* $\overbrace{\Gamma_0, \hat{\alpha}, \Gamma_1}^{\Gamma} \vdash \hat{\alpha} :\leqq A \dashv \Delta$ *or* $\overbrace{\Gamma_0, \hat{\alpha}, \Gamma_1}^{\Gamma} \vdash A \leqq: \hat{\alpha} \dashv \Delta$, *and* $\Gamma \vdash B$ *and* $\hat{\alpha} \notin FV([\Gamma]B)$, *then* $|[\Gamma]B| = |[\Delta]B|$, *where* $|C|$ *is the plain size of the term* C.

Proof. By induction on the given derivation.

- **Case**

$$\frac{\Gamma_0 \vdash \tau}{\underbrace{\Gamma_0, \hat{\alpha}, \Gamma_1}_{\Gamma} \vdash \hat{\alpha} :\leqq \tau \dashv \Gamma_0, \hat{\alpha} = \tau, \Gamma_1} \text{InstLSolve}$$

Since Δ differs from Γ only in solving $\hat{\alpha}$, and we know $\hat{\alpha} \notin FV([\Gamma]B)$, we have $[\Delta]B = [\Gamma]B$; therefore $|[\Delta]B = [\Gamma]B|$.

- **Case**

$$\frac{}{\Gamma[\hat{\alpha}][\hat{\beta}] \vdash \hat{\alpha} :\leqq \hat{\beta} \dashv \Gamma[\hat{\alpha}][\hat{\beta} = \hat{\alpha}]} \text{InstLReach}$$

Here, Δ differs from Γ only in solving $\hat{\beta}$ to $\hat{\alpha}$. However, $\hat{\alpha}$ has the same size as $\hat{\beta}$, so even if $\hat{\beta} \in FV([\Gamma]B)$, we have $|[\Delta]B = [\Gamma]B|$.

- **Case**

$$\frac{\overbrace{\Gamma_0, \hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2, \Gamma_1}^{\Gamma'} \vdash A_1 \leqq: \hat{\alpha}_1 \dashv \Theta \qquad \Theta \vdash \hat{\alpha}_2 :\leqq [\Theta]A_2 \dashv \Delta}{\underbrace{\Gamma_0, \hat{\alpha}, \Gamma_1}_{\Gamma} \vdash \hat{\alpha} :\leqq A_1 \to A_2 \dashv \Delta} \text{InstLArr}$$

We have $\Gamma \vdash B$ and $\hat{\alpha} \notin FV([\Gamma]B)$. Since $\hat{\alpha}_1, \hat{\alpha}_2 \notin \mathsf{dom}(\Gamma)$, we have $\hat{\alpha}, \hat{\alpha}_1, \hat{\alpha}_2 \notin FV([\Gamma]B)$. It follows that $[\Gamma']B = [\Gamma]B$.
By weakening, $\Gamma' \vdash B$.
By induction on the first premise, $|[\Gamma']B| = |[\Theta]B|$.
By Lemma 15 (Declaration Order Preservation), since $\hat{\alpha}_2$ is declared to the left of $\hat{\alpha}_1$ in Γ', we have that $\hat{\alpha}_2$ is declared to the left of $\hat{\alpha}_1$ in Θ.
By Lemma 33 (Left Unsolvedness Preservation), since $\hat{\alpha}_2 \in \mathsf{unsolved}(\Gamma')$, it is unsolved in Θ: that is, $\Theta = (\Theta_0, \hat{\alpha}_2, \Theta_1)$.
By Lemma 31 (Instantiation Extension), we have $\Gamma' \longrightarrow \Theta$.
By Lemma 24 (Extension Weakening), $\Theta \vdash B$.
Since $\hat{\alpha}_2 \notin FV([\Gamma']B)$, Lemma 34 (Left Free Variable Preservation) gives $\hat{\alpha}_2 \notin FV([\Theta]B)$.
By induction on the second premise, $|[\Theta]B| = |[\Delta]B|$, and by transitivity of equality, $|[\Gamma]B| = |[\Delta]B|$.

- **Case**

$$\frac{\Gamma_0, \hat{\alpha}, \Gamma_1, \beta \vdash \hat{\alpha} :\leqq A_0 \dashv \Delta, \beta, \Delta'}{\underbrace{\Gamma_0, \hat{\alpha}, \Gamma_1}_{\Gamma} \vdash \hat{\alpha} :\leqq \forall \beta.\, A_0 \dashv \Delta} \text{InstLAllR}$$

We have $\Gamma \vdash B$ and $\hat{\alpha} \notin FV([\Gamma]B)$.
By weakening, $\Gamma, \beta \vdash B$.
From the definition of substitution, $[\Gamma]B = [\Gamma, \beta]B$. Hence $\hat{\alpha} \notin FV([\Gamma, \beta]B)$.
The input context of the premise is $(\Gamma_0, \hat{\alpha}, \Gamma_1, \beta)$, which is (Γ, β), so by induction, $|[\Gamma, \beta]B| = |[\Delta, \beta, \Delta']B|$.
Suppose u is a free variable in B. Then u is declared in Γ, and so occurs before β in Γ, β.
By Lemma 15 (Declaration Order Preservation), u is declared before β in Δ, β, Δ'.
So every free variable u in B is declared in Δ.
Hence $[\Delta, \beta, \Delta']B = [\Delta]B$.
We have $[\Gamma]B = [\Gamma, \beta]B$, so $|[\Gamma]B| = |[\Gamma, \beta]B|$; by transitivity of equality, $|[\Gamma]B| = |[\Delta]B|$.

- **Case**

$$\frac{\Gamma_0 \vdash \tau}{\Gamma_0, \hat{\alpha}, \Gamma_1 \vdash \tau \leqq: \hat{\alpha} \dashv \Gamma_0, \hat{\alpha} = \tau, \Gamma_1} \text{InstRSolve}$$

Similar to the InstLSolve case.

- **Case**

$$\frac{}{\Gamma[\hat\alpha][\hat\beta] \vdash \hat\beta \leqq: \hat\alpha \dashv \Gamma[\hat\alpha][\hat\beta = \hat\alpha]}\ \mathsf{InstRReach}$$

Similar to the InstLReach case.

- **Case**

$$\frac{\overbrace{\Gamma_0, \hat\alpha_2, \hat\alpha_1, \hat\alpha = \hat\alpha_1 \to \hat\alpha_2, \Gamma_1}^{\Gamma'} \vdash \hat\alpha_1 :\leqq A_1 \dashv \Theta \qquad \Theta \vdash [\Theta]A_2 \leqq: \hat\alpha_2 \dashv \Delta}{\underbrace{\Gamma_0, \hat\alpha, \Gamma_1}_{\Gamma} \vdash A_1 \to A_2 \leqq: \hat\alpha \dashv \Delta}\ \mathsf{InstRArr}$$

Similar to the InstLArr case.

- **Case**

$$\frac{\Gamma'[\hat\alpha], \blacktriangleright_{\hat\beta}, \hat\beta \vdash [\hat\beta/\beta]A_0 \leqq: \hat\alpha \dashv \Delta, \blacktriangleright_{\hat\beta}, \Delta'}{\Gamma'[\hat\alpha] \vdash \forall\beta.\, A_0 \leqq: \hat\alpha \dashv \Delta}\ \mathsf{InstRAllL}$$

We have $\Gamma \vdash B$ and $\hat\alpha \notin FV([\Gamma]B)$.
By weakening, $\Gamma, \blacktriangleright_{\hat\beta}, \hat\beta \vdash B$.
From the definition of substitution, $[\Gamma]B = [\Gamma, \blacktriangleright_{\hat\beta}, \hat\beta]B$. Hence $\hat\alpha \notin FV([\Gamma, \blacktriangleright_{\hat\beta}, \hat\beta]B)$.
By induction, $|[\Gamma, \blacktriangleright_{\hat\beta}, \hat\beta]B| = |[\Delta, \blacktriangleright_{\hat\beta}, \Delta']B|$.
Suppose u is a free variable in B.
Then u is declared in Γ, and so occurs before $\blacktriangleright_{\hat\beta}$ in $\Gamma, \blacktriangleright_{\hat\beta}, \hat\beta$.
By Lemma 15 (Declaration Order Preservation), u is declared before $\blacktriangleright_{\hat\beta}$ in $\Delta, \blacktriangleright_{\hat\beta}, \Delta'$.
So every free variable u in B is declared in Δ.
Hence $[\Delta, \blacktriangleright_{\hat\beta}, \Delta']B = [\Delta]B$.
Since $[\Gamma]B = [\Gamma, \blacktriangleright_{\hat\beta}, \hat\beta]B$, we have $|[\Gamma]B| = |[\Gamma, \blacktriangleright_{\hat\beta}, \hat\beta]B|$; by transitivity of equality, $|[\Gamma]B| = |[\Delta]B|$. ☐

Theorem 7 (Decidability of Instantiation). *If $\Gamma = \Gamma_0[\hat\alpha]$ and $\Gamma \vdash A$ such that $[\Gamma]A = A$ and $\hat\alpha \notin FV(A)$, then:*

(1) Either there exists Δ such that $\Gamma[\hat\alpha] \vdash \hat\alpha :\leqq A \dashv \Delta$, or not.

(2) Either there exists Δ such that $\Gamma[\hat\alpha] \vdash A \leqq: \hat\alpha \dashv \Delta$, or not.

Proof. By induction on the derivation of $\Gamma \vdash A$.

(1) $\Gamma \vdash \hat\alpha :\leqq A \dashv \Delta$ is decidable.

- **Case**

$$\frac{\alpha \in \Gamma}{\underbrace{\Gamma_L, \hat\alpha, \Gamma_R}_{\Gamma} \vdash \alpha}\ \mathsf{UvarWF}$$

If $\alpha \in \Gamma_L$, then by UvarWF we have $\Gamma_L \vdash \alpha$, and by rule InstLSolve we have a derivation. Otherwise no rule matches, and so no derivation exists.

- **Case** UnitWF: By rule InstLSolve.

- **Case**

$$\frac{}{\underbrace{\Gamma_L, \hat\alpha, \Gamma_R}_{\Gamma} \vdash \hat\beta}\ \mathsf{EvarWF}$$

By inversion, we have $\hat\beta \in \Gamma$, and $[\Gamma]\hat\beta = \hat\beta$. Since $\hat\alpha \notin FV([\Gamma]\hat\beta) = FV(\hat\beta) = \{\hat\beta\}$, it follows that $\hat\alpha \neq \hat\beta$: Either $\hat\beta \in \Gamma_L$ or $\hat\beta \in \Gamma_R$.
If $\hat\beta \in \Gamma_L$, then we have a derivation by InstLSolve.
If $\hat\beta \in \Gamma_R$, then we have a derivation by InstLReach.

- **Case**

$$\frac{}{\Gamma \vdash \hat\beta}\ \mathsf{SolvedEvarWF}$$

By inversion, $(\hat\beta = \tau) \in \Gamma$, but $[\Gamma]\hat\beta = \hat\beta$ is given, so this case is impossible.

- **Case** $\dfrac{\Gamma \vdash A_1 \qquad \Gamma \vdash A_2}{\underbrace{\Gamma_L, \hat{\alpha}, \Gamma_R}_{\Gamma} \vdash A_1 \to A_2}$ ArrowWF

  By assumption, $[\Gamma](A_1 \to A_2) = A_1 \to A_2$ and $\hat{\alpha} \notin FV([\Gamma](A_1 \to A_2))$.
  The only rule matching $A_1 \to A_2$ is InstLArr.
  First, consider whether $\Gamma_L, \hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2, \Gamma_R \vdash A \stackrel{\leq}{=}: \hat{\alpha}_1 \dashv -$ is decidable.

  By definition of substitution, $[\Gamma](A_1 \to A_2) = ([\Gamma]A_1) \to ([\Gamma]A_2)$. Since $[\Gamma](A_1 \to A_2) = A_1 \to A_2$, we have $[\Gamma]A_1 = A_1$ and $[\Gamma]A_2 = A_2$.
  By weakening, $\Gamma_L, \hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2, \Gamma_R \vdash A_1 \to A_2$.
  Since $\Gamma \vdash A_1$ and $\Gamma \vdash A_2$, we have $\hat{\alpha}_1, \hat{\alpha}_2 \notin FV(A_1) \cup FV(A_2)$.
  Since $\hat{\alpha} \notin FV(A) \supseteq FV(A_1)$, it follows that $[\Gamma']A_1 = A_1$.
  By i.h., either there exists Θ such that $\Gamma_L, \hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2, \Gamma_R \vdash A_1 \stackrel{\leq}{=}: \hat{\alpha}_1 \dashv \Theta$, or not.
  If not, then no derivation exists, since the only applicable rule is InstLArr.
  If so, then we have $\Gamma_L, \hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2, \Gamma_R \vdash \hat{\alpha}_1 :\stackrel{\leq}{=} A_1 \dashv \Theta$.
  By Lemma 33 (Left Unsolvedness Preservation), we know that $\hat{\alpha}_2 \in \mathsf{unsolved}(\Theta)$.
  By Lemma 34 (Left Free Variable Preservation), we know that $\hat{\alpha}_2 \notin FV([\Theta]A_2)$.
  Clearly, $[\Theta]([\Theta]A_2) = [\Theta]A_2$.
  Hence by i.h., either there exists Δ such that $\Theta \vdash \hat{\alpha}_2 :\stackrel{\leq}{=} [\Theta]A_2 \dashv \Delta$, or not.
  If not, then no derivation exists, since the only applicable rule is InstLArr.
  If it does, then by rule InstLArr, we have $\Gamma \vdash \hat{\alpha} :\stackrel{\leq}{=} A \dashv \Delta$.

- **Case** $\dfrac{\Gamma, \alpha \vdash A_0}{\Gamma \vdash \forall \alpha.\ A_0}$ ForallWF

  We have $\forall \alpha.\ A_0 = [\Gamma](\forall \alpha.\ A_0)$. By definition of substitution, $[\Gamma](\forall \alpha.\ A_0) = \forall \alpha.\ [\Gamma]A_0$, so $A_0 = [\Gamma]A_0$.
  By definition of substitution, $[\Gamma, \alpha]A_0 = [\Gamma]A_0$.
  We have $\hat{\alpha} \notin FV([\Gamma](\forall \alpha.\ A_0))$. Therefore $\hat{\alpha} \notin FV([\Gamma]A_0) = FV([\Gamma, \alpha]A_0)$.
  By i.h., either there exists Θ such that $\Gamma, \alpha \vdash \hat{\alpha} :\stackrel{\leq}{=} A_0 \dashv \Theta$, or not.

  Suppose $\Gamma, \alpha \vdash \hat{\alpha} :\stackrel{\leq}{=} A_0 \dashv \Theta$.
  By Lemma 31 (Instantiation Extension), $\Gamma \longrightarrow \Theta$;
  by Lemma 23 (Extension Order) (i), $\Theta = \Delta, \alpha, \Delta'$.
  Hence by rule InstLAllR, $\Gamma \vdash \hat{\alpha} :\stackrel{\leq}{=} \forall \alpha.\ A_0 \dashv \Delta$.

  Suppose not.
  Then there is no derivation, since InstLAllR is the only rule matching $\forall \alpha.\ A_0$.

(2) $\Gamma \vdash A \stackrel{\leq}{=}: \hat{\alpha} \dashv \Delta$ is decidable.

- **Case** UvarWF:
  Similar to the UvarWF case in part (1), but applying rule InstRSolve instead of InstLSolve.
- **Case** UnitWF: Apply InstRSolve.
- **Case** $\dfrac{}{\underbrace{\Gamma_L, \hat{\alpha}, \Gamma_R}_{\Gamma} \vdash \hat{\beta}}$ EvarWF

  Similar to the EvarWF case in part (1), but applying InstRSolve/InstRReach instead of InstLSolve/InstLReach.
- **Case** SolvedEvarWF:
  Impossible, for exactly the same reasons as in the SolvedEvarWF case of part (1).
- **Case** $\dfrac{\Gamma \vdash A_1 \qquad \Gamma \vdash A_2}{\underbrace{\Gamma_L, \hat{\alpha}, \Gamma_R}_{\Gamma} \vdash A_1 \to A_2}$ ArrowWF

  As the ArrowWF case of part (1), except applying InstRArr instead of InstLArr.

- **Case** $\dfrac{\Gamma, \beta \vdash B}{\underbrace{\Gamma_L, \hat{\alpha}, \Gamma_R}_{\Gamma} \vdash \forall\beta.\, B}$ ForallWF

  By assumption, $[\Gamma](\forall\beta.\, B) = \forall\beta.\, B$. With the definition of substitution, we get $[\Gamma]B = B$. Hence $[\Gamma]B = B$.
  Hence $[\hat{\beta}/\beta][\Gamma]B = [\hat{\beta}/\beta]B$. Since $\hat{\beta}$ is fresh, $[\hat{\beta}/\beta][\Gamma]B = [\Gamma][\hat{\beta}/\beta]B$.
  By definition of substitution, $[\Gamma, \blacktriangleright_{\hat{\beta}}, \hat{\beta}][\hat{\beta}/\beta]B = [\Gamma][\hat{\beta}/\beta]B$, which by transitivity of equality is $[\hat{\beta}/\beta]B$.
  We have $\hat{\alpha} \notin FV([\Gamma](\forall\beta.\, B))$, so $\hat{\alpha} \notin FV([\Gamma, \blacktriangleright_{\hat{\beta}}, \hat{\beta}][\hat{\beta}/\beta]B)$.
  Therefore, by induction, either $\Gamma, \blacktriangleright_{\hat{\beta}}, \hat{\beta} \vdash [\hat{\beta}/\beta]B \leqq: \hat{\alpha} \dashv \Theta$ or not.

  Suppose $\Gamma, \blacktriangleright_{\hat{\beta}}, \hat{\beta} \vdash [\hat{\beta}/\beta]B \leqq: \hat{\alpha} \dashv \Theta$.
  By Lemma 31 (Instantiation Extension), $\Gamma, \blacktriangleright_{\hat{\beta}}, \hat{\beta} \longrightarrow \Theta$;
  by Lemma 23 (Extension Order) (ii), $\Theta = \Delta, \blacktriangleright_{\hat{\beta}}, \Delta'$.
  Hence by rule InstRAllL, $\Gamma \vdash \forall\beta.\, B \leqq: \hat{\alpha} \dashv \Delta$.

  Suppose not.
  Then there is no derivation, since InstRAllL is the only rule matching $\forall\beta.\, B$. ☐

F′ Decidability of Algorithmic Subtyping

F′.1 Lemmas for Decidability of Subtyping

Lemma 36 (Monotypes Solve Variables). *If* $\Gamma \vdash \hat{\alpha} :\leqq \tau \dashv \Delta$ *or* $\Gamma \vdash \tau \leqq: \hat{\alpha} \dashv \Delta$, *then if* $[\Gamma]\tau = \tau$ *and* $\hat{\alpha} \notin FV([\Gamma]\tau)$, *then* $|\mathsf{unsolved}(\Gamma)| = |\mathsf{unsolved}(\Delta)| + 1$.

Proof. By induction on the given derivation.

- **Case** $\dfrac{\Gamma_L \vdash \tau}{\Gamma_L, \hat{\alpha}, \Gamma_R \vdash \hat{\alpha} :\leqq \tau \dashv \underbrace{\Gamma_L, \hat{\alpha} = \tau, \Gamma_R}_{\Delta}}$ InstLSolve

  It is evident that $|\mathsf{unsolved}(\Gamma_L, \hat{\alpha}, \Gamma_R)| = |\mathsf{unsolved}(\Gamma_L, \hat{\alpha} = \tau, \Gamma_R)|$.

- **Case**
  $\dfrac{}{\Gamma[\hat{\alpha}][\hat{\beta}] \vdash \hat{\alpha} :\leqq \hat{\beta} \dashv \Gamma[\hat{\alpha}][\hat{\beta} = \hat{\alpha}]}$ InstLReach

  Similar to the previous case.

- **Case** $\dfrac{\Gamma_0[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2] \vdash \tau_1 \leqq: \hat{\alpha}_1 \dashv \Theta \qquad \Theta \vdash \hat{\alpha}_2 :\leqq [\Theta]\tau_2 \dashv \Delta}{\Gamma_0[\hat{\alpha}] \vdash \hat{\alpha} :\leqq \tau_1 \to \tau_2 \dashv \Delta}$ InstLArr

  $$\begin{aligned} |\mathsf{unsolved}(\Gamma_0[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2])| &= |\mathsf{unsolved}(\Gamma_0[\hat{\alpha}])| + 1 && \text{Immediate} \\ |\mathsf{unsolved}(\Gamma_0[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2])| &= |\mathsf{unsolved}(\Theta)| + 1 && \text{By i.h.} \\ |\mathsf{unsolved}(\Gamma)| &= |\mathsf{unsolved}(\Theta)| && \text{Subtracting 1} \\ ☞ \quad &= |\mathsf{unsolved}(\Delta)| + 1 && \text{By i.h.} \end{aligned}$$

- **Case** $\dfrac{\Gamma, \beta \vdash \hat{\alpha} :\leqq B \dashv \Delta, \beta, \Delta'}{\Gamma \vdash \hat{\alpha} :\leqq \forall\beta.\, B \dashv \Delta}$ InstLAllR

  This case is impossible, since a monotype cannot have the form $\forall\beta.\, B$.

- **Cases** InstRSolve, InstRReach: Similar to the InstLSolve and InstLReach cases.

- **Case** InstRArr: Similar to the InstLArr case.

- **Case** $\dfrac{\Gamma[\hat{\alpha}], \beta \vdash B \stackrel{\leq}{=}: \hat{\alpha} \dashv \Delta, \beta, \Delta'}{\Gamma[\hat{\alpha}] \vdash \forall\beta.\, B \stackrel{\leq}{=}: \hat{\alpha} \dashv \Delta}$ InstRAllL

  This case is impossible, since a monotype cannot have the form $\forall\beta.\, B$. ☐

Lemma 37 (Monotype Monotonicity). *If* $\Gamma \vdash \tau_1 <: \tau_2 \dashv \Delta$ *then* $|\mathsf{unsolved}(\Delta)| \leq |\mathsf{unsolved}(\Gamma)|$.

Proof. By induction on the given derivation.

- **Cases** $<:$Var, $<:$Exvar:
  In these rules, $\Delta = \Gamma$, so $\mathsf{unsolved}(\Delta) = \mathsf{unsolved}(\Gamma)$; therefore $|\mathsf{unsolved}(\Delta)| \leq |\mathsf{unsolved}(\Gamma)|$.

- **Case** $<:\to$: We have an intermediate context Θ.

  By inversion, $\tau_1 = \tau_{11} \to \tau_{12}$ and $\tau_2 = \tau_{21} \to \tau_{22}$. Therefore, we have monotypes in the first and second premises.

  By induction on the first premise, $|\mathsf{unsolved}(\Theta)| \leq |\mathsf{unsolved}(\Gamma)|$. By induction on the second premise, $|\mathsf{unsolved}(\Delta)| \leq |\mathsf{unsolved}(\Theta)|$. By transitivity of $\leq$, $|\mathsf{unsolved}(\Delta)| \leq |\mathsf{unsolved}(\Gamma)|$, which was to be shown.

- **Cases** $<:\forall$L, $<:\forall$R: We are given a derivation of subtyping on monotypes, so these cases are impossible.

- **Cases** $<:$InstantiateL, $<:$InstantiateR: The input and output contexts in the premise exactly match the conclusion, so the result follows by Lemma 36 (Monotypes Solve Variables). ☐

Lemma 38 (Substitution Decreases Size). *If* $\Gamma \vdash A$ *then* $|\Gamma \vdash [\Gamma]A| \leq |\Gamma \vdash A|$.

Proof. By induction on $|\Gamma \vdash A|$. If $A = 1$ or $A = \alpha$, or $A = \hat{\alpha}$ and $\hat{\alpha} \in \mathsf{unsolved}(\Gamma)$ then $[\Gamma]A = A$. Therefore, $|\Gamma \vdash [\Gamma]A| = |\Gamma \vdash A|$.

If $A = \hat{\alpha}$ and $(\hat{\alpha} = \tau) \in \Gamma$, then by induction hypothesis, $|\Gamma \vdash [\Gamma]\tau| \leq |\Gamma \vdash \tau|$. Of course $|\Gamma \vdash \tau| \leq |\Gamma \vdash \tau| + 1$. By definition of substitution, $[\Gamma]\tau = [\Gamma]\hat{\alpha}$, so

$$|\Gamma \vdash [\Gamma]\hat{\alpha}| \;\leq\; |\Gamma \vdash \tau| + 1$$

By the definition of type size, $|\Gamma \vdash \hat{\alpha}| = |\Gamma \vdash \tau| + 1$, so

$$|\Gamma \vdash [\Gamma]\hat{\alpha}| \;\leq\; |\Gamma \vdash \hat{\alpha}|$$

which was to be shown.

If $A = A_1 \to A_2$, the result follows via the induction hypothesis (twice).

If $A = \forall\alpha.\, A_0$, the result follows via the induction hypothesis. ☐

Lemma 39 (Monotype Context Invariance).
If $\Gamma \vdash \tau <: \tau' \dashv \Delta$ *where* $[\Gamma]\tau = \tau$ *and* $[\Gamma]\tau' = \tau'$ *and* $|\mathsf{unsolved}(\Gamma)| = |\mathsf{unsolved}(\Delta)|$ *then* $\Gamma = \Delta$.

Proof. By induction on the derivation of $\Gamma \vdash \tau <: \tau' \dashv \Delta$.

- **Cases** $<:$Var, $<:$Unit, $<:$Exvar:

  In these rules, the output context is the same as the input context, so the result is immediate.

- **Case** $\dfrac{\Gamma \vdash \tau_1' <: \tau_1 \dashv \Theta \qquad \Theta \vdash [\Theta]\tau_2 <: [\Theta]\tau_2' \dashv \Delta}{\Gamma \vdash \tau_1 \to \tau_2 <: \tau_1' \to \tau_2' \dashv \Delta}$ $<:\to$

  We have that $[\Gamma](\tau_1 \to \tau_2) = \tau_1 \to \tau_2$. By definition of substitution, $[\Gamma]\tau_1 = \tau_1$ and $[\Gamma]\tau_2 = \tau_2$. Similarly, $[\Gamma]\tau_1 = \tau_1'$ and $[\Gamma]\tau_2 = \tau_2'$.

  By i.h., $\Theta = \Gamma$.
  Since Θ is predicative, $[\Theta]\tau_2$ and $[\Theta]\tau_2'$ are monotypes.
  Substitution is idempotent: $[\Theta][\Theta]\tau_2 = [\Theta]\tau_2$ and $[\Theta][\Theta]\tau_2' = [\Theta]\tau_2'$.
  By i.h., $\Delta = \Theta$. Hence $\Delta = \Gamma$.

- **Cases** $<:\forall L$, $<:\forall R$: Impossible, since τ and τ' are monotypes.

- **Case** $$\dfrac{\hat{\alpha} \notin FV(A) \qquad \Gamma_0[\hat{\alpha}] \vdash \hat{\alpha} :\stackrel{\leq}{=} A \dashv \Delta}{\Gamma_0[\hat{\alpha}] \vdash \hat{\alpha} <: A \dashv \Delta} <:\text{InstantiateL}$$

  By Lemma 36 (Monotypes Solve Variables), $|\mathsf{unsolved}(\Delta)| < |\mathsf{unsolved}(\Gamma_0[\hat{\alpha}])|$, but it is given that $|\mathsf{unsolved}(\Gamma_0[\hat{\alpha}])| = |\mathsf{unsolved}(\Delta)|$, so this case is impossible.

- **Case** $<:$InstantiateR: Impossible, as for the $<:$InstantiateL case. ☐

F′.2 Decidability of Subtyping

Theorem 8 (Decidability of Subtyping).
Given a context Γ *and types* A, B *such that* $\Gamma \vdash A$ *and* $\Gamma \vdash B$ *and* $[\Gamma]A = A$ *and* $[\Gamma]B = B$*, it is decidable whether there exists* Δ *such that* $\Gamma \vdash A <: B \dashv \Delta$.

Proof. Let the judgment $\Gamma \vdash A <: B \dashv \Delta$ be measured lexicographically by

(S1) the number of $\forall$ quantifiers in A and B;

(S2) $|\mathsf{unsolved}(\Gamma)|$, the number of unsolved existential variables in Γ;

(S3) $|\Gamma \vdash A| + |\Gamma \vdash B|$.

For each subtyping rule, we show that every premise is smaller than the conclusion. The condition that $[\Gamma]A = A$ and $[\Gamma]B = B$ is easily satisfied at each inductive step, using the definition of substitution.

- Rules $<:$Var, $<:$Unit and $<:$Exvar have no premises.

- **Case** $$\dfrac{\Gamma \vdash B_1 <: A_1 \dashv \Theta \qquad \Theta \vdash [\Theta]A_2 <: [\Theta]B_2 \dashv \Delta}{\Gamma \vdash A_1 \to A_2 <: B_1 \to B_2 \dashv \Delta} <:\to$$

  If A_2 or B_2 has a quantifier, then the first premise is smaller by (S1). Otherwise, the first premise shares an input context with the conclusion, so it has the same (S2). The types B_1 and A_1 are subterms of the conclusion's types, so the first premise is smaller by (S3).

  If B_1 or A_1 has a quantifier, then the second premise is smaller by (S1). Otherwise, by Lemma 37 (Monotype Monotonicity) on the first premise, $|\mathsf{unsolved}(\Theta)| \leq |\mathsf{unsolved}(\Gamma)|$.

  – If $|\mathsf{unsolved}(\Theta)| < |\mathsf{unsolved}(\Gamma)|$, then the second premise is smaller by (S2).

  – If $|\mathsf{unsolved}(\Theta)| = |\mathsf{unsolved}(\Gamma)|$, we have the same (S2).
    However, by Lemma 39 (Monotype Context Invariance), $\Theta = \Gamma$, so $|\Theta \vdash [\Theta]A_2| = |\Gamma \vdash [\Gamma]A_2|$, which by Lemma 38 (Substitution Decreases Size) is less than or equal to $|\Gamma \vdash A_2|$.
    By the same logic, $|\Theta \vdash [\Theta]B_2| \leq |\Gamma \vdash B_2|$.
    Therefore,

    $$|\Theta \vdash [\Theta]A_2| \ + \ |\Theta \vdash [\Theta]B_2| \quad \leq \quad |\Gamma \vdash (A_1 \to A_2)| \ + \ |\Gamma \vdash (B_1 \to B_2)|$$

    and the second premise is smaller by (S3).

- **Cases** $<:\forall L$, $<:\forall R$: In each of these rules, the premise has one less quantifier than the conclusion, so the premise is smaller by (S1).

- **Cases** $<:$InstantiateL, $<:$InstantiateR: Follows from Theorem 7. ☐

G′ Decidability of Typing

Theorem 9 (Decidability of Typing).

(i) Checking: *Given an algorithmic context* Γ*, a term* e*, and a type* B *such that* $\Gamma \vdash B$*, it is decidable whether there is a context* Δ *such that* $\Gamma \vdash e \Leftarrow B \dashv \Delta$.

(ii) Synthesis: *Given an algorithmic context* Γ *and a term* e*, it is decidable whether there exist a type* A *and a context* Δ *such that* $\Gamma \vdash e \Rightarrow A \dashv \Delta$.

(iii) Application: *Given an algorithmic context* Γ*, a term* e*, and a type* A *such that* $\Gamma \vdash A$*, it is decidable whether there exist a type* C *and a context* Δ *such that* $\Gamma \vdash A \bullet e \Rightarrow\!\!\!\!\!\Rightarrow C \dashv \Delta$.

Proof. For rules deriving judgments of the form

$$\begin{array}{l} \Gamma \vdash e \Rightarrow - \dashv - \\ \Gamma \vdash e \Leftarrow B \dashv - \\ \Gamma \vdash A \bullet e \Rightarrow\!\!\!\!\!\Rightarrow - \dashv - \end{array}$$

(where we write "–" for parts of the judgments that are outputs), the following induction measure on such judgments is adequate to prove decidability:

$$\left\langle e, \begin{array}{ll} \Rightarrow & \\ \Leftarrow, & |\Gamma \vdash B| \\ \Rightarrow\!\!\!\!\!\Rightarrow, & |\Gamma \vdash A| \end{array} \right\rangle$$

where $\langle \ldots \rangle$ denotes lexicographic order, and where (when comparing two judgments typing terms of the same size) the synthesis judgment (top line) is considered smaller than the checking judgment (second line), which in turn is considered smaller than the application judgment (bottom line). That is,

$$\Rightarrow \;\prec\; \Leftarrow \;\prec\; \Rightarrow\!\!\!\!\!\Rightarrow$$

Note that this measure only uses the input parts of the judgments, leading to a straightforward decidability argument.

We will show that in each rule, every checking/synthesis/application premise is smaller than the conclusion.

- **Case** Var: No premises.
- **Case** Sub: The first premise has the same subject term e as the conclusion, but the judgment is smaller because the measure considers a synthesis judgment to be smaller than a checking judgment.

  The second premise is a subtyping judgment, which by Theorem 8 is decidable.
- **Case** Anno: The premise types e, and the conclusion types $(e : A)$, so the first part of the measure gets smaller.
- **Case** 1I: No premises.
- **Case** $\to$I: In the premise, the term is smaller.
- **Case** $\to$E: In both premises, the term is smaller.
- **Case** $\forall$I: Both the premise and conclusion type e, and both are checking; however, $|\Gamma, \alpha \vdash A| < |\Gamma \vdash \forall\alpha.\, A|$, so the premise is smaller.
- **Case** $\to$App: Both the premise and conclusion type e, but the premise is a checking judgment, so the premise is smaller.

- **Case** Subst$\Leftarrow$: Both the premise and conclusion type e, and both are checking; however, since we can apply this rule only when Γ has a solution for $\hat{\alpha}$—that is, when $\Gamma = \Gamma_0[\hat{\alpha} = \tau]$—we have that $|\Gamma \vdash [\Gamma]\hat{\alpha}| < |\Gamma \vdash \hat{\alpha}|$, making the last part of the measure smaller.

- **Case** SubstApp: Similar to Subst$\Leftarrow$.

- **Case** $\forall$App: Both the premise and conclusion type e, and both are application judgments; however, by the definition of $|\Gamma \vdash -|$, the size of the type in the premise $[\hat{\alpha}/\alpha]A$ is smaller than $\forall\alpha.\, A$.

- **Case** $\hat{\alpha}$App: Both the premise and conclusion type e, but we switch to checking in the premise, so the premise is smaller.

- **Case** $1\text{I}\Rightarrow$: No premises.

- **Case** $\to\text{I}\Rightarrow$: In the premise, the term is smaller. ☐

H′ Soundness of Subtyping

H′.1 Lemmas for Soundness

Lemma 41 (Variable Preservation).
If $(x : A) \in \Delta$ *or* $(x : A) \in \Omega$ *and* $\Delta \longrightarrow \Omega$ *then* $(x : [\Omega]A) \in [\Omega]\Delta$.

Proof. By mutual induction on Δ and Ω.

Suppose $(x : A) \in \Delta$. In the case where $\Delta = (\Delta', x : A)$ and $\Omega = (\Omega', x : A_\Omega)$, inversion on $\Delta \longrightarrow \Omega$ gives $[\Omega']A = [\Omega']A_\Omega$; by the definition of context application, $[\Omega', x : A_\Omega](\Delta', x : A) = [\Omega']\Delta', x : [\Omega']A_\Omega$, which contains $x : [\Omega']A_\Omega$, which is equal to $x : [\Omega']A$. By well-formedness of Ω, we know that $[\Omega']A = [\Omega]A$.

Suppose $(x : A) \in \Omega$. The reasoning is similar, because equality is symmetric. ☐

Lemma 42 (Substitution Typing). *If* $\Gamma \vdash A$ *then* $\Gamma \vdash [\Gamma]A$.

Proof. By induction on $|\Gamma \vdash A|$ (the size of A under Γ).

- **Cases** UvarWF, UnitWF: Here $A = \alpha$ or $A = 1$, so applying Γ to A does not change it: $A = [\Gamma]A$. Since $\Gamma \vdash A$, we have $\Gamma \vdash [\Gamma]A$, which was to be shown.

- **Case** EvarWF: In this case $A = \hat{\alpha}$, but $\Gamma = \Gamma_0[\hat{\alpha}]$, so applying Γ to A does not change it, and we proceed as in the UnitWF case above.

- **Case** SolvedEvarWF: In this case $A = \hat{\alpha}$ and $\Gamma = \Gamma_L, \hat{\alpha} = \tau, \Gamma_R$. Thus $[\Gamma]A = [\Gamma]\alpha = [\Gamma_L]\tau$. We assume contexts are well-formed, so all free variables in τ are declared in Γ_L. Consequently, $|\Gamma_L \vdash \tau| = |\Gamma \vdash \tau|$, which is less than $|\Gamma \vdash \hat{\alpha}|$. We can therefore apply the i.h. to τ, yielding $\Gamma \vdash [\Gamma]\tau$. By the definition of substitution, $[\Gamma]\tau = [\Gamma]\hat{\alpha}$, so we have $\Gamma \vdash [\Gamma]\hat{\alpha}$.

- **Case** ArrowWF: In this case $A = A_1 \to A_2$. By i.h., $\Gamma \vdash [\Gamma]A_1$ and $\Gamma \vdash [\Gamma]A_2$. By ArrowWF, $\Gamma \vdash ([\Gamma]A_1) \to ([\Gamma]A_2)$, which by the definition of substitution is $\Gamma \vdash [\Gamma](A_1 \to A_2)$.

- **Case** ForallWF: In this case $A = \forall\alpha.\, A_0$. By i.h., $\Gamma, \alpha \vdash [\Gamma, \alpha]A_0$. By the definition of substitution, $[\Gamma, \alpha]A_0 = [\Gamma]A_0$, so by ForallWF, $\Gamma \vdash \forall\alpha.\, [\Gamma]A_0$, which by the definition of substitution is $\Gamma \vdash [\Gamma](\forall\alpha.\, A_0)$. ☐

Lemma 43 (Substitution for Well-Formedness). *If* $\Omega \vdash A$ *then* $[\Omega]\Omega \vdash [\Omega]A$.

Proof. By induction on $|\Omega \vdash A|$, the size of A under Ω (Definition 2).

We consider cases of the well-formedness rule concluding the derivation of $\Omega \vdash A$.

- **Case**

$$\frac{}{\Omega \vdash 1}\ \mathsf{UnitWF}$$

$[\Omega]\Omega \vdash 1$	By DeclUnitWF
$[\Omega]\Omega \vdash [\Omega]1$	By definition of substitution

- **Case**

$$\frac{\alpha \in \Omega}{\Omega \vdash \alpha}\ \mathsf{UvarWF}$$

$\Omega \longrightarrow \Omega$	By Lemma 19 (Reflexivity)
$\Omega = (\Omega_L, \alpha, \Omega_R)$	By $\alpha \in \Omega$
$\alpha \in [\Omega]\Omega$	By Lemma 40 (Uvar Preservation)
$[\Omega]\Omega \vdash \alpha$	By DeclUvarWF
$[\Omega]\Omega \vdash [\Omega]\alpha$	By definition of substitution

- **Case**

$$\frac{}{\underbrace{\Omega_L, \hat{\alpha} = \tau, \Omega_R}_{\Omega} \vdash \hat{\alpha}}\ \mathsf{SolvedEvarWF}$$

$\Omega \vdash \hat{\alpha}$	Given
$\Omega \longrightarrow \Omega$	By Lemma 19 (Reflexivity)
$\Omega \vdash [\Omega]\hat{\alpha}$	By Lemma 42 (Substitution Typing)
$\lvert\Omega \vdash [\Omega]\hat{\alpha}\rvert < \lvert\Omega \vdash \hat{\alpha}\rvert$	Follows from definition of type size
$[\Omega]\Omega \vdash [\Omega][\Omega]\hat{\alpha}$	By i.h.
$[\Omega][\Omega]\hat{\alpha} = [\Omega]\hat{\alpha}$	By Lemma 17 (Substitution Extension Invariance)
$[\Omega]\Omega \vdash [\Omega]\hat{\alpha}$	Applying equality

- **Case**

$$\frac{}{\Omega_L, \hat{\alpha}, \Omega_R \vdash \hat{\alpha}}\ \mathsf{EvarWF}$$

Impossible: the grammar for Ω does not allow unsolved declarations.

- **Case**

$$\frac{\Omega \vdash A_1 \qquad \Omega \vdash A_2}{\Omega \vdash A_1 \to A_2}\ \mathsf{ArrowWF}$$

$\Omega \vdash A_1$	Subderivation
$\lvert\Omega \vdash A_1\rvert < \lvert\Omega \vdash A_1 \to A_2\rvert$	Follows from definition of type size
$[\Omega]\Omega \vdash [\Omega]A_1$	By i.h.
$[\Omega]\Omega \vdash [\Omega]A_2$	By similar reasoning on 2nd subderivation
$[\Omega]\Omega \vdash [\Omega]A_1 \to [\Omega]A_2$	By DeclArrowWF
$[\Omega]\Omega \vdash [\Omega](A_1 \to A_2)$	By definition of substitution

- **Case**

$$\frac{\Omega, \alpha \vdash A_0}{\Omega \vdash \forall\alpha.\, A_0}\ \mathsf{ForallWF}$$

$\Omega, \alpha \vdash A_0$	Subderivation
Let $\Omega' = (\Omega, \alpha)$.	
$\lvert\Omega' \vdash A_0\rvert < \lvert\Omega \vdash \forall\alpha.\, A_0\rvert$	Follows from definition of type size
$[\Omega'](\Omega, \alpha) \vdash [\Omega']A_0$	By i.h.
$[\Omega]\Omega, \alpha \vdash [\Omega']A_0$	By definition of context application
$[\Omega]\Omega, \alpha \vdash [\Omega]A_0$	By definition of substitution
$[\Omega]\Omega \vdash \forall\alpha.\, [\Omega]A_0$	By DeclForallWF
$[\Omega]\Omega \vdash [\Omega](\forall\alpha.\, A_0)$	By definition of substitution

□

Lemma 44 (Substitution Stability).
For any well-formed complete context (Ω, Ω_Z)*, if* $\Omega \vdash A$ *then* $[\Omega]A = [\Omega, \Omega_Z]A$.

Proof. By induction on Ω_Z. If $\Omega_Z = \cdot$, the result is immediate. Otherwise, use the i.h. and the fact that $\Omega \vdash A$ implies $\mathrm{FV}(A) \cap \mathsf{dom}(\Omega_Z) = \emptyset$. □

Lemma 45 (Context Partitioning).
If $\Delta, \blacktriangleright_{\hat\alpha}, \Theta \longrightarrow \Omega, \blacktriangleright_{\hat\alpha}, \Omega_Z$ *then there is a* Ψ *such that* $[\Omega, \blacktriangleright_{\hat\alpha}, \Omega_Z](\Delta, \blacktriangleright_{\hat\alpha}, \Theta) = [\Omega]\Delta, \Psi$.

Proof. By induction on the given derivation.

- **Case** $\longrightarrow$ID: Impossible: $\Delta, \blacktriangleright_{\hat\alpha}, \Theta$ cannot have the form $\cdot$.
- **Case** $\longrightarrow$Var: We have $\Omega_Z = (\Omega'_Z, x : A)$ and $\Theta = (\Theta', x : A')$. By i.h., there is Ψ' such that $[\Omega, \blacktriangleright_{\hat\alpha}, \Omega'_Z](\Delta, \blacktriangleright_{\hat\alpha}, \Theta') = [\Omega]\Delta, \Psi'$. Then by the definition of context application, $[\Omega, \blacktriangleright_{\hat\alpha}, \Omega'_Z, x : A](\Delta, \blacktriangleright_{\hat\alpha}, \Theta', x : A') = [\Omega]\Delta, \Psi', x : [\Omega']A$. Let $\Psi = (\Psi', x : [\Omega']A)$.
- **Case** $\longrightarrow$Uvar: Similar to the $\longrightarrow$Var case, with $\Psi = (\Psi', \alpha)$.
- **Cases** $\longrightarrow$Unsolved, $\longrightarrow$Solve, $\longrightarrow$Marker, $\longrightarrow$Add, $\longrightarrow$AddSolved: Broadly similar to the $\longrightarrow$Uvar case, but since the rightmost context element is soft it disappears in context application, so we let $\Psi = \Psi'$. □

Lemma 48 (Completing Stability).
If $\Gamma \longrightarrow \Omega$ *then* $[\Omega]\Gamma = [\Omega]\Omega$.

Proof. By induction on the derivation of $\Gamma \longrightarrow \Omega$.

- **Case**

$$\frac{}{\cdot \longrightarrow \cdot}\ \longrightarrow\text{ID}$$

  In this case, $\Omega = \Gamma = \cdot$.
  By definition, $[\cdot]\cdot = \cdot$, which gives us the conclusion.

- **Case**

$$\frac{\Gamma' \longrightarrow \Omega' \qquad [\Omega']A_\Gamma = [\Omega']A}{\Gamma', x : A_\Gamma \longrightarrow \Omega', x : A}\ \longrightarrow\text{Var}$$

$[\Omega']\Gamma' = [\Omega']\Omega'$ By i.h.
$[\Omega']A_\Gamma = [\Omega']A$ Premise

$$\begin{aligned}
[\Omega]\Gamma &= [\Omega', x : A](\Gamma', x : A_\Gamma) && \text{Expanding } \Omega \text{ and } \Gamma \\
&= [\Omega']\Gamma', x : [\Omega']A_\Gamma && \text{By definition of context application} \\
& && \text{(using } [\Omega']A_\Gamma = [\Omega']A) \\
&= [\Omega']\Omega', x : [\Omega']A && \text{By above equalities} \\
&= [\Omega]\Omega && \text{By definition of context application}
\end{aligned}$$

- **Case**

$$\frac{\Gamma' \longrightarrow \Omega'}{\Gamma', \alpha \longrightarrow \Omega', \alpha}\ \longrightarrow\text{Uvar}$$

$$\begin{aligned}
[\Omega]\Gamma &= [\Omega', \alpha](\Gamma', \alpha) && \text{Expanding } \Omega \text{ and } \Gamma \\
&= [\Omega']\Gamma', \alpha && \text{By definition of context application} \\
&= [\Omega']\Omega', \alpha && \text{By i.h.} \\
&= [\Omega', \alpha](\Omega', \alpha) && \text{By definition of context application} \\
&= [\Omega]\Omega && \text{By } \Omega = (\Omega', \alpha)
\end{aligned}$$

- **Case**

$$\frac{\Gamma' \longrightarrow \Omega'}{\Gamma', \blacktriangleright_{\hat\alpha} \longrightarrow \Omega', \blacktriangleright_{\hat\alpha}}\ \longrightarrow\text{Marker}$$

  Similar to the $\longrightarrow$Uvar case.

- **Case** $\dfrac{\Gamma \longrightarrow \Omega'}{\Gamma \longrightarrow \Omega', \hat{\alpha} = \tau} \longrightarrow\mathsf{AddSolved}$

$$\begin{aligned}
[\Omega]\Gamma &= [\Omega', \hat{\alpha} = \tau]\Gamma && \text{Expanding } \Omega \\
&= [\Omega']\Gamma && \text{By } \hat{\alpha} \notin \mathsf{dom}(\Gamma) \\
&= [\Omega']\Omega' && \text{By i.h.} \\
&= [\Omega', \hat{\alpha} = \tau](\Omega', \hat{\alpha} = \tau) && \text{By definition of context application} \\
&= [\Omega]\Omega && \text{By } \Omega = (\Omega', \hat{\alpha} = \tau)
\end{aligned}$$

- **Case** $\dfrac{\Gamma' \longrightarrow \Omega' \qquad [\Omega']\tau_\Gamma = [\Omega']\tau}{\Gamma', \hat{\alpha} = \tau_\Gamma \longrightarrow \Omega', \hat{\alpha} = \tau} \longrightarrow\mathsf{Solved}$

$$\begin{aligned}
[\Omega]\Gamma &= [\Omega', \hat{\alpha} = \tau](\Gamma', \hat{\alpha} = \tau_\Gamma) && \text{Expanding } \Omega \text{ and } \Gamma \\
&= [\Omega']\Gamma' && \text{By definition of context application} \\
&= [\Omega']\Omega' && \text{By i.h.} \\
&= [\Omega', \hat{\alpha} = \tau](\Omega', \hat{\alpha} = \tau) && \text{By definition of context application} \\
&= [\Omega]\Omega && \text{By } \Omega = (\Omega', \hat{\alpha} = \tau)
\end{aligned}$$

- **Case** $\dfrac{\Gamma' \longrightarrow \Omega'}{\Gamma', \hat{\alpha} \longrightarrow \Omega', \hat{\alpha} = \tau} \longrightarrow\mathsf{Solve}$

$$\begin{aligned}
[\Omega]\Gamma &= [\Omega', \hat{\alpha} = \tau](\Gamma', \hat{\alpha}) && \text{Expanding } \Omega \text{ and } \Gamma \\
&= [\Omega']\Gamma' && \text{By definition of context application} \\
&= [\Omega']\Omega' && \text{By i.h.} \\
&= [\Omega', \hat{\alpha} = \tau](\Omega', \hat{\alpha} = \tau) && \text{By definition of context application} \\
&= [\Omega]\Omega && \text{By } \Omega = (\Omega', \hat{\alpha} = \tau)
\end{aligned}$$

- **Case** $\dfrac{\Gamma \longrightarrow \Delta}{\Gamma, \hat{\alpha} \longrightarrow \Delta, \hat{\alpha}} \longrightarrow\mathsf{Unsolved}$

  Impossible: Ω cannot have the form $\Delta, \hat{\alpha}$.

- **Case** $\dfrac{\Gamma \longrightarrow \Delta}{\Gamma \longrightarrow \Delta, \hat{\alpha}} \longrightarrow\mathsf{Add}$

  Impossible: Ω cannot have the form $\Delta, \hat{\alpha}$. ☐

Lemma 49 (Finishing Types).
If $\Omega \vdash A$ *and* $\Omega \longrightarrow \Omega'$ *then* $[\Omega]A = [\Omega']A$.

Proof. By Lemma 17 (Substitution Extension Invariance), $[\Omega']A = [\Omega'][\Omega]A$.
If $\mathsf{FEV}(C) = \emptyset$ then $[\Omega']C = C$.
Since Ω is complete and $\Omega \vdash A$, we have $\mathsf{FEV}([\Omega]A) = \emptyset$. Therefore $[\Omega'][\Omega]A = [\Omega]A$. ☐

Lemma 50 (Finishing Completions).
If $\Omega \longrightarrow \Omega'$ *then* $[\Omega]\Omega = [\Omega']\Omega'$.

Proof. By induction on the given derivation of $\Omega \longrightarrow \Omega'$.

Only cases $\longrightarrow$ID, $\longrightarrow$Var, $\longrightarrow$Uvar, $\longrightarrow$Solved, $\longrightarrow$Marker and $\longrightarrow$AddSolved are possible. In all of these cases, we use the i.h. and the definition of context application; in cases $\longrightarrow$Var and $\longrightarrow$Solved, we also use the equality in the premise of the respective rule. ☐

Lemma 51 (Confluence of Completeness).
If $\Delta_1 \longrightarrow \Omega$ *and* $\Delta_2 \longrightarrow \Omega$ *then* $[\Omega]\Delta_1 = [\Omega]\Delta_2$.

Proof.

$\Delta_1 \longrightarrow \Omega$	Given
$[\Omega]\Delta_1 = [\Omega]\Omega$	By Lemma 48 (Completing Stability)
$\Delta_2 \longrightarrow \Omega$	Given
$[\Omega]\Delta_2 = [\Omega]\Omega$	By Lemma 48 (Completing Stability)
$[\Omega]\Delta_1 = [\Omega]\Delta_2$	By transitivity of equality

□

H′.2 Instantiation Soundness

Theorem 10 (Instantiation Soundness).
Given $\Delta \longrightarrow \Omega$ *and* $[\Gamma]B = B$ *and* $\hat{\alpha} \notin FV(B)$*:*

(1) If $\Gamma \vdash \hat{\alpha} :\leqq B \dashv \Delta$ *then* $[\Omega]\Delta \vdash [\Omega]\hat{\alpha} \leq [\Omega]B$.

(2) If $\Gamma \vdash B \leqq: \hat{\alpha} \dashv \Delta$ *then* $[\Omega]\Delta \vdash [\Omega]B \leq [\Omega]\hat{\alpha}$.

Proof. By induction on the given instantiation derivation.

(1)

- **Case**

$$\frac{\Gamma_0 \vdash \tau}{\underbrace{\Gamma_0, \hat{\alpha}, \Gamma_1}_{\Gamma} \vdash \hat{\alpha} :\leqq \tau \dashv \underbrace{\Gamma_0, \hat{\alpha} = \tau, \Gamma_1}_{\Delta}}\ \text{InstLSolve}$$

In this case $[\Delta]\hat{\alpha} = [\Delta]\tau$. By reflexivity of subtyping (Lemma 3 (Reflexivity of Declarative Subtyping)), $[\Omega]\Delta \vdash [\Delta]\hat{\alpha} \leq [\Delta]\tau$.

- **Case**

$$\frac{}{\Gamma[\hat{\alpha}][\hat{\beta}] \vdash \hat{\alpha} :\leqq \hat{\beta} \dashv \underbrace{\Gamma[\hat{\alpha}][\hat{\beta} = \hat{\alpha}]}_{\Delta}}\ \text{InstLReach}$$

We have $\Delta = \Gamma[\hat{\alpha}][\hat{\beta} = \hat{\alpha}]$. Therefore $[\Delta]\hat{\alpha} = \hat{\alpha} = [\Delta]\hat{\beta}$.
By reflexivity of subtyping (Lemma 3 (Reflexivity of Declarative Subtyping)), $[\Omega]\Delta \vdash [\Delta]\hat{\alpha} \leq [\Delta]\hat{\beta}$.

- **Case**

$$\frac{\overbrace{\Gamma[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2]}^{\Gamma_1} \vdash A_1 \leqq: \hat{\alpha}_1 \dashv \Gamma' \qquad \Gamma' \vdash \hat{\alpha}_2 :\leqq [\Gamma']A_2 \dashv \Delta}{\Gamma[\hat{\alpha}] \vdash \hat{\alpha} :\leqq A_1 \to A_2 \dashv \Delta}\ \text{InstLArr}$$

$[\Gamma](A_1 \to A_2) = [\Gamma_1](A_1 \to A_2)$	$\hat{\alpha} \notin FV(A_1 \to A_2)$
$\hat{\alpha}_1, \hat{\alpha}_2 \notin FV(A_1) \cup FV(A_2)$	$\hat{\alpha}_1, \hat{\alpha}_2$ fresh
$\Gamma' \vdash \hat{\alpha}_2 :\leqq [\Gamma']A_2 \dashv \Delta$	Subderivation
$\Gamma' \longrightarrow \Delta$	By Lemma 31 (Instantiation Extension)
$\Delta \longrightarrow \Omega$	Given
$\Gamma' \longrightarrow \Omega$	By Lemma 20 (Transitivity)
$\Gamma_1 \vdash A_1 \leqq: \hat{\alpha}_1 \dashv \Gamma'$	Subderivation
$[\Omega]\Delta \vdash [\Omega]A_1 \leq [\Omega]\hat{\alpha}_1$	By i.h. and Lemma 51 (Confluence of Completeness)
$\Gamma' \vdash \hat{\alpha}_2 :\leqq [\Gamma']A_2 \dashv \Delta$	Subderivation
$[\Omega]\Delta \vdash [\Omega][\Gamma']\hat{\alpha}_2 \leq [\Omega][\Gamma']A_2$	By i.h.
$\Gamma' \longrightarrow \Omega$	Above
$[\Omega]\Delta \vdash [\Omega]\hat{\alpha}_2 \leq [\Omega]A_2$	By Lemma 17 (Substitution Extension Invariance)
$[\Omega]\Delta \vdash [\Omega](\hat{\alpha}_1 \to \hat{\alpha}_2) \leq [\Omega]A_1 \to [\Omega]A_2$	By $\leq\!\to$ and definition of substitution

Since $(\hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2) \in \Gamma_1$ and $\Gamma_1 \longrightarrow \Delta$, we know that $[\Omega]\hat{\alpha} = [\Omega](\hat{\alpha}_1 \to \hat{\alpha}_2)$.
Therefore $[\Omega]\Delta \vdash [\Omega]\hat{\alpha} \leq [\Omega](A_1 \to A_2)$.

- **Case** $\dfrac{\Gamma[\hat\alpha], \beta \vdash \hat\alpha :\leqq B_0 \dashv \Delta, \beta, \Delta'}{\Gamma[\hat\alpha] \vdash \hat\alpha :\leqq \forall\beta.\, B_0 \dashv \Delta}$ InstLAllR

  We have $\Delta \longrightarrow \Omega$ and $[\Gamma[\hat\alpha]](\forall\beta.\, B_0) = \forall\beta.\, B_0$ and $\hat\alpha \notin FV(\forall\beta.\, B_0)$.
  Hence $\hat\alpha \notin FV(B_0)$ and by definition, $[\Gamma[\hat\alpha], \beta]B_0 = B_0$.
  By Lemma 47 (Filling Completes), $\Delta, \beta, \Delta' \longrightarrow \Omega, \beta, |\Delta'|$.
  By induction, $[\Omega, \beta, |\Delta'|](\Delta, \beta, \Delta') \vdash [\Omega, \beta, |\Delta'|]\hat\alpha \leq [\Omega, \beta, |\Delta'|]B_0$.
  Each free variable in $\hat\alpha$ and B_0 is declared in (Ω, β), so $\Omega, \beta, |\Delta'|$ behaves as $[\Omega, \beta]$ on $\hat\alpha$ and on B_0, yielding $[\Omega, \beta, |\Delta'|](\Delta, \beta, \Delta') \vdash [\Omega, \beta]\hat\alpha \leq [\Omega, \beta]B_0$.
  By Lemma 45 (Context Partitioning) and thinning, $[\Omega, \beta](\Delta, \beta) \vdash [\Omega, \beta]\hat\alpha \leq [\Omega, \beta]B_0$.
  By the definition of context application, $[\Omega]\Delta, \beta \vdash [\Omega, \beta]\hat\alpha \leq [\Omega, \beta]B_0$.
  By the definition of substitution, $[\Omega]\Delta, \beta \vdash [\Omega]\hat\alpha \leq [\Omega]B_0$.
  Since $\hat\alpha$ is declared to the left of β, we have $\beta \notin FV([\Omega]\hat\alpha)$.
  Applying rule $\leq\forall$L gives $[\Omega]\Delta \vdash [\Omega]\hat\alpha \leq \forall\beta.\, [\Omega]B_0$.

(2)
- **Case** $\dfrac{\Gamma_0 \vdash \tau}{\underbrace{\Gamma_0, \hat\alpha, \Gamma_1}_{\Gamma} \vdash \tau \leqq: \hat\alpha \dashv \underbrace{\Gamma_0, \hat\alpha = \tau, \Gamma_1}_{\Gamma'}}$ InstRSolve

  Similar to the InstLSolve case.

- **Case** $\dfrac{}{\Gamma[\hat\alpha][\hat\beta] \vdash \hat\beta \leqq: \hat\alpha \dashv \underbrace{\Gamma[\hat\alpha][\hat\beta = \hat\alpha]}_{\Gamma'}}$ InstRReach

  Similar to the InstLReach case.

- **Case** $\dfrac{\Gamma[\hat\alpha_2, \hat\alpha_1, \hat\alpha = \hat\alpha_1 \to \hat\alpha_2] \vdash \hat\alpha_1 :\leqq A_1 \dashv \Gamma' \qquad \Gamma' \vdash [\Gamma']A_2 \leqq: \hat\alpha_2 \dashv \Delta}{\Gamma[\hat\alpha] \vdash A_1 \to A_2 \leqq: \hat\alpha \dashv \Delta}$ InstRArr

  Similar to the InstLArr case.

- **Case** $\dfrac{\Gamma[\hat\alpha], \blacktriangleright_{\hat\beta}, \hat\beta \vdash [\hat\beta/\beta]B_0 \leqq: \hat\alpha \dashv \Delta, \blacktriangleright_{\hat\beta}, \Delta'}{\Gamma[\hat\alpha] \vdash \forall\beta.\, B_0 \leqq: \hat\alpha \dashv \Delta}$ InstRAllL

  $[\Gamma[\hat\alpha]](\forall\beta.\, B_0) = \forall\beta.\, B_0$ Given
  $[\Gamma[\hat\alpha]]B_0 = B_0$
  $[\Gamma[\hat\alpha], \blacktriangleright_{\hat\beta}, \hat\beta][\hat\beta/\beta]B_0 = [\hat\beta/\beta]B_0$

  $\Delta \longrightarrow \Omega$ Given
  $\Delta, \blacktriangleright_{\hat\beta}, \Delta' \longrightarrow \Omega, \blacktriangleright_{\hat\beta}, |\Delta'|$ By Lemma 47 (Filling Completes)

  $\hat\alpha \notin FV(\forall\beta.\, B_0)$ Given
  $\hat\alpha \notin FV(B_0)$ By definition of $FV(-)$

  $\Gamma[\hat\alpha], \blacktriangleright_{\hat\beta}, \hat\beta \vdash [\hat\beta/\beta]B_0 \leqq: \hat\alpha \dashv \Delta, \blacktriangleright_{\hat\beta}, \Delta'$ Subderivation
  $[\Omega, \blacktriangleright_{\hat\beta}, |\Delta'|](\Delta, \blacktriangleright_{\hat\beta}, \Delta') \vdash [\Omega, \blacktriangleright_{\hat\beta}, |\Delta'|][\hat\beta/\beta]B_0 \leq [\Omega, \blacktriangleright_{\hat\beta}, |\Delta'|]\hat\alpha$ By i.h.
  $\Gamma[\hat\alpha], \blacktriangleright_{\hat\beta}, \hat\beta \longrightarrow \Delta, \blacktriangleright_{\hat\beta}, \Delta'$ By Lemma 31 (Instantiation Extension)

  By Lemma 15 (Declaration Order Preservation), $\hat\alpha$ is declared before $\blacktriangleright_{\hat\beta}$, that is, in Ω.
  Thus, $[\Omega, \blacktriangleright_{\hat\beta}, |\Delta'|]\hat\alpha = [\Omega]\hat\alpha$.
  By Lemma 22 (Evar Input), we know that Δ' is soft, so by Lemma 46 (Softness Goes Away), $[\Omega, \blacktriangleright_{\hat\beta}, |\Delta'|](\Delta, \blacktriangleright_{\hat\beta}, \Delta') = [\Omega, \blacktriangleright_{\hat\beta}](\Delta, \blacktriangleright_{\hat\beta}) = [\Omega]\Delta$.
  Applying these equalities to the derivation above gives

  $$[\Omega]\Delta \vdash [\Omega, \blacktriangleright_{\hat\beta}, |\Delta'|][\hat\beta/\beta]B_0 \leq [\Omega]\hat\alpha$$

  By distributivity of substitution,

  $$[\Omega]\Delta \vdash \big[[\Omega, \blacktriangleright_{\hat\beta}, |\Delta'|]\hat\beta/\beta\big][\Omega, \blacktriangleright_{\hat\beta}, |\Delta'|]B_0 \leq [\Omega]\hat\alpha$$

Furthermore, $[\Omega, \blacktriangleright_{\hat\beta}, |\Delta'|]B_0 = [\Omega]B_0$, since B_0's free variables are either β or in Ω, giving

$$[\Omega]\Delta \vdash \big[[\Omega, \blacktriangleright_{\hat\beta}, |\Delta'|]\hat\beta/\beta\big][\Omega]B_0 \le [\Omega]\hat\alpha$$

Now apply $\le\forall$L and the definition of substitution to get $[\Omega]\Delta \vdash [\Omega](\forall\beta.\, B_0) \le [\Omega]\hat\alpha$. ☐

H′.3 Soundness of Subtyping

Theorem 11 (Soundness of Algorithmic Subtyping)**.**
If $\Gamma \vdash A <: B \dashv \Delta$ *where* $[\Gamma]A = A$ *and* $[\Gamma]B = B$ *and* $\Delta \longrightarrow \Omega$ *then* $[\Omega]\Delta \vdash [\Omega]A \le [\Omega]B$.

Proof. By induction on the derivation of $\Gamma \vdash A <: B \dashv \Delta$.

- **Case** $\dfrac{\alpha \in \Gamma}{\Gamma \vdash \alpha <: \alpha \dashv \underbrace{\Gamma}_{\Delta}}$ <:Var

$\alpha \in \Gamma$	Premise
$\alpha \in \Delta$	$\Gamma = \Delta$
$\alpha \in [\Omega]\Delta$	Follows from definition of context application
$[\Omega]\Delta \vdash \alpha \le \alpha$	By $\le$Var
$[\Omega]\Delta \vdash [\Omega]\alpha \le [\Omega]\alpha$	By def. of substitution

- **Case** <:Unit: Similar to the <:Var case, applying rule $\le$Unit instead of $\le$Var.

- **Case** $\dfrac{}{\Gamma_L, \hat\alpha, \Gamma_R \vdash \hat\alpha <: \hat\alpha \dashv \Gamma_L, \hat\alpha, \Gamma_R}$ <:Exvar

$[\Omega]\hat\alpha$ defined	Follows from definition of context application
$[\Omega]\Delta \vdash [\Omega]\hat\alpha$	Assumption that $[\Omega]\Delta$ is well-formed
$[\Omega]\Delta \vdash [\Omega]\hat\alpha \le [\Omega]\hat\alpha$	By Lemma 3 (Reflexivity of Declarative Subtyping)

- **Case** $\dfrac{\Gamma \vdash B_1 <: A_1 \dashv \Theta \qquad \Theta \vdash [\Theta]A_2 <: [\Theta]B_2 \dashv \Delta}{\Gamma \vdash \underbrace{A_1 \to A_2}_{A} <: \underbrace{B_1 \to B_2}_{B} \dashv \Delta}$ <:→

$\Gamma \vdash B_1 <: A_1 \dashv \Theta$	Subderivation
$\Delta \longrightarrow \Omega$	Given
$\Theta \longrightarrow \Omega$	By Lemma 20 (Transitivity)
$[\Omega]\Theta \vdash [\Omega]B_1 \le [\Omega]A_1$	By i.h.
$[\Omega]\Delta \vdash [\Omega]B_1 \le [\Omega]A_1$	By Lemma 51 (Confluence of Completeness)
$\Theta \vdash [\Theta]A_2 <: [\Theta]B_2 \dashv \Delta$	Subderivation
$[\Omega]\Delta \vdash [\Omega][\Theta]A_2 \le [\Omega][\Theta]B_2$	By i.h.
$[\Omega][\Theta]A_2 = [\Omega]A_2$	By Lemma 17 (Substitution Extension Invariance)
$[\Omega][\Theta]B_2 = [\Omega]B_2$	By Lemma 17 (Substitution Extension Invariance)
$[\Omega]\Delta \vdash [\Omega]A_2 \le [\Omega]B_2$	Above equations
$[\Omega]\Delta \vdash ([\Omega]A_1) \to ([\Omega]A_2) \le ([\Omega]B_1) \to ([\Omega]B_2)$	By $\le\to$
$[\Omega]\Delta \vdash [\Omega](A_1 \to A_2) \le [\Omega](B_1 \to B_2)$	By def. of substitution

- **Case** $\dfrac{\Gamma, \blacktriangleright_{\hat\alpha}, \hat\alpha \vdash [\hat\alpha/\alpha]A_0 <: B \dashv \Delta, \blacktriangleright_{\hat\alpha}, \Theta}{\Gamma \vdash \forall\alpha.\, A_0 <: B \dashv \Delta}$ <:∀L

  Let $\Omega' = (\Omega, |\blacktriangleright_{\hat\alpha}, \Theta|)$.

$$\Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} \vdash [\hat{\alpha}/\alpha]A_0 <: B \dashv \Delta, \blacktriangleright_{\hat{\alpha}}, \Theta \quad \text{Subderivation}$$

$$\Delta \longrightarrow \Omega \quad \text{Given}$$
$$(\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta) \longrightarrow \Omega' \quad \text{By Lemma 47 (Filling Completes)}$$

$$[\Omega'](\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta) \vdash [\Omega'][\hat{\alpha}/\alpha]A_0 \leq [\Omega']B \quad \text{By i.h.}$$
$$[\Omega'](\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta) \vdash [\Omega'][\hat{\alpha}/\alpha]A_0 \leq [\Omega]B \quad \text{By } [\Omega']B = [\Omega]B \text{ (Lemma 44 (Substitution Stability))}$$
$$[\Omega'](\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta) \vdash \big[[\Omega']\hat{\alpha}/\alpha\big][\Omega']A_0 \leq [\Omega]B \quad \text{By distributivity of substitution}$$

$$\Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} \vdash \hat{\alpha} \quad \text{By EvarWF}$$
$$\Gamma, \blacktriangleright_{\hat{\alpha}}, \hat{\alpha} \longrightarrow \Delta, \blacktriangleright_{\hat{\alpha}}, \Theta \quad \text{By Lemma 32 (Subtyping Extension)}$$
$$\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta \vdash \hat{\alpha} \quad \text{By Lemma 24 (Extension Weakening)}$$
$$(\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta) \longrightarrow \Omega' \quad \text{Above}$$
$$[\Omega']\Omega' \vdash [\Omega']\hat{\alpha} \quad \text{By Lemma 43 (Substitution for Well-Formedness)}$$
$$[\Omega'](\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta) \vdash [\Omega']\hat{\alpha} \quad \text{By Lemma 48 (Completing Stability)}$$

$$[\Omega'](\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta) \vdash \forall\alpha.\, [\Omega']A_0 \leq [\Omega]B \quad \text{By } {\leq}\forall\mathsf{L}$$
$$[\Omega'](\Delta, \blacktriangleright_{\hat{\alpha}}, \Theta) \vdash \forall\alpha.\, [\Omega, \alpha]A_0 \leq [\Omega]B \quad \text{By Lemma 44 (Substitution Stability)}$$
$$[\Omega]\Delta \vdash \forall\alpha.\, [\Omega, \alpha]A_0 \leq [\Omega]B \quad \text{By Lemma 45 (Context Partitioning) and thinning}$$
$$[\Omega]\Delta \vdash \forall\alpha.\, [\Omega]A_0 \leq [\Omega]B \quad \text{By def. of substitution}$$
$$[\Omega]\Delta \vdash [\Omega](\forall\alpha.\, A_0) \leq [\Omega]B \quad \text{By def. of substitution}$$

- **Case** $\dfrac{\Gamma, \alpha \vdash A <: B_0 \dashv \Delta, \alpha, \Theta}{\Gamma \vdash A <: \forall\alpha.\, B_0 \dashv \Delta} <:\forall\mathsf{R}$

$$\Gamma, \alpha \vdash A <: B_0 \dashv \Delta, \alpha, \Theta \quad \text{Subderivation}$$
Let $\Omega_Z = |\Theta|$.
Let $\Omega' = (\Omega, \alpha, \Omega_Z)$.
$$(\Delta, \alpha, \Theta) \longrightarrow \Omega' \quad \text{By Lemma 47 (Filling Completes)}$$
$$[\Omega'](\Delta, \alpha, \Theta) \vdash [\Omega']A \leq [\Omega']B_0 \quad \text{By i.h.}$$
$$[\Omega, \alpha](\Delta, \alpha) \vdash [\Omega, \alpha]A \leq [\Omega, \alpha]B_0 \quad \text{By Lemma 44 (Substitution Stability)}$$
$$[\Omega, \alpha](\Delta, \alpha) \vdash [\Omega]A \leq [\Omega]B_0 \quad \text{By def. of substitution}$$
$$[\Omega]\Delta \vdash [\Omega]A \leq \forall\alpha.\, [\Omega]B_0 \quad \text{By } {\leq}\forall\mathsf{R}$$
$$[\Omega]\Delta \vdash [\Omega]A \leq [\Omega](\forall\alpha.\, B_0) \quad \text{By def. of substitution}$$

- **Case** $\dfrac{\hat{\alpha} \notin FV(B) \qquad \Gamma \vdash \hat{\alpha} :\leqq B \dashv \Delta}{\underbrace{\Gamma}_{\Gamma_0[\hat{\alpha}]} \vdash \hat{\alpha} <: B \dashv \Delta} <:\mathsf{InstantiateL}$

$$\Gamma \vdash \hat{\alpha} :\leqq B \dashv \Delta \quad \text{Subderivation}$$
$$[\Omega]\Delta \vdash [\Omega]\hat{\alpha} \leq [\Omega]B \quad \text{By Theorem 10}$$

- **Case** <:InstantiateR: Similar to the case for <:InstantiateL. □

Corollary 52 (Soundness, Pretty Version). *If* $\Psi \vdash A <: B \dashv \Delta$, *then* $\Psi \vdash A \leq B$.

Proof. By reflexivity (Lemma 19 (Reflexivity)), $\Psi \longrightarrow \Psi$.
Since Ψ has no existential variables, it is a complete context Ω.
By Lemma 11 (Soundness of Algorithmic Subtyping), $[\Psi]\Psi \vdash [\Psi]A \leq [\Psi]B$.
Since Ψ has no existential variables, $[\Psi]\Psi = \Psi$, and $[\Psi]A = A$, and $[\Psi]B = B$.
Therefore $\Psi \vdash A \leq B$. □

I′ Typing Extension

Lemma 53 (Typing Extension).
If $\Gamma \vdash e \Leftarrow A \dashv \Delta$ *or* $\Gamma \vdash e \Rightarrow A \dashv \Delta$ *or* $\Gamma \vdash A \bullet e \Rrightarrow C \dashv \Delta$ *then* $\Gamma \longrightarrow \Delta$.

Proof. By induction on the given derivation.

- **Cases** Var, 1I, 1I⇒:

  Since $\Delta = \Gamma$, the result follows by Lemma 19 (Reflexivity).

- **Case** $$\frac{\Gamma \vdash e \Rightarrow B \dashv \Theta \qquad \Theta \vdash [\Theta]B <: [\Theta]A \dashv \Delta}{\Gamma \vdash e \Leftarrow A \dashv \Delta}\ \mathsf{Sub}$$

  $\Gamma \longrightarrow \Theta$ By i.h.

  $\Theta \longrightarrow \Delta$ By Lemma 32 (Subtyping Extension)

  ☞ $\Gamma \longrightarrow \Delta$ By Lemma 20 (Transitivity)

- **Case** $$\frac{\Gamma \vdash e \Leftarrow A \dashv \Delta}{\Gamma \vdash (e : A) \Rightarrow A \dashv \Delta}\ \mathsf{Anno}$$

  ☞ $\Gamma \longrightarrow \Delta$ By i.h.

- **Case** $$\frac{\Gamma, \alpha \vdash e \Leftarrow A_0 \dashv \Delta, \alpha, \Theta}{\Gamma \vdash e \Leftarrow \forall \alpha.\, A_0 \dashv \Delta}\ \forall\mathsf{I}$$

  $\Gamma, \alpha \longrightarrow \Delta, \alpha, \Theta$ By i.h.

  ☞ $\Gamma \longrightarrow \Delta$ By Lemma 23 (Extension Order) (i)

- **Case** $$\frac{\Gamma, \hat{\alpha} \vdash [\hat{\alpha}/\alpha]A_0 \bullet e \Rrightarrow C \dashv \Delta}{\Gamma \vdash \forall \alpha.\, A_0 \bullet e \Rrightarrow C \dashv \Delta}\ \forall\mathsf{App}$$

  $\Gamma, \hat{\alpha} \longrightarrow \Delta$ By i.h.

  $\Gamma \longrightarrow \Gamma, \hat{\alpha}$ By $\longrightarrow$Add

  ☞ $\Gamma \longrightarrow \Delta$ By Lemma 20 (Transitivity)

- **Case** $$\frac{\Gamma, x : A_1 \vdash e \Leftarrow A_2 \dashv \Delta, x : A_1, \Theta}{\Gamma \vdash \lambda x.\, e \Leftarrow A_1 \rightarrow A_2 \dashv \Delta}\ {\rightarrow}\mathsf{I}$$

  $\Gamma, x : A_1 \longrightarrow \Delta, x : A_1, \Theta$ By i.h.

  ☞ $\Gamma \longrightarrow \Delta$ By Lemma 23 (Extension Order) (v)

- **Case** $$\frac{\Gamma \vdash e_1 \Rightarrow B \dashv \Theta \qquad \Theta \vdash [\Theta]B \bullet e_2 \Rrightarrow A \dashv \Delta}{\Gamma \vdash e_1\, e_2 \Rightarrow A \dashv \Delta}\ {\rightarrow}\mathsf{E}$$

  By the i.h. on each premise, then Lemma 20 (Transitivity).

- **Case** $$\frac{\Gamma, \hat{\alpha}, \hat{\beta}, x : \hat{\alpha} \vdash e \Leftarrow \hat{\beta} \dashv \Delta, x : \hat{\alpha}, \Theta}{\Gamma \vdash \lambda x.\, e \Rightarrow \hat{\alpha} \rightarrow \hat{\beta} \dashv \Delta}\ {\rightarrow}\mathsf{I}{\Rightarrow}$$

  $\Gamma, \hat{\alpha}, \hat{\beta}, x : \hat{\alpha} \longrightarrow \Delta, x : \hat{\alpha}, \Theta$ By i.h.

  $\Gamma, \hat{\alpha}, \hat{\beta} \longrightarrow \Delta$ By Lemma 23 (Extension Order) (v)

  $\Gamma \longrightarrow \Gamma, \hat{\alpha}, \hat{\beta}$ By $\longrightarrow$Add (twice)

  ☞ $\Gamma \longrightarrow \Delta$ By Lemma 20 (Transitivity)

- **Case** $\dfrac{\Gamma \vdash e \Leftarrow A \dashv \Delta}{\Gamma \vdash A \to C \bullet e \Rrightarrow C \dashv \Delta} \to\!\mathsf{App}$

☞ $\Gamma \longrightarrow \Delta$ By i.h.

- **Case** $\dfrac{\Gamma[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2] \vdash e \Leftarrow \hat{\alpha}_1 \dashv \Delta}{\Gamma[\hat{\alpha}] \vdash \hat{\alpha} \bullet e \Rrightarrow \hat{\alpha}_2 \dashv \Delta} \hat{\alpha}\mathsf{App}$

$\Gamma[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2] \longrightarrow \Delta$ By i.h.

$\Gamma[\hat{\alpha}] \longrightarrow \Gamma[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2]$ By Lemma 26 (Solved Variable Addition for Extension) then Lemma 28 (Parallel Admissibility) (ii)

☞ $\Gamma \longrightarrow \Delta$ By Lemma 20 (Transitivity) □

J′ Soundness of Typing

Theorem 12 (Soundness of Algorithmic Typing). *Given* $\Delta \longrightarrow \Omega$*:*

(i) If $\Gamma \vdash e \Leftarrow A \dashv \Delta$ *then* $[\Omega]\Delta \vdash e \Leftarrow [\Omega]A$.

(ii) If $\Gamma \vdash e \Rightarrow A \dashv \Delta$ *then* $[\Omega]\Delta \vdash e \Rightarrow [\Omega]A$.

(iii) If $\Gamma \vdash A \bullet e \Rrightarrow C \dashv \Delta$ *then* $[\Omega]\Delta \vdash [\Omega]A \bullet e \Rrightarrow [\Omega]C$.

Proof. By induction on the given algorithmic typing derivation.

- **Case** $\dfrac{(x : A) \in \Gamma}{\Gamma \vdash x \Rightarrow A \dashv \Gamma} \mathsf{Var}$

	$(x : A) \in \Gamma$	Premise
	$(x : A) \in \Delta$	By $\Gamma = \Delta$
	$\Delta \longrightarrow \Omega$	Given
	$(x : [\Omega]A) \in [\Omega]\Gamma$	By Lemma 41 (Variable Preservation)
☞	$[\Omega]\Gamma \vdash x \Rightarrow [\Omega]A$	By DeclVar

- **Case** $\dfrac{\Gamma \vdash e \Rightarrow A \dashv \Theta \qquad \Theta \vdash [\Theta]A <: [\Theta]B \dashv \Delta}{\Gamma \vdash e \Leftarrow B \dashv \Delta} \mathsf{Sub}$

	$\Gamma \vdash e \Rightarrow A \dashv \Theta$	Subderivation
	$\Theta \vdash [\Theta]A <: [\Theta]B \dashv \Delta$	Subderivation
	$\Theta \longrightarrow \Delta$	By Lemma 53 (Typing Extension)
	$\Delta \longrightarrow \Omega$	Given
	$\Theta \longrightarrow \Omega$	By Lemma 20 (Transitivity)
	$[\Omega]\Theta \vdash e \Rightarrow [\Omega]A$	By i.h.
	$[\Omega]\Theta = [\Omega]\Delta$	By Lemma 51 (Confluence of Completeness)
	$[\Omega]\Delta \vdash e \Rightarrow [\Omega]A$	By above equalities
	$\Theta \vdash [\Theta]A <: [\Theta]B \dashv \Delta$	Subderivation
	$[\Omega]\Delta \vdash [\Omega][\Theta]A \leq [\Omega][\Theta]B$	By Lemma 11 (Soundness of Algorithmic Subtyping)
	$[\Omega][\Theta]A = [\Omega]A$	By Lemma 17 (Substitution Extension Invariance)
	$[\Omega][\Theta]B = [\Omega]B$	By Lemma 17 (Substitution Extension Invariance)
	$[\Omega]\Delta \vdash [\Omega]A \leq [\Omega]B$	By above equalities
☞	$[\Omega]\Delta \vdash e \Leftarrow [\Omega]B$	By DeclSub

- **Case** $$\frac{\Gamma \vdash e_0 \Leftarrow A \dashv \Delta}{\Gamma \vdash (e_0 : A) \Rightarrow A \dashv \Delta}\ \mathsf{Anno}$$

$\Gamma \vdash e_0 \Leftarrow A \dashv \Delta$	Subderivation
$[\Omega]\Delta \vdash e_0 \Leftarrow [\Omega]A$	By i.h.
$[\Omega]\Delta \vdash (e_0 : [\Omega]A) \Rightarrow [\Omega]A$	By DeclAnno
A contains no existential variables	Assumption about source programs
$[\Omega]A = A$	From definition of substitution
☞ $[\Omega]\Delta \vdash (e_0 : A) \Rightarrow [\Omega]A$	By above equality

- **Case** $$\frac{}{\Gamma \vdash () \Leftarrow 1 \dashv \underbrace{\Gamma}_{\Delta}}\ 1\mathsf{I}$$

$[\Omega]\Delta \vdash () \Leftarrow 1$	By Decl1I
☞ $[\Omega]\Delta \vdash () \Leftarrow [\Omega]1$	By definition of substitution

- **Case** $$\frac{\Gamma, x : A_1 \vdash e_0 \Leftarrow A_2 \dashv \Delta, x : A_1, \Theta}{\Gamma \vdash \lambda x.\, e \Leftarrow A_1 \to A_2 \dashv \Delta}\ {\to}\mathsf{I}$$

$\Delta \longrightarrow \Omega$	Given
$\Delta, x : A_1 \longrightarrow \Omega, x : [\Omega]A_1$	By $\longrightarrow$Var
$\Gamma, x : A_1 \longrightarrow \Delta, x : A_1, \Theta$	By Lemma 53 (Typing Extension)
Θ is soft	By Lemma 23 (Extension Order) (v) (with $\Gamma_R = \cdot$, which is soft)
$\underbrace{\Delta, x : A_1, \Theta}_{\Delta'} \longrightarrow \underbrace{\Omega, x : [\Omega]A_1, \lvert\Theta\rvert}_{\Omega'}$	By Lemma 47 (Filling Completes)
$\Gamma, x : A_1 \vdash e_0 \Leftarrow A_2 \dashv \Delta'$	Subderivation
$[\Omega']\Delta' \vdash e_0 \Leftarrow [\Omega']A_2$	By i.h.
$[\Omega']A_2 = [\Omega]A_2$	By Lemma 44 (Substitution Stability)
$[\Omega']\Delta' \vdash e_0 \Leftarrow [\Omega]A_2$	By above equality
$\underbrace{\Delta, x : A_1, \Theta}_{\Delta'} \longrightarrow \underbrace{\Omega, x : [\Omega]A_1, \lvert\Theta\rvert}_{\Omega'}$	Above
Θ is soft	Above
$[\Omega']\Delta' = [\Omega]\Delta, x : [\Omega]A_1$	By Lemma 46 (Softness Goes Away)
$[\Omega]\Delta, x : [\Omega]A_1 \vdash e_0 \Leftarrow [\Omega]A_2$	By above equality
$[\Omega]\Delta \vdash \lambda x.\, e_0 \Leftarrow ([\Omega]A_1) \to ([\Omega]A_2)$	By Decl$\to$I
☞ $[\Omega]\Delta \vdash \lambda x.\, e_0 \Leftarrow [\Omega](A_1 \to A_2)$	By definition of substitution

- **Case** $$\frac{\Gamma \vdash e_1 \Rightarrow A_1 \dashv \Theta \qquad \Theta \vdash A_1 \bullet e_2 \Rrightarrow A_2 \dashv \Delta}{\Gamma \vdash e_1\, e_2 \Rightarrow A_2 \dashv \Delta}\ {\to}\mathsf{E}$$

$\Gamma \vdash e_1 \Rightarrow A_1 \dashv \Theta$	Subderivation
$\Theta \vdash A_1 <: B \dashv \Delta$	Subderivation
$\Theta \longrightarrow \Delta$	By Lemma 53 (Typing Extension)
$\Delta \longrightarrow \Omega$	Given
$\Theta \longrightarrow \Omega$	By Lemma 20 (Transitivity)
$[\Omega]\Theta \vdash e_1 \Rightarrow [\Omega]A_1$	By i.h.
$[\Omega]\Theta = [\Omega]\Delta$	By Lemma 51 (Confluence of Completeness)
$[\Omega]\Delta \vdash e_1 \Rightarrow [\Omega]A_1$	By above equality
$\Theta \vdash A_1 \bullet e_2 \Rrightarrow A_2 \dashv \Delta$	Subderivation
$\Delta \longrightarrow \Omega$	Given
$[\Omega]\Delta \vdash [\Omega]A_1 \bullet e_2 \Rrightarrow [\Omega]A_2$	By i.h.
☞ $[\Omega]\Delta \vdash e_1 e_2 \Rightarrow [\Omega]A_2$	By Decl$\to$E

- **Case** $\dfrac{\Gamma, \alpha \vdash e \Leftarrow A_0 \dashv \Delta, \alpha, \Theta}{\Gamma \vdash e \Leftarrow \forall\alpha.\, A_0 \dashv \Delta}\ \forall\mathsf{I}$

(Similar to $\to$I, using a different subpart of Lemma 23 (Extension Order) and applying Decl$\forall$I; written out anyway.)

$\Delta \longrightarrow \Omega$	Given
$\Delta, \alpha \longrightarrow \Omega, \alpha$	By $\longrightarrow$Uvar
$\Gamma, \alpha \longrightarrow \Delta, \alpha, \Theta$	By Lemma 53 (Typing Extension)
Θ is soft	By Lemma 23 (Extension Order) (i) (with $\Gamma_R = \cdot$, which is soft)
$\underbrace{\Delta, \alpha, \Theta}_{\Delta'} \longrightarrow \underbrace{\Omega, \alpha, \lvert\Theta\rvert}_{\Omega'}$	By Lemma 47 (Filling Completes)
$\Gamma, \alpha \vdash e \Leftarrow A_0 \dashv \Delta'$	Subderivation
$[\Omega']\Delta' \vdash e \Leftarrow [\Omega']A_0$	By i.h.
$[\Omega']A_0 = [\Omega]A_0$	By Lemma 44 (Substitution Stability)
$[\Omega']\Delta' \vdash e \Leftarrow [\Omega]A_0$	By above equality
$\underbrace{\Delta, \alpha, \Theta}_{\Delta'} \longrightarrow \underbrace{\Omega, \alpha, \lvert\Theta\rvert}_{\Omega'}$	Above
Θ is soft	Above
$[\Omega']\Delta' = [\Omega]\Delta, \alpha$	By Lemma 46 (Softness Goes Away)
$[\Omega]\Delta, \alpha \vdash e \Leftarrow [\Omega]A_0$	By above equality
$[\Omega]\Delta \vdash e \Leftarrow \forall\alpha.\, [\Omega]A_0$	By Decl$\forall$I
☞ $[\Omega]\Delta \vdash e \Leftarrow [\Omega](\forall\alpha.\, A_0)$	By definition of substitution

- **Case** $\dfrac{\Gamma, \hat\alpha \vdash [\hat\alpha/\alpha]A_0 \bullet e \Rrightarrow C \dashv \Delta}{\Gamma \vdash \forall\alpha.\, A_0 \bullet e \Rrightarrow C \dashv \Delta}\ \forall\mathsf{App}$

$$
\begin{array}{ll}
\Gamma, \hat{\alpha} \vdash [\hat{\alpha}/\alpha]A_0 \bullet e \Rightarrow\!\!\!\Rightarrow C \dashv \Delta & \text{Subderivation} \\
\Delta \longrightarrow \Omega & \text{Given} \\
[\Omega]\Delta \vdash [\Omega][\hat{\alpha}/\alpha]A_0 \bullet e \Rightarrow\!\!\!\Rightarrow [\Omega]C & \text{By i.h.} \\
[\Omega]\Delta \vdash \big[[\Omega]\hat{\alpha} \,/\, \alpha\big]\, [\Omega]A_0 \bullet e \Rightarrow\!\!\!\Rightarrow [\Omega]C & \text{By distributivity of substitution}
\end{array}
$$

$$
\begin{array}{ll}
\Gamma, \hat{\alpha} \longrightarrow \Delta & \text{By Lemma 53 (Typing Extension)} \\
\Gamma, \hat{\alpha} \longrightarrow \Omega & \text{By Lemma 20 (Transitivity)} \\
\Gamma, \hat{\alpha} \vdash \hat{\alpha} & \text{By EvarWF} \\
\Omega \vdash \hat{\alpha} & \text{By Lemma 24 (Extension Weakening)} \\
[\Omega]\Omega \vdash [\Omega]\hat{\alpha} & \text{By Lemma 43 (Substitution for Well-Formedness)} \\
[\Omega]\Omega = [\Omega]\Delta & \text{By Lemma 48 (Completing Stability)} \\
[\Omega]\Delta \vdash [\Omega]\hat{\alpha} & \text{By above equality}
\end{array}
$$

$$
\begin{array}{lll}
& [\Omega]\Delta \vdash \forall\alpha.\, [\Omega]A_0 \bullet e \Rightarrow\!\!\!\Rightarrow [\Omega]C & \text{By Decl}\forall\text{App} \\
☞ & [\Omega]\Delta \vdash [\Omega](\forall\alpha.\, A_0) \bullet e \Rightarrow\!\!\!\Rightarrow [\Omega]C & \text{By definition of substitution}
\end{array}
$$

- **Case** $\dfrac{\Gamma \vdash e \Leftarrow B \dashv \Delta}{\Gamma \vdash B \to C \bullet e \Rightarrow\!\!\!\Rightarrow C \dashv \Delta}\;{\to}\mathsf{App}$

$$
\begin{array}{lll}
& \Gamma \vdash e \Leftarrow B \dashv \Delta & \text{Subderivation} \\
& \Delta \longrightarrow \Omega & \text{Given} \\
& [\Omega]\Delta \vdash e \Leftarrow [\Omega]B & \text{By i.h.} \\
& [\Omega]\Delta \vdash ([\Omega]B) \to ([\Omega]C) \bullet e \Rightarrow\!\!\!\Rightarrow [\Omega]C & \text{By Decl}{\to}\text{App} \\
☞ & [\Omega]\Delta \vdash [\Omega](B \to C) \bullet e \Rightarrow\!\!\!\Rightarrow [\Omega]C & \text{By definition of substitution}
\end{array}
$$

- **Case** $\dfrac{\Gamma_0[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2] \vdash e \Leftarrow \hat{\alpha}_1 \dashv \Delta}{\underbrace{\Gamma_0[\hat{\alpha}]}_{\Gamma} \vdash \hat{\alpha} \bullet e \Rightarrow\!\!\!\Rightarrow \hat{\alpha}_2 \dashv \Delta}\;\hat{\alpha}\mathsf{App}$

$$
\begin{array}{ll}
\overbrace{\Gamma_0[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2]}^{\Gamma'} \vdash e \Leftarrow \hat{\alpha}_1 \dashv \Delta & \text{Subderivation} \\
\Delta \longrightarrow \Omega & \text{Given} \\
[\Omega]\Delta \vdash e \Leftarrow [\Omega]\hat{\alpha}_1 & \text{By i.h.} \\
[\Omega]\Delta \vdash ([\Omega]\hat{\alpha}_1) \to ([\Omega]\hat{\alpha}_2) \bullet e \Rightarrow\!\!\!\Rightarrow [\Omega]\hat{\alpha}_2 & \text{By Decl}{\to}\text{App}
\end{array}
$$

$$
\begin{array}{lll}
& \Gamma' \longrightarrow \Delta & \text{By Lemma 53 (Typing Extension)} \\
& \Delta \longrightarrow \Omega & \text{Given} \\
& \Gamma' \longrightarrow \Omega & \text{By Lemma 20 (Transitivity)} \\
& [\Gamma']\hat{\alpha} = [\Gamma'](\hat{\alpha}_1 \to \hat{\alpha}_2) & \text{By definition of } [\Gamma'](-) \\
& [\Omega][\Gamma']\hat{\alpha} = [\Omega][\Gamma'](\hat{\alpha}_1 \to \hat{\alpha}_2) & \text{Applying } \Omega \text{ to both sides} \\
& [\Omega]\hat{\alpha} = [\Omega](\hat{\alpha}_1 \to \hat{\alpha}_2) & \text{By Lemma 17 (Substitution Extension Invariance), twice} \\
& \quad = ([\Omega]\hat{\alpha}_1) \to ([\Omega]\hat{\alpha}_2) & \text{By definition of substitution} \\
☞ & [\Omega]\Delta \vdash [\Omega]\hat{\alpha} \bullet e \Rightarrow\!\!\!\Rightarrow [\Omega]\hat{\alpha}_2 & \text{By above equality}
\end{array}
$$

- **Case** $\dfrac{}{\Gamma \vdash () \Rightarrow 1 \dashv \underbrace{\Gamma}_{\Delta}}\;1\mathsf{I}{\Rightarrow}$

☞ $[\Omega]\Delta \vdash () \Rightarrow [\Omega]1$ By Decl1I⇒ and definition of substitution

- **Case** $\dfrac{\Gamma, \hat{\alpha}, \hat{\beta}, x : \hat{\alpha} \vdash e_0 \Leftarrow \hat{\beta} \dashv \Delta, x : \hat{\alpha}, \Theta}{\Gamma \vdash \lambda x.\, e_0 \Rightarrow \hat{\alpha} \to \hat{\beta} \dashv \Delta}\;{\to}\mathsf{I}{\Rightarrow}$

$\Gamma, \hat\alpha, \hat\beta, x : \hat\alpha \longrightarrow \Delta, x : \hat\alpha, \Theta$	By Lemma 53 (Typing Extension)
Θ is soft	By Lemma 23 (Extension Order) (v) (with $\Gamma_R = \cdot$, which is soft)
$\Gamma, \hat\alpha, \hat\beta \longrightarrow \Delta$	"

$\Delta \longrightarrow \Omega$	Given
$\Delta, x : \hat\alpha \longrightarrow \Omega, x : [\Omega]\hat\alpha$	By $\longrightarrow$Var
$\underbrace{\Delta, x : \hat\alpha, \Theta}_{\Delta'} \longrightarrow \underbrace{\Omega, x : [\Omega]\hat\alpha, \lvert\Theta\rvert}_{\Omega'}$	By Lemma 47 (Filling Completes)

$\Gamma, \hat\alpha, \hat\beta, x : \hat\alpha \vdash e \Leftarrow \hat\beta \dashv \Delta, x : \hat\alpha, \Theta$	Subderivation

$[\Omega']\Delta' \vdash e_0 \Leftarrow [\Omega']\hat\beta$	By i.h.
$[\Omega']\hat\beta = \big[\Omega, x : [\Omega]\hat\alpha\big]\hat\beta$	By Lemma 44 (Substitution Stability)
$= [\Omega]\hat\beta$	By definition of substitution
$[\Omega']\Delta' = \big[\Omega, x : [\Omega]\hat\alpha\big]\big(\Delta, x : \hat\alpha\big)$	By Lemma 46 (Softness Goes Away)
$= [\Omega]\Delta, x : [\Omega]\hat\alpha$	By definition of context substitution
$[\Omega]\Delta, x : [\Omega]\hat\alpha \vdash e_0 \Leftarrow [\Omega]\hat\beta$	By above equalities

$\Gamma, \hat\alpha, \hat\beta \longrightarrow \Delta$	Above
$\Gamma, \hat\alpha, \hat\beta \longrightarrow \Omega$	By Lemma 20 (Transitivity)
$\Gamma, \hat\alpha, \hat\beta \vdash \hat\alpha$	By EvarWF
$\Omega \vdash \hat\alpha$	By Lemma 24 (Extension Weakening)
$[\Omega]\Delta \vdash [\Omega]\hat\alpha$	By Lemma 43 (Substitution for Well-Formedness) and Lemma 48 (Completing Stability)

$[\Omega]\Delta \vdash [\Omega]\hat\beta$	By similar reasoning
$[\Omega]\Delta \vdash ([\Omega]\hat\alpha) \to ([\Omega]\hat\beta)$	By DeclArrowWF
$[\Omega]\hat\alpha$, $[\Omega]\hat\beta$ monotypes	Ω predicative

$[\Omega]\Delta \vdash \lambda x.\, e_0 \Rightarrow ([\Omega]\hat\alpha) \to ([\Omega]\hat\beta)$	By Decl$\to$I$\Rightarrow$
☞ $[\Omega]\Delta \vdash \lambda x.\, e_0 \Rightarrow [\Omega](\hat\alpha \to \hat\beta)$	By definition of substitution ☐

K′ Completeness

K′.1 Instantiation Completeness

Theorem 13 (Instantiation Completeness).
Given $\Gamma \longrightarrow \Omega$ *and* $A = [\Gamma]A$ *and* $\hat{\alpha} \in \mathsf{unsolved}(\Gamma)$ *and* $\hat{\alpha} \notin FV(A)$*:*

(1) If $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq [\Omega]A$
then there are Δ, Ω' *such that* $\Omega \longrightarrow \Omega'$ *and* $\Delta \longrightarrow \Omega'$ *and* $\Gamma \vdash \hat{\alpha} :\leqq A \dashv \Delta$.

(2) If $[\Omega]\Gamma \vdash [\Omega]A \leq [\Omega]\hat{\alpha}$
then there are Δ, Ω' *such that* $\Omega \longrightarrow \Omega'$ *and* $\Delta \longrightarrow \Omega'$ *and* $\Gamma \vdash A \leqq: \hat{\alpha} \dashv \Delta$.

Proof. By mutual induction on the given declarative subtyping derivation.

(1) We have $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq [\Omega]A$. We now case-analyze the shape of A.

- **Case** $A = \hat{\beta}$:
  It is given that $\hat{\alpha} \notin FV(\hat{\beta})$, so $\hat{\alpha} \neq \hat{\beta}$.
  Since $A = \hat{\beta}$, we have $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq [\Omega]\hat{\beta}$.
  Since Ω is predicative, $[\Omega]\hat{\alpha} = \tau_1$ and $[\Omega]\hat{\beta} = \tau_2$, so we have $[\Omega]\Gamma \vdash \tau_1 \leq \tau_2$.
  By Lemma 9 (Monotype Equality), $\tau_1 = \tau_2$.
  We have $A = \hat{\beta}$ and $[\Gamma]A = A$, so $[\Gamma]\hat{\beta} = \hat{\beta}$. Thus $\hat{\beta} \in \mathsf{unsolved}(\Gamma)$.
  Let Ω' be Ω. By Lemma 19 (Reflexivity), $\Omega \longrightarrow \Omega$.
  Now consider whether $\hat{\alpha}$ is declared to the left of $\hat{\beta}$, or vice versa.
  - **Case** $\Gamma = (\Gamma_0, \hat{\alpha}, \Gamma_1, \hat{\beta}, \Gamma_2)$:
    Let Δ be $\Gamma_0, \hat{\alpha}, \Gamma_1, \hat{\beta} = \hat{\alpha}, \Gamma_2$.
    By rule InstLReach, $\Gamma \vdash \hat{\alpha} :\leqq \hat{\beta} \dashv \Delta$.
    It remains to show that $\Delta \longrightarrow \Omega$.
    We have $[\Omega]\hat{\alpha} = [\Omega]\hat{\beta}$. Then by Lemma 29 (Parallel Extension Solution), $\Delta \longrightarrow \Omega$.
  - **Case** $(\Gamma = \Gamma_0, \hat{\beta}, \Gamma_1, \hat{\alpha}, \Gamma_2)$:
    Let Δ be $\Gamma_0, \hat{\beta}, \Gamma_1, \hat{\alpha} = \hat{\beta}, \Gamma_2$.
    By rule InstLSolve, $\Gamma \vdash \hat{\alpha} :\leqq \hat{\beta} \dashv \Delta$.
    It remains to show that $\Delta \longrightarrow \Omega$.
    We have $[\Omega]\hat{\beta} = [\Omega]\hat{\alpha}$. Then by Lemma 29 (Parallel Extension Solution), $\Delta \longrightarrow \Omega$.
- **Case** $A = \alpha$:
  Since $A = \alpha$, we have $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq [\Omega]\alpha$.
  Since $[\Omega]\alpha = \alpha$, we have $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq \alpha$.
  By inversion, $\leq$Var was used, so $[\Omega]\hat{\alpha} = \alpha$; therefore, since Ω is well-formed, α is declared to the left of $\hat{\alpha}$ in Ω.
  We have $\Gamma \longrightarrow \Omega$.
  By Lemma 16 (Reverse Declaration Order Preservation), we know that α is declared to the left of $\hat{\alpha}$ in Γ; that is, $\Gamma = \Gamma_0[\alpha][\hat{\alpha}]$.
  Let $\Delta = \Gamma_0[\alpha][\hat{\alpha} = \alpha]$ and $\Omega' = \Omega$.
  By InstLSolve, $\Gamma_0[\alpha][\hat{\alpha}] \vdash \hat{\alpha} :\leqq \alpha \dashv \Delta$.
  By Lemma 29 (Parallel Extension Solution), $\Gamma_0[\alpha][\hat{\alpha} = \alpha] \longrightarrow \Omega$.
- **Case** $A = A_1 \rightarrow A_2$:
  By the definition of substitution, $[\Omega]A = ([\Omega]A_1) \rightarrow ([\Omega]A_2)$.
  Therefore $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq ([\Omega]A_1) \rightarrow ([\Omega]A_2)$.
  Since we have an arrow as the supertype, only $\leq\forall$L or $\leq\rightarrow$ could have been used, and the subtype $[\Omega]\hat{\alpha}$ must be either a quantifier or an arrow. But Ω is predicative, so $[\Omega]\hat{\alpha}$ cannot be a quantifier. Therefore, it is an arrow: $[\Omega]\hat{\alpha} = \tau_1 \rightarrow \tau_2$, and $\leq\rightarrow$ concluded the derivation. Inverting $\leq\rightarrow$ gives $[\Omega]\Gamma \vdash [\Omega]A_2 \leq \tau_2$ and $[\Omega]\Gamma \vdash \tau_1 \leq [\Omega]A_1$.

  Since $\hat{\alpha} \in \mathsf{unsolved}(\Gamma)$, we know that Γ has the form $\Gamma_0[\hat{\alpha}]$.
  By Lemma 27 (Unsolved Variable Addition for Extension) twice, inserting unsolved variables

$\hat{\alpha}_2$ and $\hat{\alpha}_1$ into the middle of the context extends it, that is: $\Gamma_0[\hat{\alpha}] \longrightarrow \Gamma_0[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha}]$.
Clearly, $\hat{\alpha}_1 \to \hat{\alpha}_2$ is well-formed in $(\ldots, \hat{\alpha}_2, \hat{\alpha}_1)$, so by Lemma 25 (Solution Admissibility for Extension), solving $\hat{\alpha}$ extends the context: $\Gamma_0[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha}] \longrightarrow \Gamma_0[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2]$. Then by Lemma 20 (Transitivity), $\Gamma_0[\hat{\alpha}] \longrightarrow \Gamma_0[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2]$.

Since $\hat{\alpha} \in \mathsf{unsolved}(\Gamma)$ and $\Gamma \longrightarrow \Omega$, we know that Ω has the form $\Omega_0[\hat{\alpha} = \tau_0]$. To show that we can extend this context, we apply Lemma 26 (Solved Variable Addition for Extension) twice to introduce $\hat{\alpha}_2 = \tau_2$ and $\hat{\alpha}_1 = \tau_1$, and then Lemma 25 (Solution Admissibility for Extension) to overwrite τ_0:

$$\underbrace{\Omega_0[\hat{\alpha} = \tau_0]}_{\Omega} \longrightarrow \Omega_0[\hat{\alpha}_2 = \tau_2, \hat{\alpha}_1 = \tau_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2]$$

We have $\Gamma \longrightarrow \Omega$, that is,

$$\Gamma_0[\hat{\alpha}] \longrightarrow \Omega_0[\hat{\alpha} = \tau_0]$$

By Lemma 28 (Parallel Admissibility) (i) twice, inserting unsolved variables $\hat{\alpha}_2$ and $\hat{\alpha}_1$ on both contexts in the above extension preserves extension:

$$\Gamma_0[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha}] \longrightarrow \Omega_0[\hat{\alpha}_2 = \tau_2, \hat{\alpha}_1 = \tau_1, \hat{\alpha} = \tau_0] \qquad \text{By Lemma 28 (Parallel Admissibility) (ii) twice}$$
$$\underbrace{\Gamma_0[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2]}_{\Gamma_1} \longrightarrow \underbrace{\Omega_0[\hat{\alpha}_2 = \tau_2, \hat{\alpha}_1 = \tau_1, \hat{\alpha} = \hat{\alpha}_1 \to \hat{\alpha}_2]}_{\Omega_1} \qquad \text{By Lemma 30 (Parallel Variable Update)}$$

Since $\hat{\alpha} \notin FV(A)$, it follows that $[\Gamma_1]A = [\Gamma]A = A$.
Therefore $\hat{\alpha}_1 \notin FV(A_1)$ and $\hat{\alpha}_1, \hat{\alpha}_2 \notin FV(A_2)$.
By Lemma 50 (Finishing Completions) and Lemma 49 (Finishing Types), $[\Omega_1]\Gamma_1 = [\Omega]\Gamma$ and $[\Omega_1]\hat{\alpha}_1 = \tau_1$.
By i.h., there are Δ_2 and Ω_2 such that $\Gamma_1 \vdash A_1 \stackrel{\leq}{=}: \hat{\alpha}_1 \dashv \Delta_2$ and $\Delta_2 \longrightarrow \Omega_2$ and $\Omega_1 \longrightarrow \Omega_2$.
Next, note that $[\Delta_2][\Delta_2]A_2 = [\Delta_2]A_2$.
By Lemma 33 (Left Unsolvedness Preservation), we know that $\hat{\alpha}_2 \in \mathsf{unsolved}(\Delta_2)$.
By Lemma 34 (Left Free Variable Preservation), we know that $\hat{\alpha}_2 \notin FV([\Delta_2]A_2)$.
By Lemma 20 (Transitivity), $\Omega \longrightarrow \Omega_2$.
We know $[\Omega_2]\Delta_2 = [\Omega]\Gamma$ because:

$$\begin{array}{rcll} [\Omega_2]\Delta_2 & = & [\Omega_2]\Omega_2 & \text{By Lemma 48 (Completing Stability)} \\ & = & [\Omega]\Omega & \text{By Lemma 50 (Finishing Completions)} \\ & = & [\Omega]\Gamma & \text{By Lemma 48 (Completing Stability)} \end{array}$$

By Lemma 49 (Finishing Types), we know that $[\Omega_2]\hat{\alpha}_2 = [\Omega_1]\hat{\alpha}_2 = \tau_2$.
By Lemma 49 (Finishing Types), we know that $[\Omega_2]A_2 = [\Omega]A_2$.
Hence we know that $[\Omega_2]\Delta_2 \vdash [\Omega_2]\hat{\alpha}_2 \leq [\Omega_2]A_2$.
By i.h., we have Δ and Ω' such that $\Delta_2 \vdash \hat{\alpha}_2 :\stackrel{\leq}{=} [\Delta_2]A_2 \dashv \Delta$ and $\Omega_2 \longrightarrow \Omega'$ and $\Delta \longrightarrow \Omega'$.
By rule $\mathsf{InstLArr}$, $\Gamma \vdash \hat{\alpha} :\stackrel{\leq}{=} A \dashv \Delta$.
By Lemma 20 (Transitivity), $\Omega \longrightarrow \Omega'$.

- **Case** $A = 1$:
  We have $A = 1$, so $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq [\Omega]1$.
  Since $[\Omega]1 = 1$, we have $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq 1$.
  The only declarative subtyping rules that can have 1 as the supertype in the conclusion are $\leq\forall\mathsf{L}$ and $\leq\mathsf{Unit}$. However, since Ω is predicative, $[\Omega]\hat{\alpha}$ cannot be a quantifier, so $\leq\forall\mathsf{L}$ cannot have been used. Hence $\leq\mathsf{Unit}$ was used and $[\Omega]\hat{\alpha} = 1$.
  Let $\Delta = \Gamma[\hat{\alpha} = 1]$ and $\Omega' = \Omega$.
  By $\mathsf{InstLSolve}$, $\Gamma[\hat{\alpha}] \vdash \hat{\alpha} :\stackrel{\leq}{=} 1 \dashv \Delta$.
  By Lemma 29 (Parallel Extension Solution), $\Gamma[\hat{\alpha} = 1] \longrightarrow \Omega$.

- **Case** $A = \forall\beta.\, B$:
  We have $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq [\Omega](\forall\beta.\, B)$.
  By definition of substitution, $[\Omega](\forall\beta.\, B) = \forall\beta.\, [\Omega]B$, so we have $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq \forall\beta.\, [\Omega]B$.
  The only declarative subtyping rules that can have a quantifier as supertype are $\leq\forall\mathsf{L}$ and $\leq\forall\mathsf{R}$. However, since Ω is predicative, $[\Omega]\hat{\alpha}$ cannot be a quantifier, so $\leq\forall\mathsf{L}$ cannot have been used. Hence $\leq\forall\mathsf{R}$ was used, and we have a subderivation of $[\Omega]\Gamma, \beta \vdash [\Omega]\hat{\alpha} \leq [\Omega]B$.

Let $\Omega_1 = (\Omega, \beta)$ and $\Gamma_1 = (\Gamma, \beta)$.
By $\longrightarrow$Uvar, $\Gamma_1 \longrightarrow \Omega_1$.
By the definition of substitution, $[\Omega_1]B = [\Omega]B$ and $[\Omega_1]\hat{\alpha} = [\Omega]\hat{\alpha}$.
Note that $[\Omega_1]\Gamma_1 = [\Omega]\Gamma, \beta$.
Since $\hat{\alpha} \in \mathsf{unsolved}(\Gamma)$, we have $\hat{\alpha} \in \mathsf{unsolved}(\Gamma_1)$.
Since $\hat{\alpha} \notin FV(A)$ and $A = \forall\beta.\, B$, we have $\hat{\alpha} \notin FV(B)$.
By i.h., there are Ω_2 and Δ_2 such that $\Gamma, \beta \vdash \hat{\alpha} :\stackrel{\leq}{=} B \dashv \Delta_2$ and $\Delta_2 \longrightarrow \Omega_2$ and $\Omega_1 \longrightarrow \Omega_2$.
By Lemma 31 (Instantiation Extension), $\Gamma_1 \longrightarrow \Delta_2$, that is, $\Gamma, \beta \longrightarrow \Delta_2$.
Therefore by Lemma 23 (Extension Order), $\Delta_2 = (\Delta', \beta, \Omega'')$ where $\Gamma \longrightarrow \Delta'$.
By equality, we know $\Delta', \beta, \Delta'' \longrightarrow \Omega_2$.
By Lemma 23 (Extension Order), $\Omega_2 = (\Omega', \beta, \Omega'')$ where ☞ $\Delta' \longrightarrow \Omega'$.
We have $\Omega_1 \longrightarrow \Omega_2$, that is, $\Omega, \beta \longrightarrow \Omega', \beta, \Omega''$, so Lemma 23 (Extension Order) gives ☞ $\Omega \longrightarrow \Omega'$.
By rule InstLAllR, $\Gamma \vdash \hat{\alpha} :\stackrel{\leq}{=} \forall\beta.\, B \dashv \Delta'$.

(2) $[\Omega]\Gamma \vdash [\Omega]A \leq [\Omega]\hat{\alpha}$

These cases are mostly symmetric. The one exception is the one connective that is not treated symmetrically in the declarative subtyping rules:

- **Case** $A = \forall\alpha.\, B$:
  Since $A = \forall\alpha.\, B$, we have $[\Omega]\Gamma \vdash [\Omega]\forall\beta.\, B \leq [\Omega]\hat{\alpha}$.
  By symmetric reasoning to the previous case (the last case of part (1) above), $\leq\forall$L must have been used, with a subderivation of $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq [\tau/\beta][\Omega]B$.
  Since $[\Omega]\Gamma \vdash \tau$, the type τ has no existential variables and is therefore invariant under substitution: $\tau = [\Omega]\tau$. Therefore $\big[\tau/\beta\big]\big[\Omega\big]B = \big[[\Omega]\tau/\beta\big]\big[\Omega\big]B$.
  By distributivity of substitution, this is $\big[\Omega\big][\tau/\beta]B$. Interposing $\hat{\beta}$, this is equal to $[\Omega][\tau/\hat{\beta}][\hat{\beta}/\beta]B$.
  Therefore $[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq [\Omega][\tau/\hat{\beta}][\hat{\beta}/\beta]B$.
  Let Ω_1 be $\Omega, \blacktriangleright_{\hat{\beta}}, \hat{\beta} = \tau$ and let Γ_1 be $\Gamma, \blacktriangleright_{\hat{\beta}}, \hat{\beta}$.

  - By the definition of context application, $[\Omega_1]\Gamma_1 = [\Omega]\Gamma$.
  - From the definition of substitution, $[\Omega_1]\hat{\alpha} = [\Omega]\hat{\alpha}$.
  - It follows from the definition of substitution that $[\Omega][\tau/\hat{\beta}]C = [\Omega_1]C$ for all C. Therefore $[\Omega][\tau/\hat{\beta}][\hat{\beta}/\beta]B = [\Omega_1][\hat{\beta}/\beta]B$.

  Applying these three equalities, $[\Omega_1]\Gamma_1 \vdash [\Omega_1]\hat{\alpha} \leq [\Omega_1][\hat{\beta}/\beta]B$.
  By the definition of substitution, $[\Gamma, \blacktriangleright_{\hat{\beta}}, \hat{\beta}]B = [\Gamma]B = B$, so $\hat{\alpha} \notin FV([\Gamma_1]B)$.
  Since $\hat{\alpha} \in \mathsf{unsolved}(\Gamma)$, we have $\hat{\alpha} \in \mathsf{unsolved}(\Gamma_1)$.

  By i.h., there exist Δ_2 and Ω_2 such that $\Gamma_1 \vdash B \stackrel{\leq}{=}: \hat{\alpha} \dashv \Delta_2$ and $\Omega_1 \longrightarrow \Omega_2$ and $\Delta_2 \longrightarrow \Omega_2$.
  By Lemma 31 (Instantiation Extension), $\Gamma_1 \longrightarrow \Delta_2$, which is, $\Gamma, \blacktriangleright_{\hat{\beta}}, \hat{\beta} \longrightarrow \Delta_2$.
  By Lemma 23 (Extension Order), $\Delta_2 = (\Delta', \blacktriangleright_{\hat{\beta}}, \Delta'')$ and $\Gamma \longrightarrow \Delta'$.
  By equality, $\Delta', \blacktriangleright_{\hat{\beta}}, \Delta'' \longrightarrow \Omega_2$.
  By Lemma 23 (Extension Order), $\Omega_2 = (\Omega', \blacktriangleright_{\hat{\beta}}, \Omega'')$ and ☞ $\Delta' \longrightarrow \Omega'$.
  By equality, $\Omega, \blacktriangleright_{\hat{\beta}}, \hat{\beta} = \tau \longrightarrow \Omega', \blacktriangleright_{\hat{\beta}}, \Omega''$.
  ☞ By Lemma 23 (Extension Order), $\Omega \longrightarrow \Omega'$.
  By InstRAllL, $\Gamma \vdash \forall\beta.\, B \stackrel{\leq}{=}: \hat{\alpha} \dashv \Delta'$. $\square$

K′.2 Completeness of Subtyping

Theorem 14 (Generalized Completeness of Subtyping)**.** *If* $\Gamma \longrightarrow \Omega$ *and* $\Gamma \vdash A$ *and* $\Gamma \vdash B$ *and* $[\Omega]\Gamma \vdash [\Omega]A \leq [\Omega]B$ *then there exist* Δ *and* Ω' *such that* $\Delta \longrightarrow \Omega'$ *and* $\Omega \longrightarrow \Omega'$ *and* $\Gamma \vdash [\Gamma]A <: [\Gamma]B \dashv \Delta$*.*

Proof. By induction on the derivation of $[\Omega]\Gamma \vdash [\Omega]A \leq [\Omega]B$.

We distinguish cases of $[\Gamma]B$ and $[\Gamma]A$ that are *impossible*, **fully written out**, and similar to fully-written-out cases.

$[\Gamma]A$ \ $[\Gamma]B$	$\forall\beta.\, B'$	1	α	$\hat{\beta}$	$B_1 \to B_2$
$\forall\alpha.\, A'$	**1 (B poly)**	**2.Poly**	**2.Poly**	**2.Poly**	**2.Poly**
1	**1 (B poly)**	**2.Units**	*impossible*	2.BEx.Unit	*impossible*
α	**1 (B poly)**	*impossible*	**2.Uvars**	2.BEx.Uvar	*impossible*
$\hat{\alpha}$	**1 (B poly)**	**2.AEx.Unit**	2.AEx.Uvar	**2.AEx.SameEx** **2.AEx.OtherEx**	**2.AEx.Arrow**
$A_1 \to A_2$	**1 (B poly)**	*impossible*	*impossible*	2.BEx.Arrow	**2.Arrows**

The impossibility of the "*impossible*" entries follows from inspection of the declarative subtyping rules.

We first split on $[\Gamma]B$.

- **Case 1 (B poly): $[\Gamma]B$ polymorphic:** $[\Gamma]B = \forall\beta.\, B'$:

	$B = \forall\beta.\, B_0$	Γ predicative
	$B' = [\Gamma]B_0$	Γ predicative
	$[\Omega]B = [\Omega](\forall\beta.\, B_0)$	Applying Ω to both sides
	$= \forall\beta.\, [\Omega]B_0$	By definition of substitution
$\mathcal{D}$ ::	$[\Omega]\Gamma \vdash [\Omega]A \leq [\Omega]B$	Given
$\mathcal{D}$ ::	$[\Omega]\Gamma \vdash [\Omega]A \leq \forall\beta.\, [\Omega]B_0$	By above equality
$\mathcal{D}'$ ::	$[\Omega]\Gamma, \beta \vdash [\Omega]A \leq [\Omega]B_0$	By Lemma 7 (Invertibility)
	$\mathcal{D}' < \mathcal{D}$	〃
$\mathcal{D}'$ ::	$[\Omega, \beta](\Gamma, \beta) \vdash [\Omega, \beta]A \leq [\Omega, \beta]B_0$	By definitions of substitution
	$\Gamma, \beta \vdash [\Gamma, \beta]A <: [\Gamma, \beta]B_0 \dashv \Delta'$	By i.h.
	$\Delta' \longrightarrow \Omega_0'$	〃
	$\Omega, \beta \longrightarrow \Omega_0'$	〃
	$\Gamma, \beta \vdash [\Gamma]A <: [\Gamma]B_0 \dashv \Delta'$	By definition of substitution
	$\Gamma, \beta \longrightarrow \Delta'$	By Lemma 31 (Instantiation Extension)
	$\Delta' = \Delta, \beta, \Theta$	By Lemma 23 (Extension Order) (i)
	$\Gamma \longrightarrow \Delta$	〃
	$\Delta, \beta, \Theta \longrightarrow \Omega_0'$	By $\Delta' \longrightarrow \Omega_0'$ and above equality
	$\Omega_0' = \Omega', \beta, \Omega_R$	By Lemma 23 (Extension Order) (i)
☞	$\Delta \longrightarrow \Omega'$	〃
	$\Gamma, \beta \vdash [\Gamma]A <: [\Gamma]B_0 \dashv \Delta, \beta, \Theta$	By above equality
	$\Omega, \beta \longrightarrow \Omega', \beta, \Omega_R$	By above equality
☞	$\Omega \longrightarrow \Omega'$	By Lemma 20 (Transitivity)
	$\Gamma \vdash [\Gamma]A <: \forall\beta.\, [\Gamma]B_0 \dashv \Delta$	By $<:\forall R$
☞	$\Gamma \vdash [\Gamma]A <: \forall\beta.\, B' \dashv \Delta$	By above equality

- **Cases 2.*:** $[\Gamma]B$ **not polymorphic:**

  We split on the form of $[\Gamma]A$.

  - **Case 2.Poly:** $[\Gamma]A$ **is polymorphic:** $[\Gamma]A = \forall\alpha.\, A'$:

$$
\begin{array}{rll}
A = \forall\alpha.\, A_0 & & \Gamma \text{ predicative} \\
A' = [\Gamma]A_0 & & \Gamma \text{ predicative} \\
[\Omega]A = [\Omega](\forall\alpha.\, A_0) & & \text{Applying } \Omega \text{ to both sides} \\
[\Omega]A = \forall\alpha.\, [\Omega]A_0 & & \text{By definition of substitution} \\
[\Omega]\Gamma \vdash [\Omega]A \leq [\Omega]B & & \text{Given} \\
[\Omega]\Gamma \vdash \forall\alpha.\, [\Omega]A_0 \leq [\Omega]B & & \text{By above equality} \\
[\Gamma]B \neq (\forall\beta.\cdots) & & \text{We are in the ``}[\Gamma]B\text{ not polymorphic'' subcase} \\
B \neq (\forall\beta.\ldots) & & \Gamma \text{ predicative} \\
[\Omega]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Omega]B & & \text{By inversion on } {\leq}\forall\mathsf{L} \\
[\Omega]\Gamma \vdash \tau & & '' \\
\\
\Gamma \longrightarrow \Omega & & \text{Given} \\
\Gamma, \blacktriangleright_{\hat\alpha} \longrightarrow \Omega, \blacktriangleright_{\hat\alpha} & & \text{By } {\longrightarrow}\mathsf{Marker} \\
\Gamma, \blacktriangleright_{\hat\alpha}, \hat\alpha \longrightarrow \underbrace{\Omega, \blacktriangleright_{\hat\alpha}, \hat\alpha = \tau}_{\Omega_0} & & \text{By } {\longrightarrow}\mathsf{Solve} \\
\\
[\Omega]\Gamma = [\Omega_0](\Gamma, \blacktriangleright_{\hat\alpha}, \hat\alpha) & & \text{By definition of context application (lines 16, 13)}
\end{array}
$$

$$
\begin{array}{rll}
[\Omega]\Gamma \vdash [\tau/\alpha][\Omega]A_0 \leq [\Omega]B & & \text{Above} \\
[\Omega_0](\Gamma, \blacktriangleright_{\hat\alpha}, \hat\alpha) \vdash [\tau/\alpha][\Omega]A_0 \leq [\Omega]B & & \text{By above equality} \\
[\Omega_0](\Gamma, \blacktriangleright_{\hat\alpha}, \hat\alpha) \vdash \big[[\Omega_0]\hat\alpha/\alpha\big][\Omega]A_0 \leq [\Omega]B & & \text{By definition of substitution} \\
[\Omega_0](\Gamma, \blacktriangleright_{\hat\alpha}, \hat\alpha) \vdash \big[[\Omega_0]\hat\alpha/\alpha\big][\Omega_0]A_0 \leq [\Omega_0]B & & \text{By definition of substitution} \\
[\Omega_0](\Gamma, \blacktriangleright_{\hat\alpha}, \hat\alpha) \vdash [\Omega_0][\hat\alpha/\alpha]A_0 \leq [\Omega_0]B & & \text{By distributivity of substitution} \\
\\
\Gamma, \blacktriangleright_{\hat\alpha}, \hat\alpha \vdash [\Gamma, \blacktriangleright_{\hat\alpha}, \hat\alpha][\hat\alpha/\alpha]A_0 <: [\Gamma, \blacktriangleright_{\hat\alpha}, \hat\alpha]B \dashv \Delta_0 & & \text{By i.h.} \\
\Delta_0 \longrightarrow \Omega'' & & '' \\
\Omega_0 \longrightarrow \Omega'' & & '' \\
\Gamma, \blacktriangleright_{\hat\alpha}, \hat\alpha \vdash [\Gamma][\hat\alpha/\alpha]A_0 <: [\Gamma]B \dashv \Delta_0 & & \text{By definition of substitution} \\
\Gamma, \blacktriangleright_{\hat\alpha}, \hat\alpha \longrightarrow \Delta_0 & & \text{By Lemma 32 (Subtyping Extension)} \\
\Delta_0 = (\Delta, \blacktriangleright_{\hat\alpha}, \Theta) & & \text{By Lemma 23 (Extension Order) (ii)} \\
\Gamma \longrightarrow \Delta & & '' \\
\Omega'' = (\Omega', \blacktriangleright_{\hat\alpha}, \Omega_Z) & & \text{By Lemma 23 (Extension Order) (ii)} \\
☞\quad \Delta \longrightarrow \Omega' & & '' \\
\Omega_0 \longrightarrow \Omega'' & & \text{Above} \\
\Omega, \blacktriangleright_{\hat\alpha}, \hat\alpha = \tau \longrightarrow \Omega', \blacktriangleright_{\hat\alpha}, \Omega_Z & & \text{By above equalities} \\
☞\quad \Omega \longrightarrow \Omega' & & \text{By Lemma 23 (Extension Order) (ii)} \\
\\
\Gamma, \blacktriangleright_{\hat\alpha}, \hat\alpha \vdash [\Gamma][\hat\alpha/\alpha]A_0 <: [\Gamma]B \dashv \Delta, \blacktriangleright_{\hat\alpha}, \Theta & & \text{By above equality } \Delta_0 = (\Delta, \blacktriangleright_{\hat\alpha}, \Theta) \\
\Gamma, \blacktriangleright_{\hat\alpha}, \hat\alpha \vdash [\hat\alpha/\alpha][\Gamma]A_0 <: [\Gamma]B \dashv \Delta, \blacktriangleright_{\hat\alpha}, \Theta & & \text{By def. of subst. } ([\Gamma]\hat\alpha = \hat\alpha \text{ and } [\Gamma]\alpha = \alpha) \\
\Gamma \vdash \forall\alpha.\, [\Gamma]A_0 <: [\Gamma]B \dashv \Delta & & \text{By } {<:}\forall\mathsf{L} \\
☞\quad \Gamma \vdash \forall\alpha.\, A' <: [\Gamma]B \dashv \Delta & & \text{By above equality}
\end{array}
$$

  - **Case 2.AEx:** A **is an existential variable** $[\Gamma]A = \hat\alpha$:

    We split on the form of $[\Gamma]B$.

    * **Case 2.AEx.SameEx:** $[\Gamma]B$ **is the same existential variable** $[\Gamma]B = \hat\alpha$:

	$\Gamma \vdash \hat\alpha <: \hat\alpha \dashv \Gamma$	By <:Exvar
☞	$\Gamma \vdash [\Gamma]A <: [\Gamma]B \dashv \Gamma$	By $[\Gamma]A = [\Gamma]B = \hat\alpha$
☞	$\Delta \longrightarrow \Omega$	$\Delta = \Gamma$
☞	$\Omega \longrightarrow \Omega'$	By Lemma 19 (Reflexivity) and $\Omega' = \Omega$

* **Case 2.AEx.OtherEx: $[\Gamma]B$ is a different existential variable** $[\Gamma]B = \hat\beta$ where $\hat\beta \neq \hat\alpha$:
  Either $\hat\alpha \in FV([\Gamma]\hat\beta)$, or $\hat\alpha \notin FV([\Gamma]\hat\beta)$.
  - $\hat\alpha \in FV([\Gamma]\hat\beta)$:
    We have $\hat\alpha \preceq [\Gamma]\hat\beta$.
    Therefore $\hat\alpha = [\Gamma]\hat\beta$, or $\hat\alpha \prec [\Gamma]\hat\beta$.
    But we are in Case 2.AEx.**Other**Ex, so the former is impossible.
    Therefore, $\hat\alpha \prec [\Gamma]\hat\beta$.
    Since Γ is predicative, $[\Gamma]\hat\beta$ cannot have the form $\forall\beta. \cdots$, so the only way that $\hat\alpha$ can be a proper subterm of $[\Gamma]\hat\beta$ is if $[\Gamma]\hat\beta$ has the form $B_1 \to B_2$ such that $\hat\alpha$ is a subterm of B_1 or B_2, that is: $\hat\alpha \precneqq [\Gamma]\hat\beta$.
    Then by a property of substitution, $[\Omega]\hat\alpha \precneqq [\Omega][\Gamma]\hat\beta$.
    By Lemma 17 (Substitution Extension Invariance), $[\Omega][\Gamma]\hat\beta = [\Omega]\hat\beta$, so $[\Omega]\hat\alpha \precneqq [\Omega]\hat\beta$.
    We have $[\Omega]\Gamma \vdash [\Omega]\hat\alpha \leq [\Omega]\hat\beta$, and we know that $[\Omega]\hat\alpha$ is a monotype, so we can use Lemma 8 (Occurrence) (ii) to show that $[\Omega]\hat\alpha \not\precneqq [\Omega]\hat\beta$, a contradiction.
  - $\hat\alpha \notin FV([\Gamma]\hat\beta)$:

	$\Gamma \vdash \hat\alpha :\leqq [\Gamma]\hat\beta \dashv \Delta$	By Theorem 13 (1)
☞	$\Gamma \vdash \hat\alpha <: \hat\beta \dashv \Delta$	By <:InstantiateL
☞	$\Delta \longrightarrow \Omega'$	"
☞	$\Omega \longrightarrow \Omega'$	"

* **Case 2.AEx.Unit:** $[\Gamma]B = 1$:

	$\Gamma \longrightarrow \Omega$	Given
	$1 = [\Omega]1$	By definition of substitution
	$\hat\alpha \notin FV(1)$	By definition of $FV(-)$
	$[\Omega]\Gamma \vdash [\Omega]\hat\alpha \leq [\Omega]1$	Given
	$\Gamma \vdash \hat\alpha :\leqq 1 \dashv \Delta$	By Theorem 13 (1)
☞	$\Omega \longrightarrow \Omega'$	"
☞	$\Delta \longrightarrow \Omega'$	"
	$1 = [\Gamma]1$	By definition of substitution
	$\hat\alpha \notin FV(1)$	By definition of $FV(-)$
☞	$\Gamma \vdash \hat\alpha <: 1 \dashv \Delta$	By <:InstantiateL

* **Case 2.AEx.Uvar:** $[\Gamma]B = \beta$:
  Similar to Case 2.AEx.Unit, using $\beta = [\Omega]\beta = [\Gamma]\beta$ and $\hat\alpha \notin FV(\beta)$.
* **Case 2.AEx.Arrow:** $[\Gamma]B = B_1 \to B_2$:
  Since $[\Gamma]B$ is an arrow, it cannot be exactly $\hat\alpha$.
  Suppose, for a contradiction, that $\hat\alpha \in FV([\Gamma]B)$.

$$
\begin{array}{ll}
\hat{\alpha} \preceq [\Gamma]B & \hat{\alpha} \in FV([\Gamma]B) \\
[\Omega]\hat{\alpha} \preceq [\Omega][\Gamma]B & \text{By a property of substitution} \\
\\
\Gamma \longrightarrow \Omega & \text{Given} \\
[\Omega][\Gamma]B = [\Omega]B & \text{By Lemma 17 (Substitution Extension Invariance)} \\
\\
[\Omega]\hat{\alpha} \preceq [\Omega]B & \text{By above equality} \\
\\
[\Gamma]B \neq \hat{\alpha} & \text{Given (2.AEx.Arrow)} \\
[\Omega][\Gamma]B \neq [\Omega]\hat{\alpha} & \text{By a property of substitution} \\
[\Omega]B \neq [\Omega]\hat{\alpha} & \text{By Lemma 17 (Substitution Extension Invariance)} \\
\\
[\Omega]\hat{\alpha} \prec [\Omega]B & \text{Follows from } \preceq \text{ and } \neq \\
[\Omega]\hat{\alpha} \npreceq [\Omega]B & [\Omega]A \text{ has the form } \cdots \to \cdots \\
[\Omega]\Gamma \vdash [\Omega]\hat{\alpha} \leq [\Omega]B & \text{Given} \\
[\Omega]B \text{ is a monotype} & \Omega \text{ is predicative} \\
[\Omega]\hat{\alpha} \not\preceq [\Omega]B & \text{By Lemma 8 (Occurrence) (ii)} \\
\Rightarrow\Leftarrow & \\
\hat{\alpha} \notin FV([\Gamma]B) & \text{By contradiction}
\end{array}
$$

$$
\begin{array}{lll}
 & \Gamma \vdash \hat{\alpha} :\overset{\leq}{=} [\Gamma]B \dashv \Delta & \text{By Theorem 13 (1)} \\
☞ & \Delta \longrightarrow \Omega' & '' \\
☞ & \Omega \longrightarrow \Omega' & '' \\
☞ & \Gamma \vdash \hat{\alpha} <: \underbrace{[\Gamma]}_{B_1 \to B_2} \dashv \Delta & \text{By } <:\text{InstantiateL}
\end{array}
$$

– **Case 2.BEx: $[\Gamma]A$ is not polymorphic and $[\Gamma]B$ is an existential variable:** $[\Gamma]B = \hat{\beta}$
We split on the form of $[\Gamma]A$.

 * **Case 2.BEx.Unit** ($[\Gamma]A = 1$),
 Case 2.BEx.Uvar ($[\Gamma]A = \alpha$),
 Case 2.BEx.Arrow ($[\Gamma]A = A_1 \to A_2$):
 Similar to Cases **2.AEx.Unit**, **2.AEx.Uvar** and **2.AEx.Arrow**, but using part (2) of Theorem 13 instead of part (1), and applying $<:$InstantiateR instead of $<:$InstantiateL as the final step.

– **Case 2.Units**: $[\Gamma]A = [\Gamma]B = 1$:

$$
\begin{array}{lll}
☞ & \Gamma \vdash 1 <: 1 \dashv \Gamma & \text{By } <:\text{Unit} \\
 & \Gamma \longrightarrow \Omega & \text{Given} \\
☞ & \Delta \longrightarrow \Omega & \Delta = \Gamma \\
☞ & \Omega \longrightarrow \Omega' & \text{By Lemma 19 (Reflexivity) and } \Omega' = \Omega
\end{array}
$$

– **Case 2.Uvars**: $[\Gamma]A = [\Gamma]B = \alpha$:

$$
\begin{array}{lll}
 & \alpha \in \Omega & \text{By inversion on } \leq\text{Var} \\
 & \Gamma \longrightarrow \Omega & \text{Given} \\
 & \alpha \in \Gamma & \text{By Lemma 23 (Extension Order)} \\
☞ & \Gamma \vdash \alpha <: \alpha \dashv \Gamma & \text{By } <:\text{Var} \\
☞ & \Delta \longrightarrow \Omega & \Delta = \Gamma \\
☞ & \Omega \longrightarrow \Omega' & \text{By Lemma 19 (Reflexivity) and } \Omega' = \Omega
\end{array}
$$

– **Case 2.Arrows**: $A = A_1 \to A_2$ and $B = B_1 \to B_2$:
Only rule $\leq\!\to$ could have been used.

$$
\begin{array}{lll}
& [\Omega]\Gamma \vdash [\Omega]B_1 \leq [\Omega]A_1 & \text{Subderivation} \\
& \Gamma \vdash [\Gamma]B_1 <: [\Gamma]A_1 \dashv \Theta & \text{By i.h.} \\
& \Theta \longrightarrow \Omega_0 & '' \\
& \Omega \longrightarrow \Omega_0 & '' \\
& \Gamma \longrightarrow \Omega & \text{Given} \\
& \Gamma \longrightarrow \Omega_0 & \text{By Lemma 20 (Transitivity)} \\
& \Theta \longrightarrow \Omega_0 & \text{Above} \\
& [\Omega]\Gamma = [\Omega]\Theta & \text{By Lemma 51 (Confluence of Completeness)} \\
& [\Omega]\Gamma \vdash [\Omega]A_2 \leq [\Omega]B_2 & \text{Subderivation} \\
& [\Omega]\Theta \vdash [\Omega]A_2 \leq [\Omega]B_2 & \text{By above equality} \\
& [\Omega]A_2 = [\Omega][\Gamma]A_2 & \text{By Lemma 17 (Substitution Extension Invariance)} \\
& [\Omega]B_2 = [\Omega][\Gamma]B_2 & \text{By Lemma 17 (Substitution Extension Invariance)} \\
& [\Omega]\Theta \vdash [\Omega][\Gamma]A_2 \leq [\Omega][\Gamma]B_2 & \text{By above equalities} \\
& \Theta \vdash [\Theta][\Gamma]A_2 <: [\Theta][\Gamma]B_2 \dashv \Delta & \text{By i.h.} \\
☞ & \Delta \longrightarrow \Omega' & '' \\
& \Omega_0 \longrightarrow \Omega' & '' \\
& \Gamma \vdash ([\Gamma]A_1) \to ([\Gamma]A_2) <: ([\Gamma]B_1) \to ([\Gamma]B_2) \dashv \Delta & \text{By } <:\to \\
☞ & \Gamma \vdash [\Gamma](A_1 \to A_2) <: [\Gamma](B_1 \to B_2) \dashv \Delta & \text{By definition of substitution} \\
☞ & \Omega \longrightarrow \Omega' & \text{By Lemma 20 (Transitivity)} \quad \square
\end{array}
$$

Corollary 54 (Completeness of Subtyping). *If* $\Psi \vdash A \leq B$ *then there is a* Δ *such that* $\Psi \vdash A <: B \dashv \Delta$.

Proof. Let $\Omega = \Psi$ and $\Gamma = \Psi$.
By Lemma 19 (Reflexivity), $\Psi \longrightarrow \Psi$, so $\Gamma \longrightarrow \Omega$.
By Lemma 4 (Well-Formedness), $\Psi \vdash A$ and $\Psi \vdash B$; since $\Gamma = \Psi$, we have $\Gamma \vdash A$ and $\Gamma \vdash B$.
By Theorem 14, there exists Δ such that $\Gamma \vdash [\Gamma]A <: [\Gamma]B \dashv \Delta$.
Since $\Gamma = \Psi$ and Ψ is a declarative context with no existentials, $[\Psi]C = C$ for all C, so we actually have $\Psi \vdash A <: B \dashv \Delta$, which was to be shown. $\square$

L′ Completeness of Typing

Theorem 15 (Completeness of Algorithmic Typing). *Given* $\Gamma \longrightarrow \Omega$ *and* $\Gamma \vdash A$*:*

(i) If $[\Omega]\Gamma \vdash e \Leftarrow [\Omega]A$
then there exist Δ *and* Ω'
such that $\Delta \longrightarrow \Omega'$ *and* $\Omega \longrightarrow \Omega'$ *and* $\Gamma \vdash e \Leftarrow [\Gamma]A \dashv \Delta$.

(ii) If $[\Omega]\Gamma \vdash e \Rightarrow A$
then there exist Δ, Ω', *and* A'
such that $\Delta \longrightarrow \Omega'$ *and* $\Omega \longrightarrow \Omega'$ *and* $\Gamma \vdash e \Rightarrow A' \dashv \Delta$ *and* $A = [\Omega']A'$.

(iii) If $[\Omega]\Gamma \vdash [\Omega]A \bullet e \Rrightarrow C$
then there exist Δ, Ω', *and* C'
such that $\Delta \longrightarrow \Omega'$ *and* $\Omega \longrightarrow \Omega'$ *and* $\Gamma \vdash [\Gamma]A \bullet e \Rrightarrow C' \dashv \Delta$ *and* $C = [\Omega']C'$.

Proof. By induction on the given declarative derivation.

- **Case** $\dfrac{(x : A) \in [\Omega]\Gamma}{[\Omega]\Gamma \vdash x \Rightarrow A}$ DeclVar

	$(x : A) \in [\Omega]\Gamma$	Premise
	$\Gamma \longrightarrow \Omega$	Given
	$(x : A') \in \Gamma$ where $[\Omega]A' = [\Omega]A$	From definition of context application
	Let $\Delta = \Gamma$.	
	Let $\Omega' = \Omega$.	
☞	$\Gamma \longrightarrow \Omega$	Given
☞	$\Omega \longrightarrow \Omega$	By Lemma 19 (Reflexivity)
☞	$\Gamma \vdash x \Rightarrow A' \dashv \Gamma$	By Var
	$[\Omega]A' = [\Omega]A$	Above
☞	$= A$	$\mathsf{FEV}(A) = \emptyset$

- **Case** $\dfrac{[\Omega]\Gamma \vdash e \Rightarrow B \qquad [\Omega]\Gamma \vdash B \leq [\Omega]A}{[\Omega]\Gamma \vdash e \Leftarrow [\Omega]A}$ DeclSub

	$[\Omega]\Gamma \vdash e \Rightarrow B$	Subderivation
	$\Gamma \vdash e \Rightarrow B' \dashv \Theta$	By i.h.
	$B = [\Omega]B'$	〃
	$\Theta \longrightarrow \Omega_0$	〃
	$\Omega \longrightarrow \Omega_0$	〃
	$\Gamma \longrightarrow \Omega$	Given
	$\Gamma \longrightarrow \Omega_0$	By Lemma 20 (Transitivity)
	$[\Omega]\Gamma \vdash B \leq [\Omega]A$	Subderivation
	$[\Omega]\Gamma = [\Omega]\Theta$	By Lemma 51 (Confluence of Completeness)
	$[\Omega]\Theta \vdash B \leq [\Omega]A$	By above equalities
	$\Theta \longrightarrow \Omega_0$	Above
	$\Theta \vdash [\Theta]B' <: [\Theta]A \dashv \Delta$	By Theorem 14
	$\Delta \longrightarrow \Omega'$	〃
	$\Omega_0 \longrightarrow \Omega'$	〃
☞	$\Delta \longrightarrow \Omega'$	By Lemma 20 (Transitivity)
☞	$\Omega \longrightarrow \Omega'$	By Lemma 20 (Transitivity)
☞	$\Gamma \vdash e \Leftarrow A \dashv \Delta$	By Sub

- **Case** $\dfrac{[\Omega]\Gamma \vdash e_0 \Leftarrow [\Omega]A}{[\Omega]\Gamma \vdash (e_0 : [\Omega]A) \Rightarrow [\Omega]A}$ DeclAnno

	$[\Omega]\Gamma \vdash e_0 \Leftarrow [\Omega]A$	Subderivation
	$\Gamma \vdash e_0 \Leftarrow A \dashv \Delta$	By i.h.
☞	$\Delta \longrightarrow \Omega$	"
☞	$\Omega \longrightarrow \Omega'$	"
	$\Gamma \vdash (e_0 : A) \Rightarrow A \dashv \Delta$	By Anno
	$A = [\Omega]A$	Source type annotations cannot contain evars
☞	$\Gamma \vdash (e_0 : [\Omega]A) \Rightarrow A \dashv \Delta$	By above equality

- **Case** $\dfrac{}{[\Omega]\Gamma \vdash () \Leftarrow 1}$ Decl1I

We have $[\Omega]A = 1$. Either $[\Gamma]A = 1$ or $[\Gamma]A = \hat{\alpha} \in \mathsf{unsolved}(\Gamma)$.

In the former case:

Let $\Delta = \Gamma$.

Let $\Omega' = \Omega$.

☞	$\Gamma \longrightarrow \Omega$	Given
☞	$\Omega \longrightarrow \Omega'$	By Lemma 19 (Reflexivity)
	$\Gamma \vdash () \Leftarrow 1 \dashv \Gamma$	By Anno
☞	$\Gamma \vdash () \Leftarrow [\Gamma]1 \dashv \Gamma$	$1 = [\Gamma]1$

In the latter case:

	$\Gamma \vdash () \Rightarrow 1 \dashv \Gamma$	By 1I$\Rightarrow$
	$[\Omega]\Gamma \vdash 1 \le 1$	By $\le$Unit
	$1 = [\Omega]1$	By definition of substitution
	$= [\Omega][\Gamma]\hat{\alpha}$	By $[\Omega]A = 1$
	$= [\Omega]\hat{\alpha}$	By Lemma 17 (Substitution Extension Invariance)
	$[\Omega]\Gamma \vdash [\Omega]1 \le [\Omega]\hat{\alpha}$	By above equalities
	$\Gamma \vdash 1 <: \hat{\alpha} \dashv \Delta$	By Theorem 13 (1)
	$1 = [\Gamma]1$	By definition of substitution
	$\hat{\alpha} = [\Gamma]\hat{\alpha}$	$\hat{\alpha} \in \mathsf{unsolved}(\Gamma)$
	$\Gamma \vdash [\Gamma]1 <: [\Gamma]\hat{\alpha} \dashv \Delta$	By above equalities
☞	$\Omega \longrightarrow \Omega'$	"
☞	$\Delta \longrightarrow \Omega'$	"
	$\Gamma \vdash () \Leftarrow \hat{\alpha} \dashv \Delta$	By Sub
☞	$\Gamma \vdash () \Leftarrow [\Gamma]A \dashv \Delta$	By $[\Gamma]A = \hat{\alpha}$

- **Case** $\dfrac{[\Omega]\Gamma, \alpha \vdash e \Leftarrow A_0}{[\Omega]\Gamma \vdash e \Leftarrow \forall \alpha.\, A_0}$ Decl∀I

$[\Omega]A = \forall \alpha.\, A_0$	Given
$= \forall \alpha.\, [\Omega]A'$	By def. of subst. and predicativity of Ω
$A_0 = [\Omega]A'$	Follows from above equality
$[\Omega]\Gamma, \alpha \vdash e \Leftarrow [\Omega]A'$	Subderivation and above equality
$\Gamma \longrightarrow \Omega$	Given
$\Gamma, \alpha \longrightarrow \Omega, \alpha$	By $\longrightarrow$Uvar
$[\Omega]\Gamma, \alpha = [\Omega, \alpha](\Gamma, \alpha)$	By definition of context substitution
$[\Omega, \alpha](\Gamma, \alpha) \vdash e \Leftarrow [\Omega]A'$	By above equality
$[\Omega, \alpha](\Gamma, \alpha) \vdash e \Leftarrow [\Omega, \alpha]A'$	By definition of substitution

$$\begin{array}{lll}
& \Gamma, \alpha \vdash e \Leftarrow [\Gamma, \alpha]A' \dashv \Delta' & \text{By i.h.} \\
& \Delta' \longrightarrow \Omega_0' & '' \\
& \Omega, \alpha \longrightarrow \Omega_0' & '' \\
& \Gamma, \alpha \longrightarrow \Delta' & \text{By Lemma 53 (Typing Extension)} \\
& \Delta' = \Delta, \alpha, \Theta & \text{By Lemma 23 (Extension Order) (i)} \\
& \Delta, \alpha, \Theta \longrightarrow \Omega_0' & \text{By above equality} \\
& \Omega_0' = \Omega', \alpha, \Omega_Z & \text{By Lemma 23 (Extension Order) (i)} \\
☞ & \Delta \longrightarrow \Omega' & '' \\
☞ & \Omega \longrightarrow \Omega' & \text{By Lemma 23 (Extension Order) on } \Omega, \alpha \longrightarrow \Omega_0'
\end{array}$$

$$\begin{array}{lll}
& \Gamma, \alpha \vdash e \Leftarrow [\Gamma, \alpha]A' \dashv \Delta, \alpha, \Theta & \text{By above equality} \\
& \Gamma, \alpha \vdash e \Leftarrow [\Gamma]A' \dashv \Delta, \alpha, \Theta & \text{By definition of substitution} \\
& \Gamma \vdash e \Leftarrow \forall\alpha.\, [\Gamma]A' \dashv \Delta & \text{By } \forall\mathsf{I} \\
☞ & \Gamma \vdash e \Leftarrow [\Gamma](\forall\alpha.\, A') \dashv \Delta & \text{By definition of substitution}
\end{array}$$

- **Case** $\dfrac{[\Omega]\Gamma \vdash \tau \qquad [\Omega]\Gamma \vdash [\tau/\alpha]A_0 \bullet e \Rrightarrow C}{[\Omega]\Gamma \vdash \underbrace{\forall\alpha.\, A_0}_{[\Omega]A} \bullet e \Rrightarrow C}\;\mathsf{Decl}\forall\mathsf{App}$

$$\begin{array}{rl}
[\Omega]\Gamma \vdash \tau & \text{Subderivation}
\end{array}$$

$$\begin{array}{rll}
[\Omega]A & = \forall\alpha.\, A_0 & \text{Given} \\
& = \forall\alpha.\, [\Omega]A' & \text{By def. of subst. and predicativity of } \Omega \\
\multicolumn{2}{r}{[\Omega]\Gamma \vdash [\tau/\alpha][\Omega]A' \bullet e \Rrightarrow C} & \text{Subderivation and above equality} \\
\multicolumn{2}{r}{\Gamma \longrightarrow \Omega} & \text{Given} \\
\multicolumn{2}{r}{\Gamma, \hat\alpha \longrightarrow \Omega, \hat\alpha = \tau} & \text{By } {\longrightarrow}\mathsf{Solve}
\end{array}$$

$$\begin{array}{rl}
[\Omega]\Gamma = [\Omega, \hat\alpha = \tau](\Gamma, \hat\alpha) & \text{By definition of context application} \\
[\Omega, \hat\alpha = \tau](\Gamma, \hat\alpha) \vdash [\tau/\alpha][\Omega]A' \bullet e \Rrightarrow C & \text{By above equality} \\
[\Omega, \hat\alpha = \tau](\Gamma, \hat\alpha) \vdash [\tau/\alpha][\Omega, \hat\alpha = \tau]A' \bullet e \Rrightarrow C & \text{By def. of subst.} \\
\big([[\Omega]\tau/\alpha]\,[\Omega, \hat\alpha = \tau]A'\big) = \big([\Omega, \hat\alpha = \tau][\hat\alpha/\alpha]A'\big) & \text{By distributivity of substitution} \\
\tau = [\Omega]\tau & \mathsf{FEV}(\tau) = \emptyset \\
\big([\tau/\alpha]\,[\Omega, \hat\alpha = \tau]A'\big) = \big([\Omega, \hat\alpha = \tau][\hat\alpha/\alpha]A'\big) & \text{By above equality} \\
[\Omega, \hat\alpha = \tau](\Gamma, \hat\alpha) \vdash [\Omega, \hat\alpha = \tau][\hat\alpha/\alpha]A' \bullet e \Rrightarrow C & \text{By above equality}
\end{array}$$

$$\begin{array}{lll}
& \Gamma, \hat\alpha \vdash [\hat\alpha/\alpha]A' \bullet e \Rrightarrow C' \dashv \Delta & \text{By i.h.} \\
☞ & C = [\Omega]C' & '' \\
☞ & \Delta \longrightarrow \Omega' & '' \\
☞ & \Omega \longrightarrow \Omega' & ''
\end{array}$$

$$\begin{array}{lll}
☞ & \Gamma \vdash \forall\alpha.\, A' \bullet e \Rrightarrow C' \dashv \Delta & \text{By } \forall\mathsf{App}
\end{array}$$

- **Case** $\dfrac{[\Omega]\Gamma, x : A_1' \vdash e_0 \Leftarrow A_2'}{[\Omega]\Gamma \vdash \lambda x.\, e_0 \Leftarrow A_1' \rightarrow A_2'}\;\mathsf{Decl}{\rightarrow}\mathsf{I}$

  We have $[\Omega]A = A_1' \rightarrow A_2'$. Either $[\Gamma]A = A_1 \rightarrow A_2$ where $A_1' = [\Omega]A_1$ and $A_2' = [\Omega]A_2$—or $[\Gamma]A = \hat\alpha$ and $[\Omega]\hat\alpha = A_1' \rightarrow A_2'$.

  In the former case:

	$[\Omega]\Gamma, x : A_1' \vdash e_0 \Leftarrow A_2'$	Subderivation
	$A_1' = [\Omega]A_1$	Known in this subcase
	$= [\Omega][\Gamma]A_1$	By Lemma 17 (Substitution Extension Invariance)
	$[\Omega]A_1' = [\Omega][\Omega][\Gamma]A_1$	Applying Ω on both sides
	$= [\Omega][\Gamma]A_1$	By idempotence of substitution
	$[\Omega]\Gamma, x : A_1' = [\Omega, x : A_1'](\Gamma, x : [\Gamma]A_1)$	By definition of context application
	$[\Omega, x : A_1'](\Gamma, x : [\Gamma]A_1) \vdash e_0 \Leftarrow A_2'$	By above equality
	$\Gamma \longrightarrow \Omega$	Given
	$\Gamma, x : [\Gamma]A_1 \longrightarrow \Omega, x : A_1'$	By $\longrightarrow$Var
	$\Gamma, x : [\Gamma]A_1 \vdash e_0 \Leftarrow A_2 \dashv \Delta'$	By i.h.
	$\Delta' \longrightarrow \Omega_0'$	"
	$\Omega, x : A_1' \longrightarrow \Omega_0'$	"
	$\Omega_0' = \Omega', x : A_1', \Theta$	By Lemma 23 (Extension Order) (v)
☞	$\Omega \longrightarrow \Omega'$	"
	$\Gamma, x : [\Gamma]A_1 \longrightarrow \Delta'$	By Lemma 53 (Typing Extension)
	$\Delta' = \Delta, x : \cdots, \Theta$	By Lemma 23 (Extension Order) (v)
	$\Delta, x : \cdots, \Theta \longrightarrow \Omega', x : A_1', \Theta$	By above equalities
☞	$\Delta \longrightarrow \Omega'$	By Lemma 23 (Extension Order) (v)
	$\Gamma, x : [\Gamma]A_1 \vdash e_0 \Leftarrow [\Gamma]A_2 \dashv \Delta, \alpha, \Theta$	By above equality
	$\Gamma \vdash \lambda x.\, e_0 \Leftarrow ([\Gamma]A_1) \to ([\Gamma]A_2) \dashv \Delta$	By $\to$I
☞	$\Gamma \vdash \lambda x.\, e_0 \Leftarrow [\Gamma](A_1 \to A_2) \dashv \Delta$	By definition of substitution

In the latter case:

	$[\Omega]\hat\alpha = A_1' \to A_2'$	Known in this subcase
	$[\Omega]\Gamma, x : A_1' \vdash e_0 \Leftarrow A_2'$	Subderivation
	$\Gamma \longrightarrow \Omega$	Given
	$\Gamma, \hat\alpha, \hat\beta \longrightarrow \Omega, \hat\alpha = A_1', \hat\beta = A_2'$	By $\longrightarrow$Solve twice
	$[\Omega]\hat\alpha = [\Omega]A_1'$	By definition of substitution
	$\Gamma, \hat\alpha, \hat\beta, x : \hat\alpha \longrightarrow \Omega, \hat\alpha = A_1', \hat\beta = A_2', x : A_1'$	By $\longrightarrow$Var
	$[\Omega]\Gamma, x : A_1' = \big[\Omega, \hat\alpha = A_1', \hat\beta = A_2', x : A_1'\big](\Gamma, \hat\alpha, \hat\beta, x : \hat\alpha)$	By definition of context application
	Let $\Omega_0 = (\Omega, \hat\alpha = A_1', \hat\beta = A_2', x : A_1')$.	
	$[\Omega_0](\Gamma, \hat\alpha, \hat\beta, x : \hat\alpha) \vdash e_0 \Leftarrow A_2'$	By above equality
	$\Gamma, \hat\alpha, \hat\beta, x : \hat\alpha \vdash e_0 \Leftarrow \hat\beta \dashv \Delta'$	By i.h. with Ω_0
	$\Delta' \longrightarrow \Omega_0'$	"
	$\Omega_0 \longrightarrow \Omega_0'$	"
	$\Gamma, \hat\alpha, \hat\beta, x : \hat\alpha \longrightarrow \Delta'$	By Lemma 53 (Typing Extension)
	$\Delta' = \Delta, x : \hat\alpha, \Theta$	By Lemma 23 (Extension Order) (v)
	$\Delta, x : \hat\alpha, \Theta \longrightarrow \Omega_0'$	By above equality
	$\Omega_0' = \Omega'', x : \cdots, \Omega_Z$	By Lemma 53 (Typing Extension)
☞	$\Delta \longrightarrow \Omega''$	"
	$\Gamma, \hat\alpha, \hat\beta \longrightarrow \Delta$	"
	$\Omega_0 \longrightarrow \underbrace{\Omega'', x : \cdots, \Omega_Z}_{\Omega_0'}$	By above equality
	$\Omega, \hat\alpha = A_1', \hat\beta = A_2', x : A_1' \longrightarrow \Omega'', x : \cdots, \Omega_Z$	By def. of Ω_0
	$\Omega'' = \Omega', \hat\alpha = \ldots, \ldots$	By Lemma 23 (Extension Order) (iii)
☞	$\Omega \longrightarrow \Omega'$	"

	$\Gamma, \hat{\alpha}, \hat{\beta}, x : \hat{\alpha} \vdash e_0 \Leftarrow \hat{\beta} \dashv \Delta, x : \hat{\alpha}, \Theta$	By above equality
	$\Gamma \vdash \lambda x.\, e_0 \Leftarrow \hat{\alpha} \to \hat{\beta} \dashv \Delta$	By $\to$I$\Rightarrow$
	$[\Gamma]\hat{\alpha} = \hat{\alpha}$	By definition of substitution
	$[\Gamma]\hat{\beta} = \hat{\beta}$	By definition of substitution
	$\Gamma \vdash \lambda x.\, e_0 \Leftarrow ([\Gamma]\hat{\alpha}) \to ([\Gamma]\hat{\beta}) \dashv \Delta$	By above equalities
☞	$\Gamma \vdash \lambda x.\, e_0 \Leftarrow [\Gamma](\hat{\alpha} \to \hat{\beta}) \dashv \Delta$	By definition of substitution

- **Case** $\dfrac{[\Omega]\Gamma \vdash e_1 \Rightarrow B \qquad [\Omega]\Gamma \vdash B \bullet e_2 \Rightarrow\!\!\!\Rightarrow A}{[\Omega]\Gamma \vdash e_1\, e_2 \Rightarrow A}$ Decl$\to$E

	$[\Omega]\Gamma \vdash e_1 \Rightarrow B$	Subderivation
	$\Gamma \longrightarrow \Omega$	Given
	$\Gamma \vdash e_1 \Rightarrow B' \dashv \Theta$	By i.h.
	$B = [\Omega]B'$	"
	$\Theta \longrightarrow \Omega'_0$	"
	$\Omega \longrightarrow \Omega'_0$	"
	$[\Omega]\Gamma \vdash B \bullet e_2 \Rightarrow\!\!\!\Rightarrow A$	Subderivation
	$[\Omega]\Gamma \vdash [\Omega]B' \bullet e_2 \Rightarrow\!\!\!\Rightarrow A$	By above equality
	$\Gamma \longrightarrow \Omega'_0$	By Lemma 20 (Transitivity)
	$[\Omega]\Gamma = [\Omega]\Omega$	By Lemma 48 (Completing Stability)
	$= [\Omega'_0]\Omega'_0$	By Lemma 50 (Finishing Completions)
	$= [\Omega'_0]\Gamma$	By Lemma 48 (Completing Stability)
	$= [\Omega'_0]\Theta$	By Lemma 51 (Confluence of Completeness)
	$[\Omega'_0]\Theta \vdash [\Omega]B' \bullet e_2 \Rightarrow\!\!\!\Rightarrow A$	By above equality
	$[\Omega]B' = [\Omega'_0]B'$	By Lemma 49 (Finishing Types)
	$[\Omega'_0]B' = [\Omega'_0][\Theta]B'$	By Lemma 17 (Substitution Extension Invariance)
	$[\Omega'_0]\Theta \vdash [\Omega][\Theta]B' \bullet e_2 \Rightarrow\!\!\!\Rightarrow A$	By above equalities
	$\Theta \vdash [\Theta]B' \bullet e_2 \Rightarrow\!\!\!\Rightarrow A' \dashv \Delta$	By i.h. with Ω'_0
☞	$A = [\Omega]A'$	"
☞	$\Delta \longrightarrow \Omega'$	"
	$\Omega'_0 \longrightarrow \Omega'$	"
	$\Omega \longrightarrow \Omega'$	By Lemma 20 (Transitivity)
☞	$\Gamma \vdash e_1\, e_2 \Rightarrow A' \dashv \Delta$	By $\to$E

- **Case** $$\dfrac{[\Omega]\Gamma \vdash e \Leftarrow B}{[\Omega]\Gamma \vdash \underbrace{B \rightarrow C}_{[\Omega]A} \bullet e \Rrightarrow C}\ \textsf{Decl}{\rightarrow}\textsf{App}$$

We have $[\Omega]A = B \rightarrow C$. Either $[\Gamma]A = B_0 \rightarrow C_0$ where $B = [\Omega]B_0$ and $C = [\Omega]C_0$—or $[\Gamma]A = \hat{\alpha}$ where $\hat{\alpha} \in \mathsf{unsolved}(\Gamma)$ and $[\Omega]\hat{\alpha} = B \rightarrow C$.

In the former case:

	$[\Omega]\Gamma \vdash e \Leftarrow B$	Subderivation
	$B = [\Omega]B_0$	Known in this subcase
	$\Gamma \longrightarrow \Omega$	Given
	$\Gamma \vdash e \Leftarrow [\Gamma]B_0 \dashv \Delta$	By i.h.
	$\Gamma \vdash ([\Gamma]B_0) \rightarrow ([\Gamma]C_0) \bullet e \Rrightarrow [\Gamma]C_0 \dashv \Delta$	By $\rightarrow$App
☞	$\Delta \longrightarrow \Omega'$	"
☞	$\Omega \longrightarrow \Omega'$	"
	Let $C' = [\Gamma]C_0$.	
	$C = [\Omega]C_0$	Known in this subcase
	$= [\Omega][\Gamma]C_0$	By Lemma 17 (Substitution Extension Invariance)
☞	$= [\Omega]C'$	$[\Gamma]C_0 = C'$
☞	$\Gamma \vdash [\Gamma](B_0 \rightarrow C_0) \bullet e \Rrightarrow [\Gamma]C_0 \dashv \Delta$	By definition of substitution

In the latter case, $\hat{\alpha} \in \mathsf{unsolved}(\Gamma)$, so the context Γ must have the form $\Gamma_0[\hat{\alpha}]$.

	$\Gamma \longrightarrow \Omega$	Given
	$\Gamma_0[\hat{\alpha}] \longrightarrow \Omega$	$\Gamma = \Gamma_0[\hat{\alpha}]$
	$[\Omega]A = B \rightarrow C$	Above
	$[\Omega]\hat{\alpha} = B \rightarrow C$	$A = \hat{\alpha}$
	$\Omega = \Omega_0[\hat{\alpha} = A_0]$ and $[\Omega]A_0 = B \rightarrow C$	Follows from $[\Omega]\hat{\alpha} = B \rightarrow C$
	Let $\Gamma' = \Gamma_0[\hat{\alpha}_2, \hat{\alpha}_1, \hat{\alpha} = \hat{\alpha}_1 \rightarrow \hat{\alpha}_2]$.	
	Let $\Omega_0' = \Omega_0[\hat{\alpha}_2 = [\Omega]C, \hat{\alpha}_1 = [\Omega]B, \hat{\alpha} = \hat{\alpha}_1 \rightarrow \hat{\alpha}_2]$.	
	$\Gamma' \longrightarrow \Omega_0'$	By Lemma 28 (Parallel Admissibility) (ii) twice
	$[\Omega]\Gamma \vdash e \Leftarrow B$	Subderivation
	$\Omega \longrightarrow \Omega_0'$	By Lemma 26 (Solved Variable Addition for Extension) then Lemma 28 (Parallel Admissibility) (iii)
	$[\Omega]\Gamma = [\Omega]\Omega$	By Lemma 48 (Completing Stability)
	$= [\Omega_0']\Omega_0'$	By Lemma 50 (Finishing Completions)
	$= [\Omega_0']\Gamma'$	By Lemma 51 (Confluence of Completeness)
	$B = [\Omega_0']\hat{\alpha}_1$	By definition of Ω_0'
	$[\Omega_0']\Gamma' \vdash e \Leftarrow [\Omega_0']\hat{\alpha}_1$	By above equalities
	$\Gamma' \vdash e \Leftarrow [\Gamma']\hat{\alpha}_1 \dashv \Delta$	By i.h.
☞	$\Delta \longrightarrow \Omega'$	"
	$\Omega_0' \longrightarrow \Omega'$	"
☞	$\Omega \longrightarrow \Omega'$	By Lemma 20 (Transitivity)
	$[\Gamma']\hat{\alpha}_1 = \hat{\alpha}_1$	$\hat{\alpha}_1 \in \mathsf{unsolved}(\Gamma')$
	$\Gamma' \vdash e \Leftarrow \hat{\alpha}_1 \dashv \Delta$	By above equality

$$\Gamma \vdash \hat{\alpha} \bullet e \Rrightarrow \hat{\alpha}_2 \dashv \Delta \qquad \text{By } \hat{\alpha}\text{App}$$

Let $C' = \hat{\alpha}_2$.

$$\begin{aligned} C &= [\Omega_0']\hat{\alpha}_2 && \text{By definition of } \Omega_0' \\ &= [\Omega']\hat{\alpha}_2 && \text{By Lemma 49 (Finishing Types)} \\ ☞ \quad &= [\Omega']C' && \text{By above equality} \end{aligned}$$

☞ $\Gamma \vdash [\Gamma]A \bullet e \Rrightarrow C' \dashv \Delta$ — $\hat{\alpha} = [\Gamma]A$ and $\hat{\alpha}_2 = C'$

- **Case**

$$\frac{}{[\Omega]\Gamma \vdash () \Rightarrow 1}\text{Decl1I}{\Rightarrow}$$

$1 = A$ — Given

$\Gamma \vdash () \Rightarrow 1 \dashv \Gamma$ — By $1\text{I}{\Rightarrow}$

Let $\Delta = \Gamma$.

Let $\Omega' = \Omega$.

$\Gamma \longrightarrow \Omega$ — Given

☞ $\Delta \longrightarrow \Omega$ — By above equality

☞ $\Omega \longrightarrow \Omega'$ — By Lemma 19 (Reflexivity)

Let $A' = 1$.

☞ $\Gamma \vdash () \Rightarrow A' \dashv \Delta$ — By above equalities

☞ $1 = [\Omega]A'$ — By definition of substitution

- **Case**

$$\frac{[\Omega]\Gamma \vdash \sigma \to \tau \qquad [\Omega]\Gamma, x : \sigma \vdash e_0 \Leftarrow \tau}{[\Omega]\Gamma \vdash \lambda x.\, e_0 \Rightarrow \sigma \to \tau}\text{Decl}{\to}\text{I}{\Rightarrow}$$

$(\sigma \to \tau) = A$ — Given

$[\Omega]\Gamma, x : \sigma \vdash e_0 \Leftarrow \tau$ — Subderivation

Let $\Gamma' = (\Gamma, \hat{\alpha}, \hat{\beta}, x : \hat{\alpha})$.

Let $\Omega_0 = (\Omega, \hat{\alpha} = \sigma, \hat{\beta} = \tau, x : \sigma)$.

$\Gamma \longrightarrow \Omega$ — Given

$\Gamma' \longrightarrow \Omega_0$ — By $\longrightarrow$Solve twice, then $\longrightarrow$Var

$[\Omega_0]\Gamma' = \big([\Omega]\Gamma, x : \sigma\big)$ — By definition of context application

$\tau = [\Omega_0]\hat{\beta}$ — By definition of Ω_0

$[\Omega_0]\Gamma' \vdash e_0 \Leftarrow [\Omega_0]\hat{\beta}$ — By above equalities

$\Gamma' \vdash e_0 \Leftarrow \hat{\beta} \dashv \Delta'$ — By i.h.

$\Delta' \longrightarrow \Omega_0'$ — "

$\Omega_0 \longrightarrow \Omega_0'$ — "

$\Delta' = (\Delta, x : \hat{\alpha}, \Theta)$ — By Lemma 23 (Extension Order) (v)

$\Gamma, \hat{\alpha}, \hat{\beta}, x : \hat{\alpha} \vdash e_0 \Leftarrow \hat{\beta} \dashv \Delta, x : \hat{\alpha}, \Theta$ — By above equalities

$(\Delta, x : \hat{\alpha}, \Theta) \longrightarrow \Omega_0'$ — By above equality

$\Omega_0' = \Omega', x : \sigma, \Omega_Z$ — By Lemma 23 (Extension Order) (v)

☞ $\Delta \longrightarrow \Omega'$ — "

$\Gamma \vdash \lambda x.\, e_0 \Rightarrow \hat{\alpha} \to \hat{\beta} \dashv \Delta$ — By $\to\text{I}{\Rightarrow}$

Let $A' = (\hat{\alpha} \to \hat{\beta})$.

☞	$\Gamma \vdash \lambda x.\, e_0 \Rightarrow A' \dashv \Delta$	By above equality
	$\sigma \to \tau = ([\Omega_0]\hat{\alpha}) \to ([\Omega_0]\hat{\beta})$	By definition of Ω_0
	$\sigma \to \tau = [\Omega_0](\hat{\alpha} \to \hat{\beta})$	By definition of substitution
	$A = [\Omega_0]A'$	By above equalities
☞	$A = [\Omega']A'$	By Lemma 49 (Finishing Types)
	$\Gamma' \longrightarrow \Delta'$	By Lemma 53 (Typing Extension)
☞	$\Omega \longrightarrow \Omega'$	By Lemma 20 (Transitivity) ☐

References

Frank Pfenning. Structural cut elimination. In *LICS*, 1995.