

A Linear-time Algorithm for
Testing the Truth of Certain Quantified
Boolean Formulas

by

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Presentation of

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The Problem

Evaluate the formula:

$$Q_1x_1Q_2x_2 \cdots Q_nx_n \quad C$$

where each Q_i is either \forall or \exists

and C is in conjunctive normal form

with at most two literals per clause.

For example

$$\exists a \exists b \forall c \exists d (a \vee b) \wedge (b \vee \bar{c}) \wedge (\bar{b} \vee \bar{d}) \wedge (b \vee d) \wedge (d \vee a)$$

Note

$a \vee b$ is equivalent to $\bar{a} \longrightarrow b$
or $\bar{b} \longrightarrow a$

$\bar{a} \vee b$ is equivalent to $a \longrightarrow b$
or $\bar{b} \longrightarrow \bar{a}$

$a \vee \bar{b}$ is equivalent to $\bar{a} \longrightarrow \bar{b}$
or $b \longrightarrow a$

$\bar{a} \vee \bar{b}$ is equivalent to $a \longrightarrow \bar{b}$
or $b \longrightarrow \bar{a}$

Graph Formation

Without loss of generality assume that all clauses of C have two literals, since $(x) = (x \vee x)$.

Assuming the formula contains n variables v_1, \dots, v_n , create $2n$ vertices named v_1, \dots, v_n and $\bar{v}_1, \dots, \bar{v}_n$.

For each clause of the form: $(a \vee b)$, add edges $\bar{a} \rightarrow b$ and $\bar{b} \rightarrow a$.

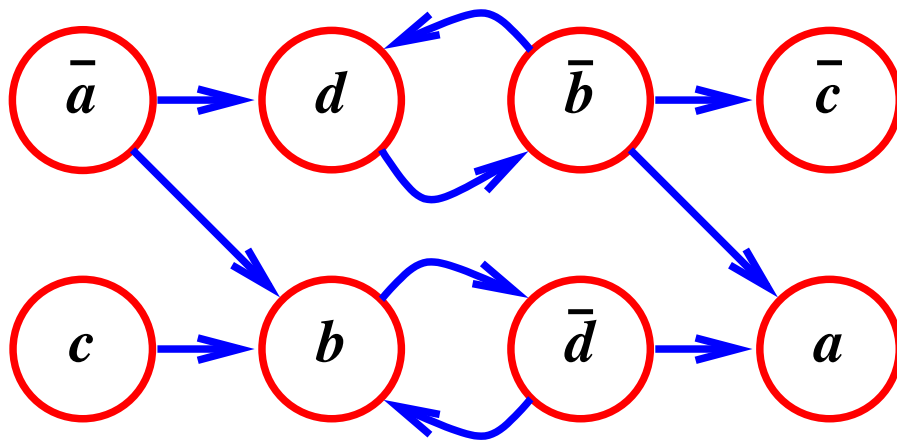
Add appropriate edges for clauses of the form:
 $(a \vee \bar{b})$, $(\bar{a} \vee b)$ and $(\bar{a} \vee \bar{b})$.

Example

If C is

$$(a \vee b) \wedge (b \vee \bar{c}) \wedge (\bar{b} \vee \bar{d}) \wedge (b \vee d) \wedge (d \vee a)$$

The graph is:



Duality Property

The constructed graph is isomorphic to one obtained by reversing all edges and complementing the names of the vertices.

Strongly Connected Components

A **strongly connected component** is a maximal subset of the vertices for which paths exists from any vertex to any other vertex.

All vertices in a strongly connected component must be assigned the same truth value ($x \longrightarrow y \longrightarrow \dots \longrightarrow x$ implies $x = y$).

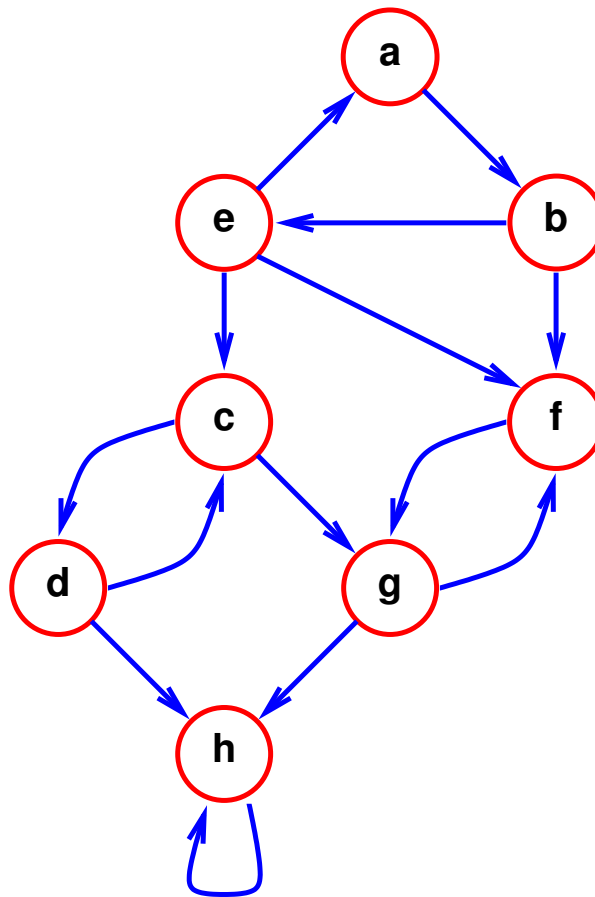
If a strongly connected component contain a variable x and its complement \bar{x} then there is an inconsistency.

All the strongly connected components can be found in linear time.

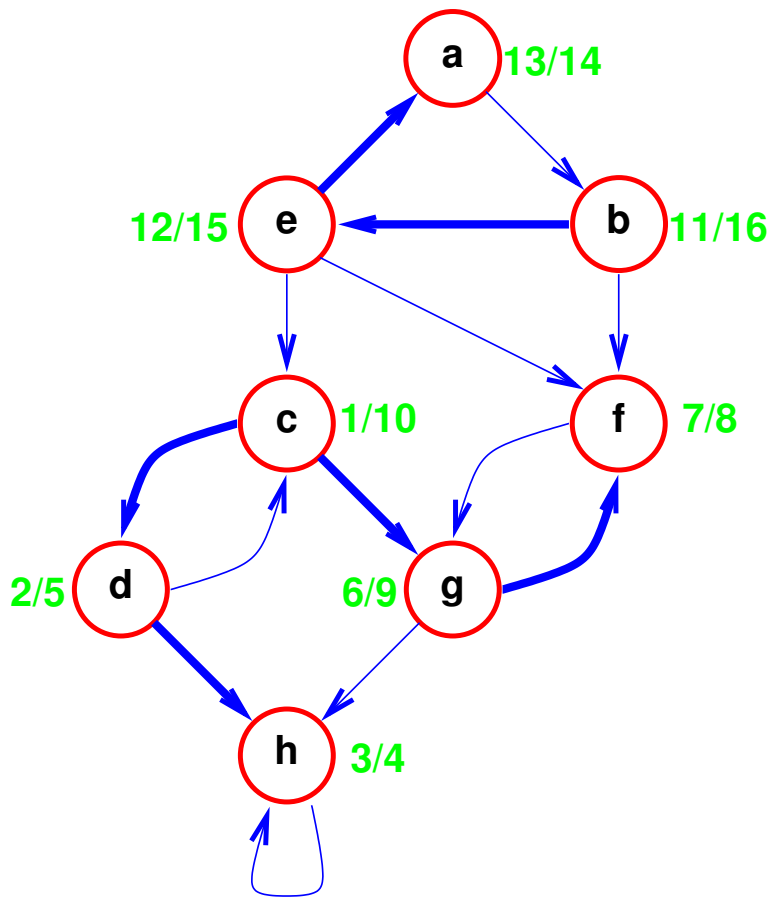
Components Algorithm

1) Perform depth first search on graph $G(V,E)$, attaching the **discovery time** ($d[u]$) and **finishing time** ($f[u]$) to every vertex (u) of G .

For example, consider



After DFS



Forefather

Define $\phi(u) = w$ (the forefather of u)

where $w \in V$

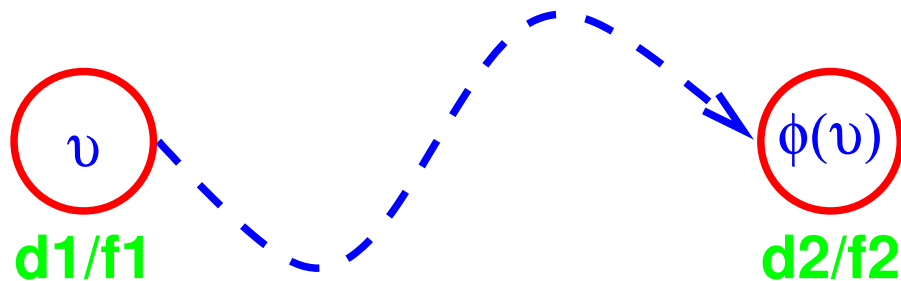
and $u \longrightarrow \dots \longrightarrow w$

and $\forall w' (u \longrightarrow \dots \longrightarrow w' \Rightarrow f[w'] \leq f[w])$

Clearly, since $u \longrightarrow \dots \longrightarrow u$

$$f[u] \leq f[\phi(u)] \quad (1)$$

Forefather Property



Either $u = \phi(u)$

or $u \neq \phi(u)$

if $d2 < f2 < d1 < f1$

or $d1 < d2 < f2 < f1$

contradicts of (1)

if $d1 < f1 < d2 < f2$

contradicts $u \longrightarrow \dots \longrightarrow \phi(u)$

so $d2 < d1 < f1 < f2$

ie $\phi(u) \longrightarrow \dots \longrightarrow u$

so u and $\phi(u)$ are in the same

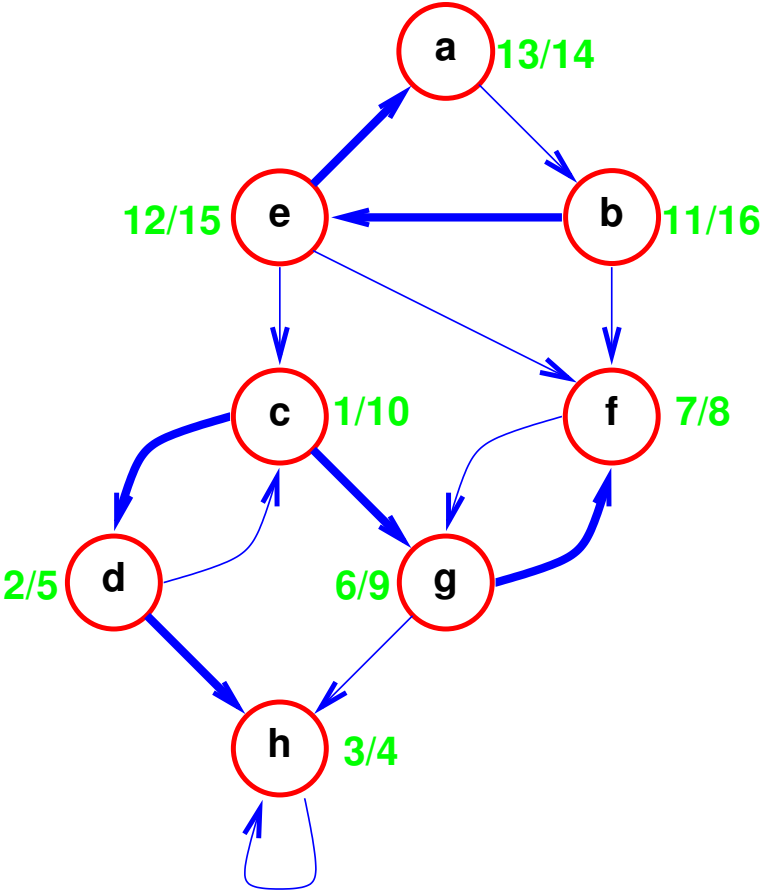
strongly connected component.

Algorithm Continued

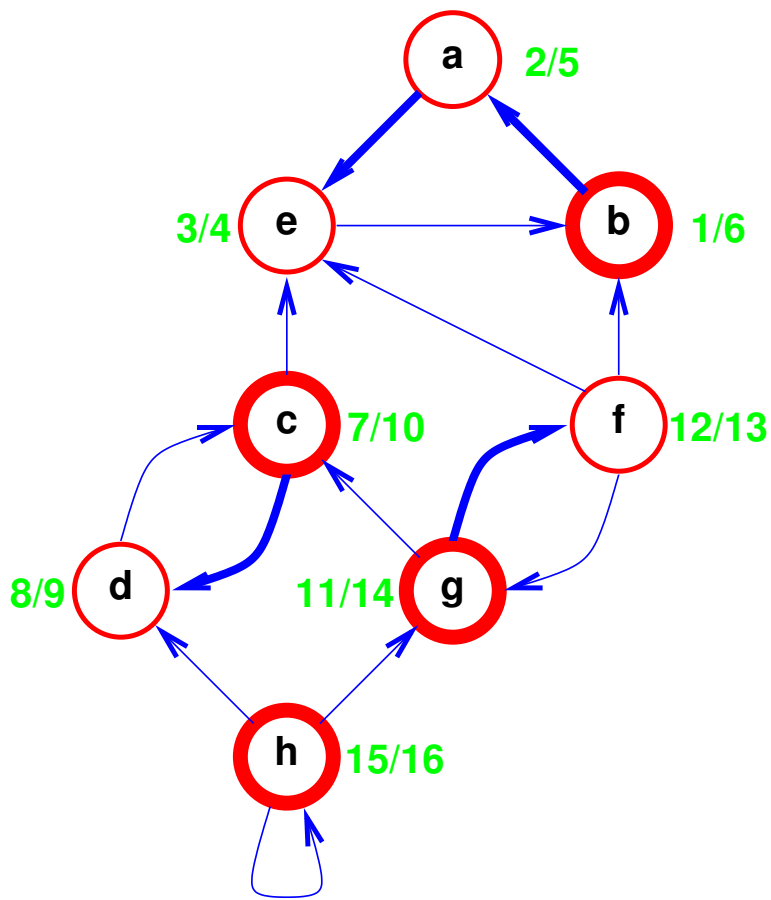
- 2) Find the vertex r with largest $f[r]$ that is not in any strongly connected component so far identified. Note that r is a forefather.
- 3) Form the set of vertices $\{u | \phi(u) = r\}$ – i.e. the strongly connected component containing r . This set is the same as $\{u | u \longrightarrow \dots \longrightarrow r\}$. This set is the set of vertices reachable from r in the graph $G^T = G$ with all its edges reversed. This set can be found using DFS on G^T .
- 4) Repeat from (2) until all components have been found.

The complexity is $O(|V| + |E|)$.

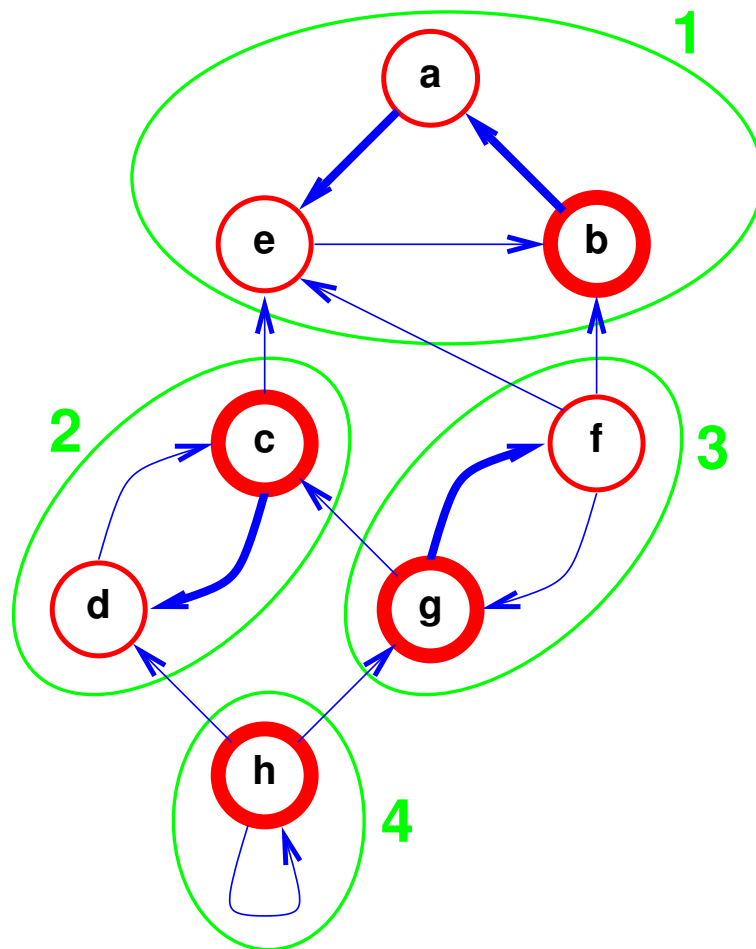
Recall



After DFS on G^T



Components



Satisfiability

If all the quantifiers are \exists , then we just have to determine whether it is possible to assign truth values to the vertices with the following properties:

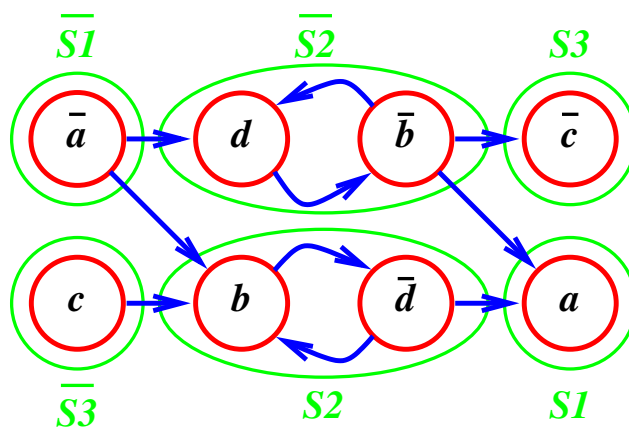
- Complementary vertices must be assigned complementary truth values,
- no edge $u \rightarrow v$ can have u assigned true and v assigned false.

Example (again)

If C is

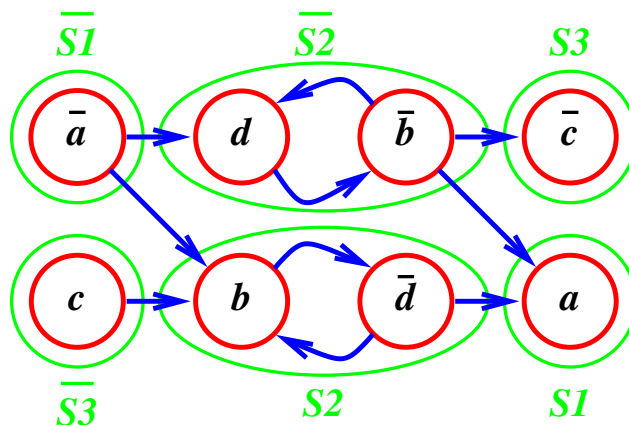
$$(a \vee b) \wedge (b \vee \bar{c}) \wedge (\bar{b} \vee \bar{d}) \wedge (b \vee d) \wedge (d \vee a)$$

The graph is:



C is satisfiable if and only if no vertex x is in the same strong component as \bar{x} .

Assignment Algorithm



Consider strongly connected components in reverse topological order:

- $S1$ Mark $S1$ true and $\bar{S1}$ false,
- $S2$ Mark $S2$ true and $\bar{S2}$ false,
- $S3$ Mark $S3$ true and $\bar{S3}$ false,
- $\bar{S2}$ Already marked,
- $\bar{S1}$ Already marked,
- $\bar{S3}$ Already marked.

General Case

We call a vertex **universal** if the corresponding variable is universally qualified, and **existential** otherwise.

A formula F is true if and only if none of the following conditions holds:

1. A vertex x is in the same strongly connected component as its complement \bar{x} .
2. An existential vertex x occurs in the same strongly connected component as a universally declared vertex y , with x declared before y . For example: $\dots \exists x \dots \forall y \dots C$.
3. There is a path from a universal vertex x to another universal vertex y . For example: $\dots \forall x \dots \forall y \dots C$.

The Algorithm

Initially all the strongly connected components are unmarked, but become marked **true**, **false** or **contingent** as the algorithm proceeds.

- 1) **Let** S **be** the next unmarked strongly connected component, chosen in reverse topological order. **If** there is no such S **return** “**F is true**”. **If** S contains a variable x and its complement \bar{x} , **return** “**F is false**”.
- 2) **If** S has some **false** or **contingent** successor,
 - 2.1) **If** S contains at least one universal vertex, **return** “**F is false**”.
 - 2.2) **If** $\bar{S} \rightarrow \dots \rightarrow S$, **return** “**F is false**”.
Mark S **false** and \bar{S} **true** then **goto** (1).
- 3) // All successors, if any, are marked **true**.
If S contains two or more universal vertices, **return** “**F is false**”.

The Algorithm (cont.)

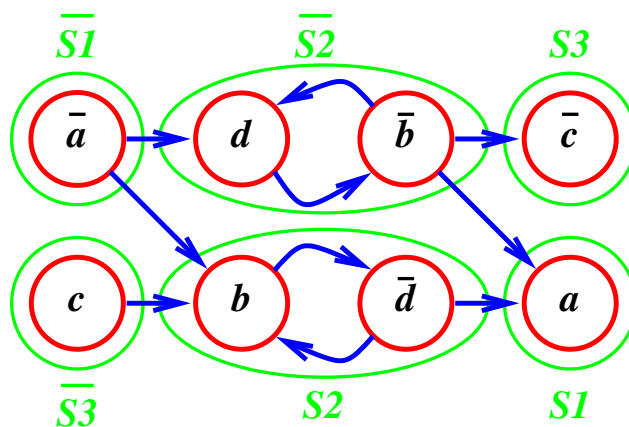
- 4) // All successors, if any, are marked **true** and
// S contains less than two universal vertices.
If S contains no universal vertices, **mark** S
true and \bar{S} **false** then **goto** (1).
- 5) // S has just one universal vertex, y say.
If S also contains an existential vertex x
declared before the universal vertex y ,
(i.e. $\dots \exists x \dots \forall y \dots C$), **return** “F is false”.
If $\bar{S} \rightarrow \dots \rightarrow S$, **return** “F is false”.
Mark S and \bar{S} **contingent** and **goto** (1).

Example (again)

If the formula is

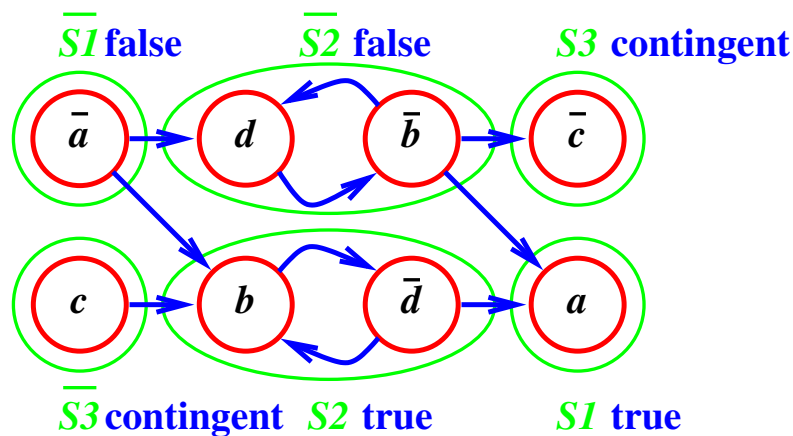
$$\exists a \exists b \forall c \exists d (a \vee b) \wedge (b \vee \bar{c}) \wedge (\bar{b} \vee \bar{d}) \wedge (b \vee d) \wedge (d \vee a)$$

The graph is:



Algorithm Trace

Formula: $\exists a \exists b \forall c \exists d \quad C$



Consider strongly connected components in reverse topological order:

- $S1$ Mark $S1$ true and $\bar{S}1$ false,
- $S2$ Mark $S2$ true and $\bar{S}2$ false,
- $S3$ Mark $S3$ and $\bar{S}3$ contingent,
- $\bar{S}2, \bar{S}1, \bar{S}3$ Already marked.

So the formula is true.