Algebraic Theories and Control Effects, Back and Forth

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an example of
the unreasonable effectiveness of mathematics
in computer science and logic

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Universal Algebra and Computational Effects

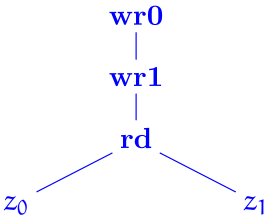
Universal Algebra

[Birkhoff (1935)]

- Signatures

  $\mathbf{wr0} : 1\ ,\ \mathbf{wr1} : 1\ ,\ \mathbf{rd} : 2$

- Free algebras



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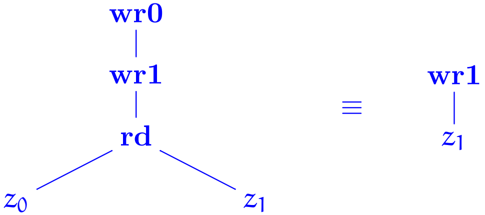


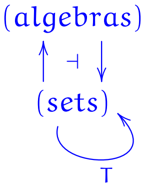
- Equational theories

$$\mathbf{wr0}\big(\mathbf{wr1}(z)\big) \equiv \mathbf{wr1}(z) \quad , \quad \mathbf{wr1}\big(\mathbf{wr0}(z)\big) \equiv \mathbf{wr0}(z)$$
$$\mathbf{wr0}\big(\mathbf{rd}(z_0, z_1)\big) \equiv \mathbf{wr0}(z_0) \quad , \quad \mathbf{wr1}\big(\mathbf{rd}(z_0, z_1)\big) \equiv \mathbf{wr1}(z_1)$$

Notions of Computation

[Moggi (1990); Plotkin, Power (2002)]

- Free-algebra monads



- Computational metalanguage

$$\frac{\Gamma \vdash_{\mathsf{c}} t : \tau}{\Gamma \vdash_{\mathsf{c}} \mathbf{wr0}_{\tau}(t) : \tau} \qquad \frac{\Gamma \vdash_{\mathsf{c}} t : \tau}{\Gamma \vdash_{\mathsf{c}} \mathbf{wr1}_{\tau}(t) : \tau}$$

$$\frac{\Gamma \vdash_{\mathsf{c}} t_0 : \tau \qquad \Gamma \vdash_{\mathsf{c}} t_1 : \tau}{\Gamma \vdash_{\mathsf{c}} \mathbf{rd}_{\tau}(t_0, t_1) : \tau}$$

- Denotational semantics

$$[\![\, \Gamma \vdash_c t : \tau \,]\!] : [\![\, \Gamma \,]\!] \to T[\![\, \tau \,]\!]$$

- $[\![\, \mathbf{wr0}_\tau(t) \,]\!] = [\![\, \Gamma \,]\!] \xrightarrow{[\![\, t \,]\!]} T[\![\, \tau \,]\!] \xrightarrow{[\![\, \mathbf{wr0} \,]\!]} T[\![\, \tau \,]\!]$
- $[\![\, \mathbf{wr1}_\tau(t) \,]\!] = [\![\, \Gamma \,]\!] \xrightarrow{[\![\, t \,]\!]} T[\![\, \tau \,]\!] \xrightarrow{[\![\, \mathbf{wr1} \,]\!]} T[\![\, \tau \,]\!]$
- $[\![\, \mathbf{rd}_\tau(t_0, t_1) \,]\!] = [\![\, \Gamma \,]\!] \xrightarrow{\langle [\![\, t_0 \,]\!], [\![\, t_1 \,]\!] \rangle} T[\![\, \tau \,]\!] \times T[\![\, \tau \,]\!] \xrightarrow{[\![\, \mathbf{rd} \,]\!]} T[\![\, \tau \,]\!]$

- Equational theory
  - $\mathbf{let}\ x = \mathbf{wr0}_\sigma(s)\ \mathbf{in}\ t[x] \ \equiv\ \mathbf{wr0}_\tau\big(\mathbf{let}\ x = s\ \mathbf{in}\ t[x]\big)$
  - $\mathbf{let}\ x = \mathbf{wr1}_\sigma(s)\ \mathbf{in}\ t[x] \ \equiv\ \mathbf{wr1}_\tau\big(\mathbf{let}\ x = s\ \mathbf{in}\ t[x]\big)$
  - $\mathbf{let}\ x = \mathbf{rd}_\sigma(s_0, s_1)\ \mathbf{in}\ t[x]$
    $\equiv\ \mathbf{rd}_\tau\big(\mathbf{let}\ x = s_0\ \mathbf{in}\ t[x]\ ,\ \mathbf{let}\ x = s_1\ \mathbf{in}\ t[x]\big)$

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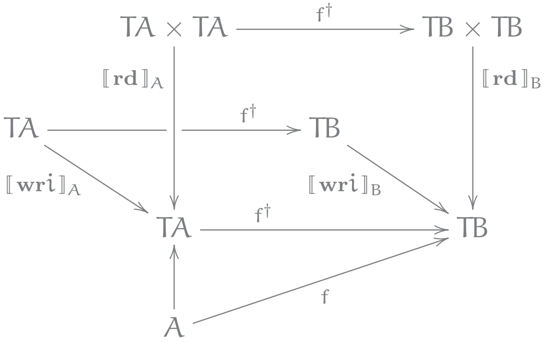

Second-Order Algebraic Theories

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- Binding signatures [Aczel (1978)]

$\mathbf{app} : (0, 0)\ ,\ \ \mathbf{abs} : (1)$

Second-Order Algebraic Theories

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- Free algebras [Fiore, Plotkin, Turi (1999); Hamana (2004); Fiore (2008)]

  $$t \ ::= \ \underline{x} \ | \ \mathsf{M}[t_1, \ldots, t_n] \ | \ \mathbf{app}(t_1, t_2) \ | \ \mathbf{abs}(x.t')$$

  $\rightsquigarrow$ substitution structure

[Fiore, Hur (2010); Fiore, Mahmoud (2010)]

- Second-order equational theories

$$(\beta) \quad \mathbf{app}\Big(\mathbf{abs}(x.\,\mathsf{M}[\underline{x}]),\ \mathsf{N}[\,]\Big) \ \equiv\ \mathsf{M}\Big[\mathsf{N}[\,]\Big]$$

$$(\eta) \quad \mathbf{abs}\Big(x.\,\mathbf{app}(\mathsf{F}[\,],\ \underline{x})\Big) \ \equiv\ \mathsf{F}[\,]$$

[Fiore, Hur (2010); Fiore, Mahmoud (2010)]

- Second-order equational theories

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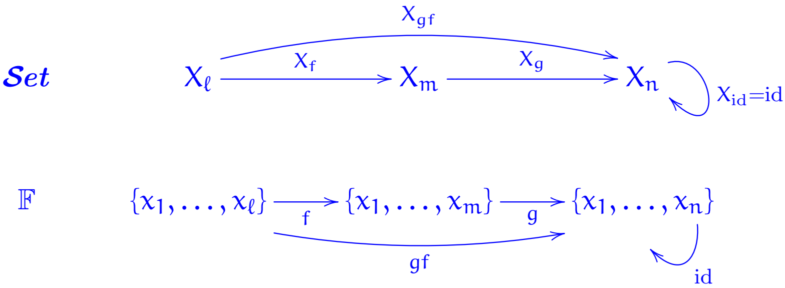
$$(\eta) \quad \mathbf{abs}\Big(x.\,\mathbf{app}(\mathsf{F}[\,],\ \underline{x})\Big) \ \equiv\ \mathsf{F}[\,]$$

- Free-algebra monads

$$\begin{array}{c} (\text{second-order algebras}) \\ \uparrow \ \dashv \ \downarrow \\ (\text{context-indexed sets}) \\ \circlearrowright \end{array}$$

$\mathcal{S}et^{\mathbb{F}}$

- Context-indexed sets



$\mathcal{S}et^{\mathbb{F}}$

- Context-indexed sets

$$\mathcal{S}et \qquad X_\ell \xrightarrow{X_f} X_m \xrightarrow{X_g} X_n \qquad X_\ell \xrightarrow{X_{gf}} X_n \qquad X_n \xrightarrow{X_{id}=id} X_n$$

$$\mathbb{F} \qquad \{x_1,\dots,x_\ell\} \xrightarrow[f]{} \{x_1,\dots,x_m\} \xrightarrow[g]{} \{x_1,\dots,x_n\} \qquad \{x_1,\dots,x_\ell\} \xrightarrow[gf]{} \{x_1,\dots,x_n\} \qquad \{x_1,\dots,x_n\} \xrightarrow[id]{} \{x_1,\dots,x_n\}$$

★ **Examples**

$$\vec{S} \qquad S = S = \cdots \qquad S \qquad \cdots$$

$$V \qquad \emptyset \longrightarrow \{x_1\} \rightrightarrows \cdots \qquad \{x_1,\dots,x_n\} \qquad \cdots$$

- Renaming-invariant functions

$$
\begin{array}{ccccc}
Y & \qquad & Y_m & \xrightarrow{Y_g} & Y_n \\
\downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi_m} & & \downarrow{\scriptstyle \varphi_n} \\
X & & X_m & \xrightarrow{X_g} & X_n
\end{array}
$$

$$\{x_1, \ldots, x_m\} \xrightarrow[g]{} \{x_1, \ldots, x_n\}$$

- Renaming-invariant functions

$$
\begin{array}{c}
Y \\
\varphi \downarrow \\
X
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y_m & \xrightarrow{Y_g} & Y_n \\
\varphi_m \downarrow & & \downarrow \varphi_n \\
X_m & \xrightarrow{X_g} & X_n
\end{array}
$$

$$\{x_1, \ldots, x_m\} \xrightarrow[g]{} \{x_1, \ldots, x_n\}$$

- **NB**

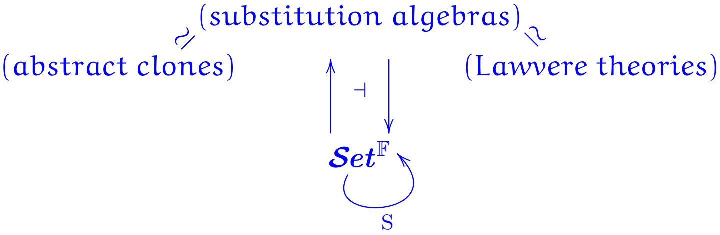
$$
\begin{array}{lccccccc}
1 & \{*\} & = & \{*\} & = & \cdots & \{*\} & \cdots \\
\nexists \downarrow & & & & & & & \\
V & \emptyset & \longrightarrow & \{x_1\} & \rightrightarrows & \cdots & \{x_1, \ldots, x_n\} & \cdots \\
\nexists \downarrow & & & & & & & \\
0 & \emptyset & = & \emptyset & = & \cdots & \emptyset & \cdots
\end{array}
$$

Substitution Algebras

Substitution Algebras

[Fiore, Plotkin, Turi (1999)]

- Free substitution-algebra monad



Substitution Algebras

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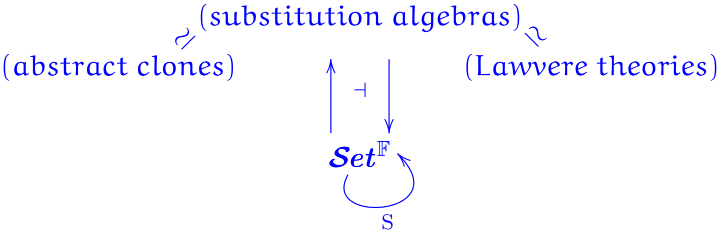

★ **Definition**

A *substitution algebra* is a structure

$$V \xrightarrow{\mathbf{var}} A \xleftarrow{\mathbf{sub}} A^V \times A \quad \text{in } \mathcal{S}et^{\mathbb{F}}$$

subject to four axioms: substitution, weakening, extensionality, and associativity.

▸ Substitution Algebra Axioms

Computational Interpretation

of

Substitution Algebras

Computational Interpretation

[Fiore, Staton (2014)]

- Computational metalanguage

$$\frac{\Gamma \vdash_{\mathsf{v}} e : \mathbf{V}}{\Gamma \vdash_{\mathsf{c}} \mathbf{var}_{\tau}(e) : \tau} \qquad \frac{\Gamma, a : \mathbf{V} \vdash_{\mathsf{c}} t : \tau \quad \Gamma \vdash_{\mathsf{c}} u : \tau}{\Gamma \vdash_{\mathsf{c}} \mathbf{sub}(a.\,t, u) : \tau}$$

Computational Interpretation

[Fiore, Staton (2014)]

- Computational metalanguage

$$\frac{\Gamma \vdash_{\mathsf{v}} e : \mathbf{V}}{\Gamma \vdash_{\mathsf{c}} \mathbf{var}_{\tau}(e) : \tau} \qquad \frac{\Gamma, a : \mathbf{V} \vdash_{\mathsf{c}} t : \tau \quad \Gamma \vdash_{\mathsf{c}} u : \tau}{\Gamma \vdash_{\mathsf{c}} \mathbf{sub}(a.\, t, u) : \tau}$$

- Denotational semantics

computations values

$$\mathrm{Kl}(S) \underset{\longleftarrow}{\overset{\longrightarrow}{\top}} \boldsymbol{\mathcal{S}et}^{\mathbb{F}} \circlearrowleft S$$

- Equational theory (I)
  - $\mathbf{let}\ x = \mathbf{sub}\big(\, a.\, \mathsf{M}[a]\,,\ \mathsf{N}[\,]\,\big)\ \mathbf{in}\ \mathsf{P}[x]$

    $\equiv\ \mathbf{sub}\big(\, a.\, \mathbf{let}\ x = \mathsf{M}[a]\ \mathbf{in}\ \mathsf{P}[x]\ ,\ \mathbf{let}\ x = \mathsf{N}[\,]\ \mathbf{in}\ \mathsf{P}[x]\ \big)$

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  - Weakening axiom

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    $\mathbf{sub}\big(\, a.\, \mathsf{M}[\,]\,,\ \mathsf{N}[\,]\,\big) \equiv \mathsf{M}[\,]$
  - Substitution axiom
    $\mathbf{sub}\big(\, a.\, \mathbf{var}(a)\,,\ \mathsf{N}[\,]\,\big) \equiv \mathsf{N}[\,]$

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  - Substitution axiom
    $$\textbf{sub}\big(\, a.\, \textbf{var}(a)\,,\ \mathsf{N}[\,]\,\big) \equiv \mathsf{N}[\,]$$
  - Abort law
    $$\big(\textbf{let}\ x = \textbf{var}(a)\ \textbf{in}\ \mathsf{P}[x]\big)\ \equiv\ \textbf{var}(a)$$

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  - Substitution axiom
    $\mathbf{sub}(\, a.\, \mathbf{var}(a)\,,\ \mathsf{N}[\,]\,\big) \equiv \mathsf{N}[\,]$
  - Abort law
    $\big(\,\mathbf{let}\ x = \mathbf{var}(a)\ \mathbf{in}\ \mathsf{P}[x]\big)\ \equiv\ \mathbf{var}(a)$
- Operational intuition

  In $\mathbf{sub}(a.t[a], u)$ the *jump point* a is declared and the computation proceeds as in $t[a]$ leading to a value if this produces one, or *aborting and restarting* with the computation of u if $\mathbf{var}(a)$ is invoked.

- Equational theory (II)
  - Extensionality axiom

    $$\mathbf{sub}\big(\, a.\,\mathsf{M}[a]\,,\, \mathbf{var}(b)\,\big) \;\equiv\; \mathsf{M}[b]$$

  - Associativity axiom

    $$\mathbf{sub}\big(\, a.\,\mathbf{sub}\big(\, b.\,\mathsf{L}[a,b]\,,\, \mathsf{M}[a]\,\big)\,,\, \mathsf{N}[\,]\,\big)$$
    $$\equiv\; \mathbf{sub}\big(\, b.\,\mathbf{sub}(\, a.\,\mathsf{L}[a,b]\,,\, \mathsf{N}[\,]\,)\,,\, \mathbf{sub}(\, a.\,\mathsf{M}[a]\,,\, \mathsf{N}[\,]\,)\,\big)$$

★ Substitution, Jumps, and Algebraic Effects

[Fiore, Staton (2014)]

- Computational interpretation of substitution algebras as a code-jumping mechanism.
- Adequate denotational semantics, equational theory, and abstract machine.
- Representation of first-order virtual effects.
- Extension incorporating effect handlers.

CPS Interpretation

of

Substitution Algebras

Algebraic CPS Semantics

★ **Theorem** [Lawvere (1969), Kock (1970)]

In the context of strong monads,

$$\frac{\mathsf{T}\text{-algebra structures on objects } A}{\text{monad morphisms } \mathsf{T} \to \mathrm{K}_A}$$

for K_A the double-dualization or continuation monad relative to A.

Algebraic CPS Semantics

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for K_A the double-dualization or continuation monad relative to A.

- Algebraic CPS semantics:

$$\frac{\mathrm{V} \text{ is the initial substituion algebra}}{\text{monad morphism } \mathrm{S} \to \mathrm{K}_\mathrm{V}}$$

CPS Interpretation

$$\mathbf{var}_X \longmapsto \lambda a : V.\, \lambda k : V^X.\, a$$

$$\mathbf{sub}_X \longmapsto \lambda \langle M : (K_V X)^V, N : K_V X \rangle.\, \lambda k : V^X.\, M\,(N\,k)\,k$$

$$S \longrightarrow K_V$$

CPS Interpretations

$$\mathbf{var}_X \longmapsto \lambda a : V.\, \lambda k : V^X.\, a$$

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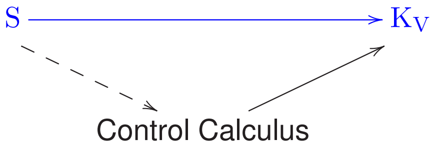

CPS Interpretation in ML

```
signature SubstitutionAlgebra
= sig
    type V
    val var : V -> 'a
    val sub : ( V -> 'a ) * 'a -> 'a
end
```

CPS Interpretation in ML

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signature SubstitutionAlgebra
= sig
    type V
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end

structure sa :> SubstitutionAlgebra
= struct
    type R = unit cont
    type V = unit -> R
    fun var( a ) = throw (a()) ()
    fun sub( M , N )
      = callcc( fn k =>  M( fn() => throw k N ) )
  end
```

```
signature ParameterisedSubstitutionAlgebra
= sig
    type 'p V
    val var : 'p V * 'p -> 'a
    val sub : ( 'p V -> 'a ) * ('p -> 'a )  -> 'a
end

structure psa :> ParameterisedSubstitutionAlgebra
= struct
    type R = unit cont
    type 'p V = 'p -> R
    fun var( a , p ) = throw (a p) ()
    fun sub( M , N )
      = callcc( fn k =>  M( fn p => throw k (N p) ) )
  end
```

Inception Algebras

Inception Algebras

★ **Definition**

An (L, P)-*inception algebra* is a structure

$$\mathsf{L} \times \mathsf{P} \xrightarrow{\textbf{rec}\text{all}} A \xleftarrow{\textbf{inc}\text{ept}} A^{\mathsf{L}} \times A^{\mathsf{P}}$$

Inception Algebras

★ **Definition**

An (L, P)-*inception algebra* is a structure

$$L \times P \xrightarrow{\mathbf{recall}} A \xleftarrow{\mathbf{incept}} A^L \times A^P$$

subject to:

- Substitution axiom

  $$\mathbf{inc}\big(\ell.\,\mathbf{rec}(\ell, \mathsf{P}[\,])\,,\ x.\,\mathsf{L}[x]\,\big) \equiv \mathsf{L}\big[\mathsf{P}[\,]\big]$$

- Extensionality axiom

  $$\mathbf{inc}\big(\ell.\,\mathsf{P}[\ell]\,,\ x.\,\mathbf{rec}(k, x)\,\big) \equiv \mathsf{P}[k]$$

- Weakening axiom

  $$\mathbf{inc}\big(\ell.\,\mathsf{M}[\,]\,,\ x.\,\mathsf{L}[x]\,\big) \equiv \mathsf{M}[\,]$$

- Associativity axiom

  $$\mathbf{inc}\big(\ell.\,\mathbf{inc}\big(k.\,\mathsf{P}[\ell, k]\,,\ x.\,\mathsf{K}[\ell, x]\,\big)\,,\ y.\,\mathsf{L}[y]\,\big)$$
  $$\equiv\ \mathbf{inc}\big(k.\,\mathbf{inc}(\,\ell.\,\mathsf{P}[\ell, k]\,,\ y.\,\mathsf{L}[y]\,)\,,\ x.\,\mathbf{inc}(\,\ell.\,\mathsf{K}[\ell, x]\,,\ y.\,\mathsf{L}[y]\,)\,\big)$$

★ Examples

- Substitution algebras = $(\mathrm{V}, 1)$-inception algebras

★ Examples

- Substitution algebras = $(V, 1)$-inception algebras
- More generally, for a set D, $(V, \vec{D})$-inception algebras model a <u>*code-jumping with data-passing*</u> mechanism:

$$\frac{\Gamma \vdash_{\mathsf{v}} e : \mathbf{V} \qquad \Gamma \vdash_{\mathsf{v}} d : \mathbf{D}}{\Gamma \vdash_{\mathsf{c}} \mathbf{rec}_{\tau}(e, d) : \tau}$$

$$\frac{\Gamma, \ell : \mathbf{V} \vdash_{\mathsf{c}} t : \tau \qquad \Gamma, x : \mathbf{D} \vdash_{\mathsf{c}} u : \tau}{\Gamma \vdash_{\mathsf{c}} \mathbf{inc}_{\tau}\big(\ell.\, t\,, x.\, u\big) : \tau}$$

Inception Algebras in Logic

- $L = \neg P$

Negation Elimination	Excluded Middle
$$\frac{\neg P \qquad P}{A}$$	$$\frac{\begin{matrix}[\neg P] \\ \vdots \\ A\end{matrix} \qquad \begin{matrix}[P] \\ \vdots \\ A\end{matrix}}{A}$$

[de Groote (1995)]

NB Logical inceptions are not local ML exceptions.

▸ Counterexample

- Algebraic CPS semantics
  - (R^P, P)-inception algebra structure on R:

$$\begin{cases} \textbf{rec} & \mapsto & \lambda\langle \ell : R^P, p : P\rangle.\, \ell\, p \\ \textbf{inc} & \mapsto & \lambda\langle M : R^{(R^P)}, N : R^P\rangle.\, M\, N \end{cases}$$

- Algebraic CPS semantics
  - (R^P, P)-inception algebra structure on R:

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  - Induced (R^P, P)-inception algebra structure on $K_R X$:

$$\begin{cases} \mathbf{rec}_X & \mapsto & \lambda\langle \ell : R^P, p : P\rangle.\, \lambda k : R^X.\, \ell\, p \\ \mathbf{inc}_X & \mapsto & \lambda\langle M : (K_R X)^{(R^P)}, N : (K_R X)^P\rangle.\, \lambda k : R^X. \\ & & \quad M\, (\lambda p : P.\, N\, p\, k)\, k \end{cases}$$

- Programming idiom

```
signature LogicalInceptionAlgebra
= sig
    type 'p linc
    val rec : 'p linc * 'p -> 'a
    val inc : ( 'p linc -> 'a ) * ( 'p -> 'a )  -> 'a
  end
```

▸ Logical Inception Algebra Structure

- Programming idiom

```
signature LogicalInceptionAlgebra
= sig
    type 'p linc
    val rec : 'p linc * 'p -> 'a
    val inc : ( 'p linc -> 'a ) * ( 'p -> 'a )  -> 'a
  end
```

- **NB** `inc` generalises and introduces a level of abstraction over `callcc`:

$$\mathtt{callcc}(f) = \mathtt{inc}\big(f, \mathrm{id}\big)$$

$$\mathtt{inc}\big(f, h\big) = \mathtt{callcc}\big(f \circ \neg h\big)$$

- Programming idiom

```
signature LogicalInceptionAlgebra
= sig
    type 'p linc
    val rec : 'p linc * 'p -> 'a
    val inc : ( 'p linc -> 'a ) * ( 'p -> 'a )  -> 'a
  end
```

- **Applications**
  - Classical logic De Morgan
  - Encoding of local and global exception mechanisms
  - Safe exception handling in program modules
  - Coroutines

Algebraically,

CPS $=$ NE $+$ EM

Untyped Inception Algebras

- $(\mathrm{V}, \mathrm{V}^n)$-inception algebras

$$\mathrm{V} \times \mathrm{V}^n \xrightarrow{\ \mathbf{rec}\ } A \xleftarrow{\ \mathbf{inc}\ } A^{\mathrm{V}} \times A^{\mathrm{V}^n}$$

Untyped Inception Algebras

- $(\mathrm{V}, \mathrm{V}^n)$-inception algebras

$$\mathrm{V} \times \mathrm{V}^n \xrightarrow{\mathbf{rec}} A \xleftarrow{\mathbf{inc}} A^{\mathrm{V}} \times A^{\mathrm{V}^n}$$

model the *untyped* CPS calculus[Appel (1992); Thielecke (1997)].

- Conversion table:

$\mathbf{rec}(\ell, \vec{x})$	$\ell\langle\vec{x}\rangle$
$\mathbf{inc}(\ell.\, \mathsf{P}[\ell],\, \vec{x}.\, \mathsf{L}[\vec{x}])$	$\mathsf{P}[\ell]\,\{\,\ell\langle\vec{x}\rangle = \mathsf{L}[\vec{x}]\,\}$

Sorted Inception Algebras

- Sorted sets [Milner (1991)]

$$|-| : S \rightharpoonup S^*$$

- Structures

$$V_\sigma \times \prod_i V_{|\sigma|_i} \longrightarrow A \longleftarrow A^{V_\sigma} \times A^{\prod_i V_{|\sigma|_i}} \quad \text{in } \mathcal{S}et^{\mathbb{F}[S]}$$

subject to the inception algebra axioms. ‣ Inception Algebra Axioms

★ **Example**

For a set of sorts S, the sorted set

$$T_S \rightharpoonup {T_S}^*, \text{ where } T_S = \mu X.\, S + X^*$$

yields the typed CPS calculus, and hence sorted $\otimes\neg$-categories[Thielecke (1997)].

Back to Control Effects

Some Developments

- Untyped inception algebras ‣ Untyped Inception Algebras
  - Recursion. ‣ Recursion
  - Algebraic interpretation of the lambda calculus in the untyped CPS calculus. ‣ CPS Lambda Structure
- Right Lambda Algebras ‣ Right Lambda Algebras
  - Computational interpretation as a mechanism for stack manipulation of code pointers. [Fiore, Staton (2014)]
  - Stack abstract machine for the untyped CPS calculus, and hence also for the lambda calculus. ‣ Stack Abstract Machine
- Left Lambda Algebras ‣ Left Lambda Algebras
  - Computational interpretation as a synchronous coroutine mechanism. ‣ Coroutine Mechanism
  - Producer-Consumer programming pattern. ‣ Producer-Consumer

Final Remarks

Final Remarks

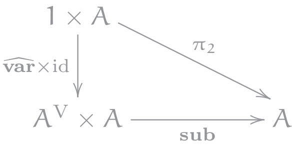
- Conclusions
  - The first algebraic axiomatisation of a variety of control effects.
  - Foundational analysis of the principles of programming with algebraic effects.
  - Inception algebras
- Directions
  - Semantic models
  - Classical/intermediate logics
  - Algebraic theories
  - Programming

Appendix

Substitution Algebra Axioms

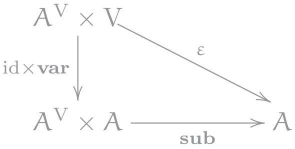
- Substitution

$$\mathbf{sub}\big(\, a.\,\mathbf{var}(a)\,,\ \mathsf{M}[\,]\,\big) \ \equiv\ \mathsf{M}[\,]$$



- Extensionality

$$\mathbf{sub}\big(\, a.\,\mathsf{M}[a]\,,\ \mathbf{var}(b)\,\big) \ \equiv\ \mathsf{M}[b]$$



- Weakening

$$\mathbf{sub}\big(\, a.\, \mathsf{M}[\,]\, ,\, \mathsf{N}[\,]\,\big) \;\equiv\; \mathsf{M}[\,]$$

$$\begin{array}{ccc}
A^{1} \times A & \xrightarrow{\;\cong\;} & A \times A \\
{\scriptstyle A^{!}\times \mathrm{id}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_1} \\
A^{V} \times A & \xrightarrow[\mathbf{sub}]{} & A
\end{array}$$

- Associativity

$$\mathbf{sub}\big(\, a.\, \mathbf{sub}\big(\, b.\, \mathsf{L}[a,b]\, ,\, \mathsf{M}[a]\,\big)\, ,\, \mathsf{N}[\,]\,\big)$$

$$\equiv\; \mathbf{sub}\big(\, b.\, \mathbf{sub}(\, a.\, \mathsf{L}[a,b]\, ,\, \mathsf{N}[\,]\,)\, ,\, \mathbf{sub}(\, a.\, \mathsf{M}[a]\, ,\, \mathsf{N}[\,]\,)\,\big)$$

$$\begin{array}{ccccc}
(A^{V} \times A)^{V} \times A & \longrightarrow & (A^{V} \times A)^{V} \times (A^{V} \times A) & \xrightarrow{\;\mathbf{sub}^{V} \times \mathbf{sub}\;} & A^{V} \times A \\
{\scriptstyle \mathbf{sub}^{V}\times \mathrm{id}}\big\downarrow & & & & \big\downarrow{\scriptstyle \mathbf{sub}} \\
A^{V} \times A & & \xrightarrow[\mathbf{sub}]{\qquad\qquad} & & A
\end{array}$$

Substitution Algebras

Contraction Laws

- $\mathbf{sub}\big(\, a.\, \mathsf{M}[a, b]\,,\ \mathbf{var}(b)\,\big) \ \equiv\ \mathsf{M}[b, b]$

- $\mathbf{sub}\big(\, a.\, \mathbf{sub}(\, b.\, \mathsf{M}[a, b]\,,\ \mathsf{N}[\,]\,)\,,\ \mathsf{N}[\,]\,\big) \ \equiv\ \mathbf{sub}\big(\, c.\, \mathsf{M}[c, c]\,,\ \mathsf{N}[\,]\,\big)$

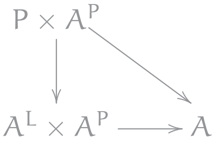
- $\mathbf{sub}\big(\, c.\, \mathbf{let}\ x = \mathsf{M}[c]\ \mathbf{in}\ \mathsf{N}[x, c]\ \ ,\ \mathsf{L}[\,]\,\big)$

  $\equiv\ \ \mathbf{sub}\big(\, a.\, \mathbf{let}\ x = \mathsf{M}[a]\ \mathbf{in}\ \mathbf{sub}\big(b.\, \mathsf{N}[x, b], \mathsf{L}[\,]\big)\ \ ,\ \mathsf{L}[\,]\,)$

Inception Algebra Axioms

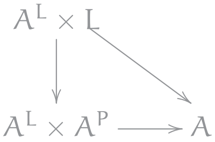
- Substitution

$$\mathbf{inc}\big(\ell.\,\mathbf{rec}(\ell, \mathsf{P}[\,])\,,\, x.\,\mathsf{L}[x]\big) \;\equiv\; \mathsf{L}[\mathsf{P}]$$



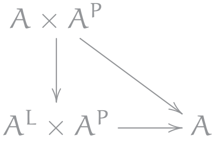
- Extensionality

$$\mathbf{inc}\big(\ell.\,\mathsf{P}[\ell]\,,\, x.\,\mathbf{rec}(k, x)\big) \;\equiv\; \mathsf{P}[k]$$



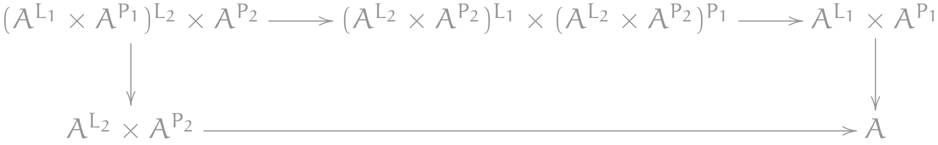
- **Weakening**

$$\mathbf{inc}\big(\ell.\,\mathsf{M}[\,]\,,\ x.\,\mathsf{L}[x]\,\big) \;\equiv\; \mathsf{M}[\,]$$



- **Associativity**

$$\mathbf{inc}\big(\,\ell.\,\mathbf{inc}\big(\,k.\,\mathsf{P}[\ell,k]\,,\ x.\,\mathsf{K}[\ell,x]\,\big)\,,\ y.\,\mathsf{L}[y]\,\big)$$
$$\equiv\ \mathbf{inc}\big(\,k.\,\mathbf{inc}(\,\ell.\,\mathsf{P}[\ell,k]\,,\ y.\,\mathsf{L}[y]\,)\,,\ x.\,\mathbf{inc}(\,\ell.\,\mathsf{K}[\ell,x]\,,\ y.\,\mathsf{L}[y]\,)\,\big)$$



Inceptions Are Not Local Exceptions

- **NB**

$$\mathbf{sub}\Big(e_1.\,\mathbf{let}\;\ell = \mathbf{sub}(\,e_2.\,\mathbf{ret}\,e_2\,,\;\mathbf{ret}\,e_1\,)\;\mathbf{in}\;\mathbf{var}\,\ell\;\;,\;\langle\,\rangle\Big)\;\equiv\;\langle\,\rangle$$

however

```
let exception e1
in ( let val l
          = let exception e2
            in e2
            handle e2 => e1 end
      in
        raise l
      end )
handle e1 => () end
```

outputs `uncaught exception e2`.

Logical Inception Algebra Structure

```
signature LogicalInceptionAlgebra
= sig
    type 'p linc
    val rec : 'p linc * 'p -> 'a
    val inc : ( 'p linc -> 'a ) * ( 'p -> 'a )  -> 'a
  end

structure lia :> LogicalInceptionAlgebra
= struct
    type R = unit cont
    type 'p linc = 'p -> R
    fun rec( a , p ) = throw (a p) ()
    fun inc( M , N )
      = callcc( fn k => M( fn p => throw k (N p) ) )
  end
```

De Morgan

```
val DeMorgan : ( 'a * 'b ) linc -> ('a linc,'b linc) sum
  = fn c =>
    inc( fn a => left a ,
         fn x => inc( fn b => right b ,
                      fn y => rec( c , (x,y) ) ) )
```

▸ Programming Idiom Applications

Nullary/Unary Untyped Inception Algebras

- Self recall

$$\mathbf{inc}\big(\,\ell.\,\mathbf{rec}(\ell,\ell)\,,\,x.\,\mathsf{M}[x]\,\big)\;\equiv\;\mathbf{inc}\big(\,\ell.\,\mathsf{M}[\ell]\,,\,x.\,\mathsf{M}[x]\,\big)$$

Nullary/Unary Untyped Inception Algebras

- Self recall

$$\mathbf{inc}\big(\,\ell.\,\mathbf{rec}(\ell,\ell)\,,\;x.\,\mathsf{M}[x]\,\big)\;\equiv\;\mathbf{inc}\big(\,\ell.\,\mathsf{M}[\ell]\,,\;x.\,\mathsf{M}[x]\,\big)$$

- Recursion

  For

$$Y_t\;=\;\mathbf{inc}\big(\,\ell.\,\mathbf{rec}(\ell,\ell)\,,\;\ell.\,\mathbf{sub}\big(\,a.\,t[a]\,,\;\mathbf{rec}(\ell,\ell)\big)\,\big)$$

  we have

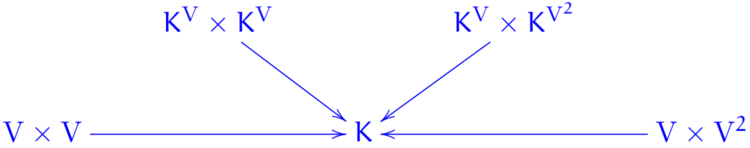
$$Y_t\;\equiv\;\mathbf{sub}\big(\,a.\,t[a]\,,\;Y_t\,\big)$$

Developments

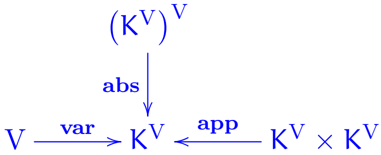
Unary/Binary Untyped Inception Algebras

- CPS lambda structure

  The initial unary/binary untyped inception algebra



provides a CPS lambda structure



thereby inducing an homomorphic CPS interpretation

$$\Lambda \to K^V$$

of lambda terms.

- CPS interpretation

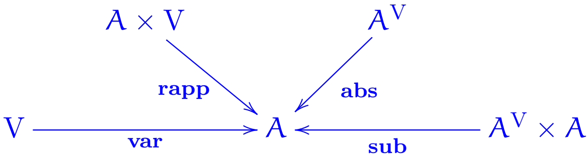
  - $\mathbf{var} : x[e] \mapsto \mathbf{rec}(\, e \,,\, \langle x \rangle \,)$

  - $\mathbf{abs} : \mathsf{M}[x][e] \mapsto \mathbf{inc}(\ \ f.\, \mathbf{rec}(\, e \,,\, \langle f \rangle \,)\ \ ,$
    $\langle a, e \rangle.\, \mathsf{M}[a][e]\ \ )$

  - $\mathbf{app} : \mathsf{M}[e], \mathsf{N}[e] \mapsto \mathbf{inc}(\ \ m.\mathsf{M}[m]\ \ ,$
    $\langle f \rangle.\, \mathbf{inc}(\ \ n.\mathsf{N}[n]\ \ ,$
    $\langle a \rangle.\, \mathbf{rec}(f, \langle a, e \rangle)\ \ )\ \ )$

Developments

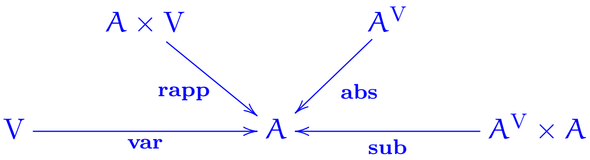
Right Lambda Algebras



$$(\beta_r) \quad \mathbf{rapp}\big(\mathbf{abs}(x.\,\mathsf{M}[x])\,,\,a\big) \;\equiv\; \mathsf{M}[a]$$

[Hirschowitz, Maggesi (2010); Hyland (2013)]

Right Lambda Algebras



$$\underbrace{(\beta_r)}_{\text{summon}} \qquad \overbrace{\mathbf{rapp}\big(\underbrace{\mathbf{abs}(x.\,\mathsf{M}[x])}_{\text{pop in } x},\ a\big)}^{\text{push } a} \equiv \mathsf{M}[a]$$

- Computational interpretation: [Fiore, Staton (2014)]

  A mechanism for stack manipulation of code pointers.

- Application:

  Stack abstract machine for CPS calculus.

- Stack CPS structure

- $$\begin{array}{rcl} V \times V^n & \longrightarrow & A \\ a, \langle a_1, \ldots, a_n \rangle & \mapsto & \mathbf{push}(\cdots \mathbf{push}(\mathbf{var}(a), a_1) \cdots, a_n) \end{array}$$

- $$\begin{array}{rcl} A^V \times A^{V^n} & \longrightarrow & A \\ \mathsf{M}[a], \mathsf{N}[a_1, \ldots, a_n] & \mapsto & \mathbf{sub}(\; a.\,\mathsf{M}[a] \;, \\ & & \quad \mathbf{pop}(a_1.\cdots \mathbf{pop}(a_n.\mathsf{N}[a_1, \ldots, a_n]))\;) \end{array}$$

▸ Developments

- Parameterised fixpoint
  - For

$$\mathsf{T}[f, x] = \mathbf{in}(\mathbf{yield}(x, \mathbf{thunk}(z.\, \mathbf{push}(z, \mathbf{push}(f, \mathbf{var}\ f)))))$$

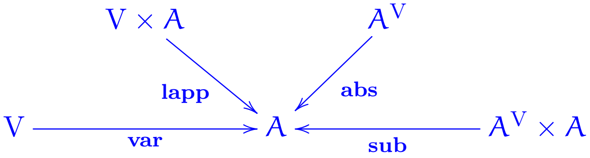
and

$$\mathbf{In}[x] = \mathbf{sub}\big(\, f.\, \mathsf{T}[f, x]\,,\ \mathbf{pop}\big(\, f.\, \mathbf{pop}(x.\, t[f, x])\,\big)\,\big)$$

we have

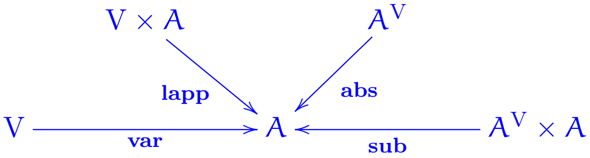
$$\mathbf{In}[x] \ \equiv\ \mathbf{in}(\mathbf{yield}(x, \mathbf{thunk}(z.\, \mathbf{In}[z])))$$

Left Lambda Algebras



$$(\beta_{\mathfrak{l}})\quad \mathbf{sub}\big(\,\ell.\,\mathbf{lapp}(\ell,\mathsf{N}[\,])\,,\ \mathbf{abs}(\,\mathsf{k}.\,\mathsf{M}[\mathsf{k}]\,)\big)\ \equiv\ \mathbf{sub}\big(\,\mathsf{k}.\,\mathsf{M}[\mathsf{k}]\,,\ \mathsf{N}[\,]\,\big)$$

Left Lambda Algebras



$$\underbrace{(\beta_l)}_{\text{resume}} \quad \mathbf{sub}\Big(\ell.\ \underbrace{\mathbf{lapp}(\ell, \mathsf{N}[\,])}_{\text{yield } \mathsf{N}},\ \underbrace{\mathbf{abs}(\,\mathrm{k}.\,\mathsf{M}[\mathrm{k}]\,)}_{\text{thunk } \mathsf{M}}\Big) \equiv \mathbf{sub}\Big(\mathrm{k}.\,\mathsf{M}[\mathrm{k}]\,,\ \mathsf{N}[\,]\Big)$$

- Computational interpretation:

  A synchronous coroutine mechanism

- Application:

  Producer-Consumer

▸ Producer-Consumer

▸ Developments

- Axioms
  - $\mathbf{sub}\big(\ell.\,\mathbf{yield}(\ell, \mathsf{N}[\,])\,,\ \mathbf{thunk}(\,\mathsf{k}.\,\mathsf{M}[\mathsf{k}]\,)\big) \;\equiv\; \mathbf{sub}\big(\,\mathsf{k}.\,\mathsf{M}[\mathsf{k}]\,,\ \mathsf{N}[\,]\big)$
  - $\mathbf{sub}\big(\,\ell.\,\mathbf{thunk}\big(x.\,\mathsf{M}[\ell, x]\big)\,,\ \mathsf{N}[\,]\,\big)$

    $\equiv\; \mathbf{thunk}\big(\,x.\,\mathbf{sub}\big(\,\ell.\,\mathsf{M}[\ell, x]\,,\ \mathsf{N}[\,]\,\big)\,\big)$
  - $\mathbf{sub}\big(\,e.\,\mathbf{yield}\big(\mathsf{L}[e], \mathsf{M}[e]\big)\,,\ \mathsf{N}[\,]\,\big)$

    $\equiv\; \mathbf{sub}\big(\ \ell.\,\mathbf{yield}(\,\ell\,,\ \mathbf{sub}\big(e.\,\mathsf{M}[e], \mathsf{N}[\,]\big)\,)\ ,$

    $\mathbf{sub}\big(\,e.\,\mathbf{var}(\mathsf{L}[e]), \mathsf{N}[\,]\,\big)\ \big)$

- Fixpoints
  - For a computation $t(\cdot)$,

    $$Y_t = \mathbf{sub}\big(\, x.\, t\big(\mathbf{yield}(x, \mathbf{var}\ x)\big)\ ,\ \mathbf{thunk}\big(\, x.\, t\big(\mathbf{yield}(x, \mathbf{var}\ x)\big)\ \big)\big)$$

    we have

    $$Y_t \equiv t(Y_t)$$

  - For a parameterised computation $t[x](c.\mathsf{M}[c])$, let

    $$\mathsf{T} = \mathbf{thunk}(f.\, \mathbf{thunk}(x.\; t[x]\big(\, c.\, \mathbf{sub}(p.\, \mathbf{yield}(p, \mathbf{var}\ c), \mathbf{yield}(f, \mathbf{var}\ f))\ \big)))$$

    For

    $$Y_t[x] = \mathbf{sub}\big(\, f.\, t[x](c.\, \mathbf{sub}(p.\mathbf{yield}(p, \mathbf{var}\ c), \mathbf{yield}(f, \mathsf{T})))\ ,\ \mathsf{T}\,\big)$$

    we have

    $$Y_t[x] \;\equiv\; t[x]\big(\, c.\, Y_t[c]\,\big)$$

- Producer-Consumer: one in, two out

  Let

$$\begin{aligned} \mathbf{In}[a] &\equiv \mathbf{in}(\mathbf{yield}(a, \mathbf{thunk}(x.\mathbf{In}[x]))) \\ \mathbf{Out} &\equiv \mathbf{thunk}(y.\mathbf{out}(\mathbf{out}(\mathbf{yield}(y, \mathbf{Out})))) \end{aligned}$$

  Then

$$\mathbf{sub}\big(\, a.\,\mathbf{In}[a]\,,\ \mathbf{Out}\,\big) \equiv \mathbf{in}(\mathbf{out}(\mathbf{out}(\mathbf{sub}(x.\mathbf{In}[x], \mathbf{Out}))))$$

▸ Developments