Algebraic Theories and Control Effects, Back and Forth

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an example of
the unreasonable effectiveness of mathematics
in computer science and logic

ALCOP
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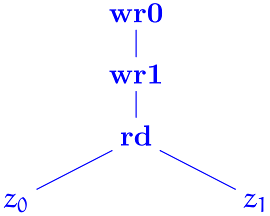
Universal Algebra

[Birkhoff (1935)]

- Signatures

  $\mathbf{wr0} : 1\ ,\ \mathbf{wr1} : 1\ ,\ \mathbf{rd} : 2$

- Free algebras



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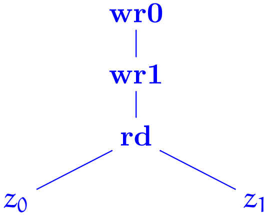

- Equational theories

$$\mathbf{wr0}\big(\mathbf{wr1}(z)\big) \equiv \mathbf{wr1}(z) \quad , \quad \mathbf{wr1}\big(\mathbf{wr0}(z)\big) \equiv \mathbf{wr0}(z)$$
$$\mathbf{wr0}\big(\mathbf{rd}(z_0, z_1)\big) \equiv \mathbf{wr0}(z_0) \quad , \quad \mathbf{wr1}\big(\mathbf{rd}(z_0, z_1)\big) \equiv \mathbf{wr1}(z_1)$$

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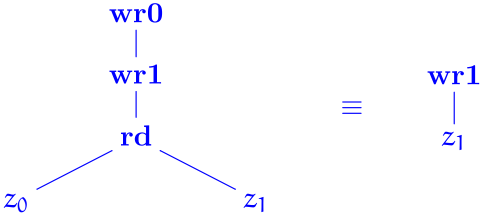

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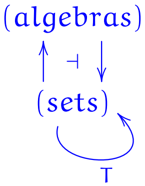
$$\mathbf{wr0}\big(\mathbf{rd}(z_0, z_1)\big) \equiv \mathbf{wr0}(z_0) \quad , \quad \mathbf{wr1}\big(\mathbf{rd}(z_0, z_1)\big) \equiv \mathbf{wr1}(z_1)$$

Notions of Computation

[Moggi (1990); Plotkin, Power (2002)]

- Free-algebra monads



- Computational metalanguage

$$\frac{\Gamma \vdash_{c} t : \tau}{\Gamma \vdash_{c} \mathbf{wr0}_{\tau}(t) : \tau} \qquad \frac{\Gamma \vdash_{c} t : \tau}{\Gamma \vdash_{c} \mathbf{wr1}_{\tau}(t) : \tau}$$

$$\frac{\Gamma \vdash_{c} t_0 : \tau \qquad \Gamma \vdash_{c} t_1 : \tau}{\Gamma \vdash_{c} \mathbf{rd}_{\tau}(t_0, t_1) : \tau}$$

- Denotational semantics

$$[\![\Gamma \vdash_c t : \tau]\!] : [\![\Gamma]\!] \to T[\![\tau]\!]$$

- $[\![\mathbf{wr0}_\tau(t)]\!] = [\![\Gamma]\!] \xrightarrow{[\![t]\!]} T[\![\tau]\!] \xrightarrow{[\![\mathbf{wr0}]\!]} T[\![\tau]\!]$
- $[\![\mathbf{wr1}_\tau(t)]\!] = [\![\Gamma]\!] \xrightarrow{[\![t]\!]} T[\![\tau]\!] \xrightarrow{[\![\mathbf{wr1}]\!]} T[\![\tau]\!]$
- $[\![\mathbf{rd}_\tau(t_0, t_1)]\!] = [\![\Gamma]\!] \xrightarrow{\langle [\![t_0]\!], [\![t_1]\!] \rangle} T[\![\tau]\!] \times T[\![\tau]\!] \xrightarrow{[\![\mathbf{rd}]\!]} T[\![\tau]\!]$

- Equational theory
  - $\mathbf{let}\ x = \mathbf{wr0}_\sigma(s)\ \mathbf{in}\ t[x] \ \equiv\ \mathbf{wr0}_\tau\big(\mathbf{let}\ x = s\ \mathbf{in}\ t[x]\big)$
  - $\mathbf{let}\ x = \mathbf{wr1}_\sigma(s)\ \mathbf{in}\ t[x] \ \equiv\ \mathbf{wr1}_\tau\big(\mathbf{let}\ x = s\ \mathbf{in}\ t[x]\big)$
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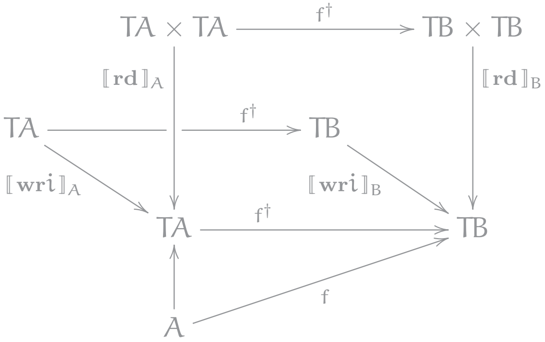

Second-Order Algebraic Theories

- Binding signatures [Aczel (1978)]

  $\mathbf{app} : (0, 0)$, $\mathbf{abs} : (1)$

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- Free algebras [Fiore, Plotkin, Turi (1999); Hamana (2004); Fiore (2008)]

  $t \quad ::= \quad \mathbf{var}(x) \ \mid\ \mathsf{M}[t_1, \ldots, t_n] \ \mid\ \mathbf{app}(t_1, t_2) \ \mid\ \mathbf{abs}(x.t')$

[Fiore, Hur (2010); Fiore, Mahmoud (2010)]

- Second-order equational theories

$$\mathbf{app}\big(\mathbf{abs}(x.\,\mathsf{M}[\,\mathbf{var}(x)\,]),\,\mathsf{N}[\,]\,\big) \equiv \mathsf{M}\big[\mathsf{N}[\,]\big]$$

$$\mathbf{abs}\big(\,x.\,\mathbf{app}(\,\mathsf{F}[\,],\,\mathbf{var}(x)\,)\,\big) \equiv \mathsf{F}[\,]$$

[Fiore, Hur (2010); Fiore, Mahmoud (2010)]

- Second-order equational theories

$$\mathbf{app}\big(\mathbf{abs}(x.\,\mathsf{M}[\,\mathbf{var}(x)\,])\,,\ \mathsf{N}[\,]\big) \quad \equiv \quad \mathsf{M}\big[\mathsf{N}[\,]\big]$$

$$\mathbf{abs}\big(x.\,\mathbf{app}(\,\mathsf{F}[\,]\,,\ \mathbf{var}(x)\,)\big) \quad \equiv \quad \mathsf{F}[\,]$$

- Free-algebra monads

$$\begin{array}{c} (\text{second-order algebras}) \\ \uparrow \;\dashv\; \downarrow \\ (\text{context-indexed sets}) \end{array}$$

$\mathcal{S}et^{\mathbb{F}}$

- Context-indexed sets

$$X_m$$

$$\{x_1, \ldots, x_m\}$$

$\mathcal{S}et^{\mathbb{F}}$

- Context-indexed sets

$$X_m \xrightarrow{X_g} X_n$$

$$\{x_1, \ldots, x_m\} \xrightarrow[g]{} \{x_1, \ldots, x_n\}$$

$\mathcal{S}et^{\mathbb{F}}$

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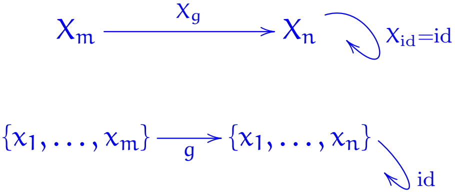

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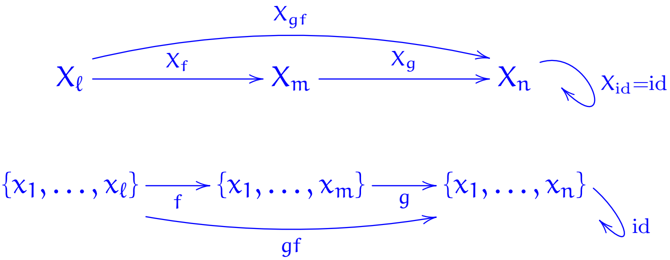


$\mathcal{S}et^{\mathbb{F}}$

- Renaming-invariant functions

$$
\begin{array}{c}
Y \\
\downarrow \varphi \\
X
\end{array}
\qquad\qquad
\begin{array}{ccc}
Y_m & \xrightarrow{Y_g} & Y_n \\
\downarrow \varphi_m & & \downarrow \varphi_n \\
X_m & \xrightarrow{X_g} & X_n
\end{array}
$$

$$\{x_1, \ldots, x_m\} \xrightarrow[g]{} \{x_1, \ldots, x_n\}$$

$\mathcal{S}et^{\mathbb{F}}$

- Renaming-invariant functions

$$\begin{array}{c} Y \\ \varphi \downarrow \\ X \end{array} \qquad\qquad \begin{array}{ccc} Y_m & \xrightarrow{\;Y_g\;} & Y_n \\ \varphi_m \downarrow & & \downarrow \varphi_n \\ X_m & \xrightarrow{\;X_g\;} & X_n \end{array}$$

$$\{x_1, \ldots, x_m\} \xrightarrow[g]{} \{x_1, \ldots, x_n\}$$

★ **Examples**

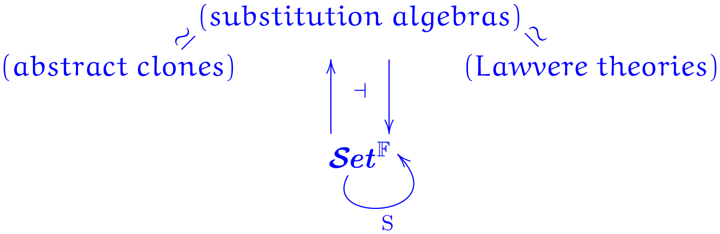
$$1 \qquad \{*\} = \{*\} = \cdots \qquad \{*\} \qquad \cdots$$

$$V \qquad \{\} \longrightarrow \{x_1\} \rightrightarrows \cdots \qquad \{x_1, \ldots, x_n\} \qquad \cdots$$

Substitution Algebras

[Fiore, Plotkin, Turi (1999)]

- Free substitution-algebra monad



Substitution Algebras

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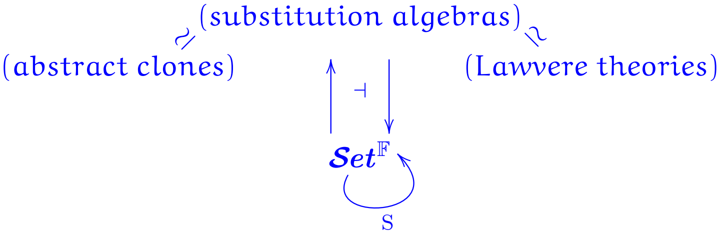

★ **Definition**

A *substitution algebra* is a structure

$$\mathrm{V} \xrightarrow{\ \mathbf{var}\ } A \xleftarrow{\ \mathbf{sub}\ } A^{\mathrm{V}} \times A$$

subject to four axioms: substitution, weakening, extensionality, and associativity.

▸ Substitution Algebra Axioms

Computational Interpretation

[Fiore, Staton (2014)]

- Computational metalanguage

$$\frac{\Gamma \vdash_{\mathsf{v}} e : \mathbf{V}}{\Gamma \vdash_{\mathsf{c}} \mathbf{var}_{\tau}(e) : \tau} \qquad \frac{\Gamma, a : \mathbf{V} \vdash_{\mathsf{c}} t : \tau \quad \Gamma \vdash_{\mathsf{c}} u : \tau}{\Gamma \vdash_{\mathsf{c}} \mathbf{sub}(a.\, t, u) : \tau}$$

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- Denotational semantics

computations values

$$\mathrm{Kl}(S) \underset{\longleftarrow}{\overset{\longrightarrow}{\top}} \mathcal{S}\boldsymbol{et}^{\mathbb{F}}$$

- Equational theory (I)
  - $\mathbf{let}\ x = \mathbf{sub}(\,a.\,\mathsf{M}[a]\,,\,\mathsf{N}[\,]\,)\ \mathbf{in}\ \mathsf{P}[x]$

    $\equiv\ \mathbf{sub}(\,a.\,\mathbf{let}\ x = \mathsf{M}[a]\ \mathbf{in}\ \mathsf{P}[x]\ ,\ \mathbf{let}\ x = \mathsf{N}[\,]\ \mathbf{in}\ \mathsf{P}[x]\ )$

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  - Weakening axiom

    $\mathbf{sub}(\,a.\,\mathsf{M}[\,]\,,\,\mathsf{N}[\,]\,) \equiv \mathsf{M}[\,]$

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    $\mathbf{sub}\big(\,a.\,\mathsf{M}[\,]\,,\ \mathsf{N}[\,]\,\big) \equiv \mathsf{M}[\,]$
  - Substitution axiom

    $\mathbf{sub}\big(\,a.\,\mathbf{var}(a)\,,\ \mathsf{N}[\,]\,\big) \equiv \mathsf{N}[\,]$

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  - Substitution axiom

    $\mathbf{sub}(\, a.\, \mathbf{var}(a)\, ,\ \mathsf{N}[\,]\,) \equiv \mathsf{N}[\,]$
  - Abort law

    $(\,\mathbf{let}\ x = \mathbf{var}(a)\ \mathbf{in}\ \mathsf{P}[x]\,)\ \equiv\ \mathbf{var}(a)$

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  - $\textbf{let } x = \textbf{sub}\big(\, a.\, \mathsf{M}[a]\,,\ \mathsf{N}[\,]\,\big) \textbf{ in } \mathsf{P}[x]$

    $\equiv\ \textbf{sub}\big(\, a.\, \textbf{let } x = \mathsf{M}[a] \textbf{ in } \mathsf{P}[x]\ ,\ \textbf{let } x = \mathsf{N}[\,] \textbf{ in } \mathsf{P}[x]\ \big)$
  - Weakening axiom

    $\textbf{sub}\big(\, a.\, \mathsf{M}[\,]\,,\ \mathsf{N}[\,]\,\big) \equiv \mathsf{M}[\,]$
  - Substitution axiom

    $\textbf{sub}\big(\, a.\, \textbf{var}(a)\,,\ \mathsf{N}[\,]\,\big) \equiv \mathsf{N}[\,]$
  - Abort law

    $\big(\textbf{let } x = \textbf{var}(a) \textbf{ in } \mathsf{P}[x]\big)\ \equiv\ \textbf{var}(a)$
- Operational intuition

  In $\textbf{sub}(a.t[a], u)$ the *jump point* a is declared and the computation proceeds as in $t[a]$ leading to a value if this produces one, or *aborting and restarting* with the computation of u if $\textbf{var}(a)$ is invoked.

- Equational theory (II)
  - Extensionality axiom

$$\mathbf{sub}\big(\, a.\, \mathsf{M}[a]\,,\, \mathbf{var}(b)\,\big) \;\equiv\; \mathsf{M}[b]$$

  - Associativity axiom

$$\mathbf{sub}\big(\, a.\, \mathbf{sub}\big(\, b.\, \mathsf{L}[a,b]\,,\, \mathsf{M}[a]\,\big)\,,\, \mathsf{N}[\,]\,\big)$$
$$\equiv\; \mathbf{sub}\big(\, b.\, \mathbf{sub}(\, a.\, \mathsf{L}[a,b]\,,\, \mathsf{N}[\,]\,)\,,\, \mathbf{sub}(\, a.\, \mathsf{M}[a]\,,\, \mathsf{N}[\,]\,)\,\big)$$

★ Substitution, Jumps, and Algebraic Effects

[Fiore, Staton (2014)]

- Computational interpretation of substitution algebras as a code-jumping mechanism.
- Adequate denotational semantics, equational theory, and abstract machine.
- Representation of first-order virtual effects.
- Extension incorporating effect handlers.

CPS Semantics

★ **Theorem** [Lawvere (1969), Kock (1970)]

In the context of strong monads, T-algebra structures on objects A are in bijective correspondence with monad morphisms $\mathrm{T} \to \mathrm{K}_A$ for K_A the double-dualization or continuation monad relative to A.

- CPS translation: $\mathrm{S} \to \mathrm{K}_\mathrm{V}$

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In the context of strong monads, T-algebra structures on objects A are in bijective correspondence with monad morphisms $T \to K_A$ for K_A the double-dualization or continuation monad relative to A.

- CPS translation: $S \to K_V$
  - $\mathbf{var}_X \mapsto \lambda a : V.\, \lambda k : V^X.\, a$
  - $\mathbf{sub}_X \mapsto \lambda\langle M : (K_V X)^V, N : K_V X\rangle.\, \lambda k : V^X.\, M\,(N\,k)\,k$

Direct-Style CPS Semantics

```
signature SubstitutionAlgebra
= sig
    type V
    val var : V -> 'a
    val sub : ( V -> 'a ) * 'a  -> 'a
end
```

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end

structure sa :> SubstitutionAlgebra
= struct
    type V = unit -> (unit cont)
    fun var( a ) = throw (a()) ()
    fun sub( M , N )
      = callcc( fn k => ( M( fn() => throw k N ) ))
  end
```

Inception Algebras

★ **Definition**

An (L, P)-*inception algebra* is a structure

$$L \times P \xrightarrow{\textbf{recall}} A \xleftarrow{\textbf{incept}} A^L \times A^P$$

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An (L, P)-*inception algebra* is a structure

$$\mathrm{L} \times \mathrm{P} \xrightarrow{\mathbf{recall}} A \xleftarrow{\mathbf{incept}} A^{\mathrm{L}} \times A^{\mathrm{P}}$$

subject to:

- Substitution axiom

$$\mathbf{inc}\big(\,\ell.\,\mathbf{rec}(\ell, \mathsf{P}[\,])\,,\ x.\,\mathsf{L}[x]\,\big) \ \equiv\ \mathsf{L}[\mathsf{P}]$$

- Extensionality axiom

$$\mathbf{inc}\big(\,\ell.\,\mathsf{P}[\ell]\,,\ x.\,\mathbf{rec}(k, x)\,\big) \ \equiv\ \mathsf{P}[k]$$

- Weakening axiom

$$\mathbf{inc}\big(\,\ell.\,\mathsf{M}[\,]\,,\ x.\,\mathsf{L}[x]\,\big) \ \equiv\ \mathsf{M}[\,]$$

- Associativity axiom

$$\mathbf{inc}\big(\,\ell.\,\mathbf{inc}\big(\,k.\,\mathsf{P}[\ell, k]\,,\ x.\,\mathsf{K}[\ell, x]\,\big)\,,\ y.\,\mathsf{L}[y]\,\big)$$
$$\equiv\ \mathbf{inc}\big(\,k.\,\mathbf{inc}(\,\ell.\,\mathsf{P}[\ell, k]\,,\ y.\,\mathsf{L}[y]\,)\,,\ x.\,\mathbf{inc}(\,\ell.\,\mathsf{K}[\ell, x]\,,\ y.\,\mathsf{L}[y]\,)\,\big)$$

★ Examples

- Substitution algebras = $(\mathrm{V}, 1)$-inception algebras

★ Examples

- Substitution algebras = $(V, 1)$-inception algebras

- More generally, for a set D, (V, D)-inception algebras model a *code-jumping with data-passing* mechanism:

$$\frac{\Gamma \vdash_{\mathsf{v}} e : \mathbf{V} \qquad \Gamma \vdash_{\mathsf{v}} d : \mathbf{D}}{\Gamma \vdash_{\mathsf{c}} \mathbf{rec}_{\tau}(e, d) : \tau}$$

$$\frac{\Gamma, \ell : \mathbf{V} \vdash_{\mathsf{c}} t : \tau \qquad \Gamma, x : \mathbf{D} \vdash_{\mathsf{c}} u : \tau}{\Gamma \vdash_{\mathsf{c}} \mathbf{inc}_{\tau}\big(\ell.\, t\,,\, x.\, u\big) : \tau}$$

Inception Algebras in Logic

- $L = \neg P$

¬ elimination

$$\frac{\neg P \qquad P}{A}$$

excluded middle

$$\frac{\begin{matrix}[\neg P] & & [P] \\ \vdots & & \vdots \\ A & & A\end{matrix}}{A}$$

[de Groote (1995)]

- Inceptions are not local exceptions.

Counterexample

- CPS semantics

Logical Inception Algebras CPS Semantics

- Programming idiom

```
signature LogicalInceptionAlgebra
= sig
    type 'a li
    val rec : 'a li * 'a -> 'b
    val inc : ( 'a li -> 'b ) * ( 'a -> 'b )  -> 'b
  end
```

▸ Logical Inception Algebra Structure

- Programming idiom

```
signature LogicalInceptionAlgebra
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    type 'a li
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  end
```

- **NB** `inc` generalises and introduces a level of abstraction over `callcc`:

$$\mathtt{callcc}(\mathrm{f}) = \mathtt{inc}\big(\mathrm{f}, \mathrm{id}\big)$$

$$\mathtt{inc}\big(\mathrm{f}, \mathrm{h}\big) = \mathtt{callcc}\big(\mathrm{f} \circ \neg\, \mathrm{h}\big)$$

- Programming idiom

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  end
```

★ **Example**

```
val DeMorgan : ( 'a * 'b ) li -> ('a li,'b li) sum
  = fn c =>
    inc( fn a => left a ,
         fn x => inc( fn b => right b ,
                      fn y => rec( c , (x,y) ) ) )
```

- Programming idiom

```
signature LogicalInceptionAlgebra
= sig
    type 'a li
    val rec : 'a li * 'a -> 'b
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  end
```

- **Applications**
  - Encoding of local and global exception mechanisms
  - Safe exception handling in program modules
  - Coroutines

Untyped Inception Algebras

- $(\mathrm{V}, \mathrm{V}^n)$-inception algebras

$$\mathrm{V} \times \mathrm{V}^n \xrightarrow{\mathbf{rec}} A \xleftarrow{\mathbf{inc}} A^{\mathrm{V}} \times A^{\mathrm{V}^n}$$

Untyped Inception Algebras

- (V, V^n)-inception algebras

$$V \times V^n \xrightarrow{\mathbf{rec}} A \xleftarrow{\mathbf{inc}} A^V \times A^{V^n}$$

model the *untyped* CPS calculus[Appel (1992); Thielecke (1997)].

- Conversion table:

$\mathbf{rec}(\ell, \vec{x})$	$\ell\langle\vec{x}\rangle$
$\mathbf{inc}(\ell.\,\mathsf{P}[\ell]\,,\,\vec{x}.\,\mathsf{L}[\vec{x}]\,)$	$\mathsf{P}[\ell]\,\{\,\ell\langle\vec{x}\rangle = \mathsf{L}[\vec{x}]\,\}$

▸ Sorted Inception Algebras

Nullary/Unary Untyped Inception Algebras

- Self recall

$$\mathbf{inc}\big(\,\ell.\,\mathbf{rec}(\ell,\ell)\,,\,x.\,\mathsf{M}[x]\,\big)\;\equiv\;\mathbf{inc}\big(\,\ell.\,\mathsf{M}[\ell]\,,\,x.\,\mathsf{M}[x]\,\big)$$

Nullary/Unary Untyped Inception Algebras

- Self recall

$$\mathbf{inc}\big(\,\ell.\,\mathbf{rec}(\ell,\ell)\,,\;x.\,\mathsf{M}[x]\,\big)\;\equiv\;\mathbf{inc}\big(\,\ell.\,\mathsf{M}[\ell]\,,\;x.\,\mathsf{M}[x]\,\big)$$

- Recursion

  For

$$Y_t\;=\;\mathbf{inc}\big(\,\ell.\,\mathbf{rec}(\ell,\ell)\,,\;\ell.\,\mathbf{sub}\big(\,a.\,t[a]\,,\;\mathbf{rec}(\ell,\ell)\big)\,\big)$$

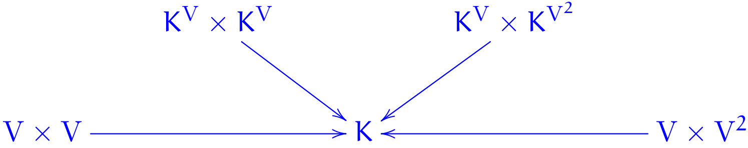
  we have

$$Y_t\;\equiv\;\mathbf{sub}\big(\,a.\,t[a]\,,\;Y_t\,\big)$$

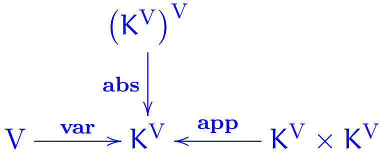
Unary/Binary Untyped Inception Algebras

- CPS lambda structure

  The initial unary/binary untyped inception algebra



provides a CPS lambda structure



thereby inducing an homomorphic CPS interpretation

$$\Lambda \to K^V$$

of lambda terms.

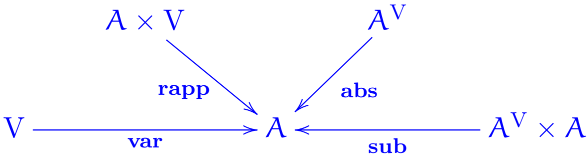
- CPS interpretation

  - $\mathbf{var} : x[e] \mapsto \mathbf{rec}(\, e\, ,\ \langle x \rangle\,)$

  - $$\mathbf{abs} : \mathsf{M}[x][e] \mapsto \mathbf{inc}(\ \ f.\, \mathbf{rec}(\, e\, ,\ \langle f \rangle\,)\ \ ,\ \ \langle a, e \rangle.\, \mathsf{M}[a][e]\ \ )$$

  - $$\mathbf{app} : \mathsf{M}[e], \mathsf{N}[e] \mapsto \mathbf{inc}(\ \ m.\mathsf{M}[m]\ \ ,\ \ \langle f \rangle.\, \mathbf{inc}(\ \ n.\mathsf{N}[n]\ \ ,\ \ \langle a \rangle.\, \mathbf{rec}(f, \langle a, e \rangle)\ \ )\ \ )$$

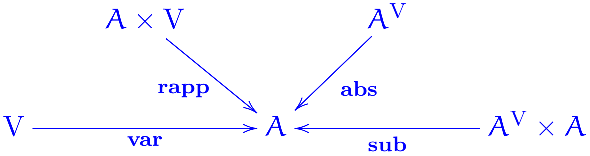
Right Lambda Algebras



$$(\beta_r) \quad \mathbf{rapp}\big(\mathbf{abs}(x.\, \mathsf{M}[x]),\, a\big) \;\equiv\; \mathsf{M}[a]$$

[Hirschowitz, Maggesi (2010); Hyland (2013)]

Right Lambda Algebras



$$\underbrace{(\beta_r)}_{\text{summon}} \qquad \overbrace{\mathbf{rapp}\big(\underbrace{\mathbf{abs}(x.\,\mathsf{M}[x])}_{\text{pop in } x},\ a\big)}^{\text{push } a} \equiv \mathsf{M}[a]$$

- Computational interpretation:

  [Fiore, Staton (2014)]

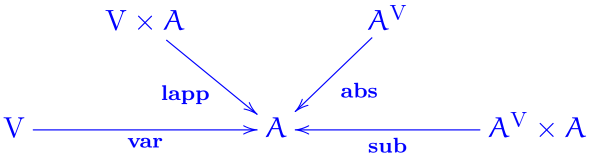
  A mechanism for stack manipulation of code pointers.

- Application:

  Stack abstract machine for CPS calculus.

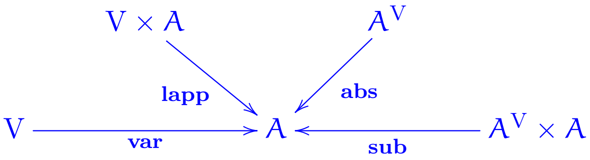
▸ CPS Structure

Left Lambda Algebras



$$(\beta_l) \quad \mathbf{sub}\big(\,\ell.\,\mathbf{lapp}(\ell, \mathsf{N}[\,])\,,\,\mathbf{abs}(\,\mathsf{k}.\,\mathsf{M}[\mathsf{k}]\,)\big) \;\equiv\; \mathbf{sub}\big(\,\mathsf{k}.\,\mathsf{M}[\mathsf{k}]\,,\,\mathsf{N}[\,]\,\big)$$

Left Lambda Algebras



$$\underbrace{(\beta_l)}_{\text{resume}} \qquad \mathbf{sub}\Big(\ell.\ \underbrace{\mathbf{lapp}(\ell, \mathsf{N}[\,])}_{\text{yield } \mathsf{N}},\ \underbrace{\mathbf{abs}(\,k.\,\mathsf{M}[k]\,)}_{\text{thunk } \mathsf{M}}\Big) \equiv \mathbf{sub}\Big(k.\,\mathsf{M}[k]\,,\ \mathsf{N}[\,]\Big)$$

- Computational interpretation:

  A synchronous coroutine mechanism

- Application:

  Producer-Consumer

▸ Producer-Consumer

Final Remarks

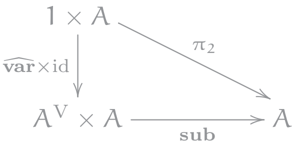
- Conclusions
  - The first algebraic axiomatisation of control effects.
  - Foundational analysis of the principles of programming with algebraic effects.
- Directions
  - Models
  - Classical/intermediate logics
  - Algebraic theories

Appendix

Substitution Algebra Axioms

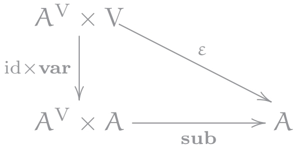
- Substitution

$$\mathbf{sub}\big(\, a.\,\mathbf{var}(a)\,,\ \mathsf{M}[\,]\,\big) \ \equiv\ \mathsf{M}[\,]$$



- Extensionality

$$\mathbf{sub}\big(\, a.\,\mathsf{M}[a]\,,\ \mathbf{var}(b)\,\big) \ \equiv\ \mathsf{M}[b]$$

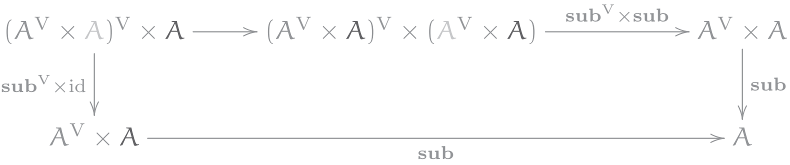


- Weakening

$$\mathbf{sub}\big(\, a.\, \mathsf{M}[\,]\, ,\ \mathsf{N}[\,]\,\big) \;\equiv\; \mathsf{M}[\,]$$

- Associativity

$$\mathbf{sub}\big(\, a.\, \mathbf{sub}\big(\, b.\, \mathsf{L}[a, b]\, ,\ \mathsf{M}[a]\,\big)\, ,\ \mathsf{N}[\,]\,\big)$$
$$\equiv\; \mathbf{sub}\big(\, b.\, \mathbf{sub}(\, a.\, \mathsf{L}[a, b]\, ,\ \mathsf{N}[\,]\,)\, ,\ \mathbf{sub}(\, a.\, \mathsf{M}[a]\, ,\ \mathsf{N}[\,]\,)\,\big)$$



▸ Substitution Algebras

Contraction Laws

- $\mathbf{sub}\big(\, a.\, \mathsf{M}[a, b]\,,\ \mathbf{var}(b)\,\big) \ \equiv\ \mathsf{M}[b, b]$

- $\mathbf{sub}\big(\, a.\, \mathbf{sub}(\, b.\, \mathsf{M}[a, b]\,,\ \mathsf{N}[\,]\,)\,,\ \mathsf{N}[\,]\,\big) \ \equiv\ \mathbf{sub}\big(\, c.\, \mathsf{M}[c, c]\,,\ \mathsf{N}[\,]\,\big)$

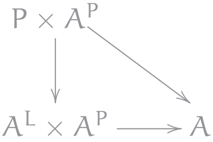
- $\mathbf{sub}\big(\, c.\, \mathbf{let}\ x = \mathsf{M}[c]\ \mathbf{in}\ \mathsf{N}[x, c]\ \ ,\ \mathsf{L}[\,]\,\big)$

  $\equiv\ \ \mathbf{sub}\big(\, a.\, \mathbf{let}\ x = \mathsf{M}[a]\ \mathbf{in}\ \mathbf{sub}\big(b.\, \mathsf{N}[x, b], \mathsf{L}[\,]\big)\ \ ,\ \mathsf{L}[\,]\,\big)$

Inception Algebra Axioms

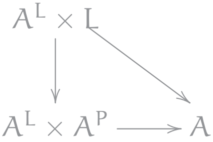
- Substitution

$$\mathbf{inc}\Big(\,\ell.\,\mathbf{rec}(\ell, \mathsf{P}[\,])\,,\, x.\,\mathsf{L}[x]\,\Big) \;\equiv\; \mathsf{L}[\mathsf{P}]$$



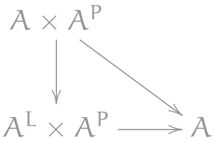
- Extensionality

$$\mathbf{inc}\Big(\,\ell.\,\mathsf{P}[\ell]\,,\, x.\,\mathbf{rec}(k, x)\,\Big) \;\equiv\; \mathsf{P}[k]$$



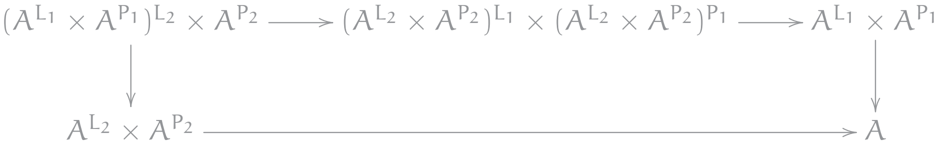
- Weakening

$$\mathbf{inc}\big(\,\ell.\,\mathsf{M}[\,]\,,\;x.\,\mathsf{L}[x]\,\big)\;\equiv\;\mathsf{M}[\,]$$



- Associativity

$$\begin{aligned}&\mathbf{inc}\big(\,\ell.\,\mathbf{inc}\big(\,k.\,\mathsf{P}[\ell,k]\,,\;x.\,\mathsf{K}[\ell,x]\,\big)\,,\;y.\,\mathsf{L}[y]\,\big)\\ &\quad\equiv\;\mathbf{inc}\big(\,k.\,\mathbf{inc}(\,\ell.\,\mathsf{P}[\ell,k]\,,\;y.\,\mathsf{L}[y]\,)\,,\;x.\,\mathbf{inc}(\,\ell.\,\mathsf{K}[\ell,x]\,,\;y.\,\mathsf{L}[y]\,)\,\big)\end{aligned}$$



Inceptions Are Not Local Exceptions

- **NB**

$$\mathbf{sub}\Big(\,e_1.\,\mathbf{let}\;\ell = \mathbf{sub}(\,e_2.\,\mathbf{ret}\;e_2\,,\;\mathbf{ret}\;e_1\,)\;\mathbf{in}\;\mathbf{var}\;\ell\;\;,\;\langle\,\rangle\,\Big)\;\equiv\;\langle\,\rangle$$

however

```
let exception e1
in ( let val l
         = let exception e2
           in e2
           handle e2 => e1 end
     in
       raise l
     end )
handle e1 => () end
```

outputs `uncaught exception e2`.

Logical Inception Algebras CPS Semantics

- (R^P, P)-inception algebra structure on R
  - $\mathbf{rec} \mapsto \lambda\langle \ell : R^P, p : P\rangle.\, \ell\, p$
  - $\mathbf{inc} \mapsto \lambda\langle M : R^{(R^P)}, N : R^P\rangle.\, M\, N$
- Induced (R^P, P)-inception algebra structure on $K_R X$
  - $\mathbf{rec}_X \mapsto \lambda\langle \ell : R^P, p : P\rangle.\, \lambda k : R^X.\, \ell\, p$
  - $\mathbf{inc}_X \mapsto \lambda\langle M : (K_R X)^{(R^P)}, N : (K_R X)^P\rangle.\, M\, (\lambda p : P.\, N\, p\, k)\, k$

▸ Inception Algebras in Logic

Logical Inception Algebra Structure

```
signature LogicalInceptionAlgebra
= sig
    type 'a li
    val rec : 'a li * 'a -> 'b
    val inc : ( 'a li -> 'b ) * ( 'a -> 'b )  -> 'b
  end

structure lia :> LogicalInceptionAlgebra
= struct
    type 'a li = 'a -> (unit cont)
    fun rec( a , x ) = throw (a x) ()
    fun inc( M , N )
      = callcc( fn k => ( M( fn x => throw k (N x) ) ))
  end
```

Sorted Inception Algebras

[Milner (1991)]

- Sorted sets

  $|-| : S \rightharpoonup S^*$

- Structures

$$V_\sigma \times \prod_i V_{|\sigma|_i} \longrightarrow A \longleftarrow A^{V_\sigma} \times A^{\prod_i V_{|\sigma|_i}} \quad \text{in } \mathcal{S}et^{\mathbb{F}[S]}$$

  subject to the inception algebra axioms.

▸ Inception Algebra Axioms

★ **Example**

For a set of sorts S, the sorted set

$$T_S \rightharpoonup {T_S}^*, \text{ where } T_S = \mu X. S + X^*$$

yields the typed CPS-calculus, and hence sorted $\otimes\neg$-categories[Thielecke (1997)].

Right Lambda Algebras

- Stack CPS structure

- $$\begin{array}{rcl} V \times V^n & \longrightarrow & A \\ a, \langle a_1, \ldots, a_n \rangle & \mapsto & \mathbf{push}(\cdots \mathbf{push}(\mathbf{var}(a), a_1) \cdots, a_n) \end{array}$$

- $$\begin{array}{rcl} A^V \times A^{V^n} & \longrightarrow & A \\ \mathsf{M}[a], \mathsf{N}[a_1, \ldots, a_n] & \mapsto & \mathbf{sub}(\ a.\, \mathsf{M}[a]\ , \\ & & \quad \mathbf{pop}(a_1. \cdots \mathbf{pop}(a_n.\mathsf{N}[a_1, \ldots, a_n]))\ ) \end{array}$$

- Parameterised fixpoint
  - For

$$\mathsf{T}[f, x] = \mathbf{in}(\mathbf{yield}(x, \mathbf{thunk}(z.\, \mathbf{push}(z, \mathbf{push}(f, \mathbf{var}\ f)))))$$

and

$$\mathbf{In}[x] = \mathbf{sub}\Big(\, f.\, \mathsf{T}[f, x]\,,\ \mathbf{pop}\Big(\, f.\, \mathbf{pop}(x.\, t[f, x])\,\Big)\,\Big)$$

we have

$$\mathbf{In}[x] \ \equiv\ \mathbf{in}(\mathbf{yield}(x, \mathbf{thunk}(z.\, \mathbf{In}[z])))$$

Left Lambda Algebras

- Axioms
  - $$\mathbf{sub}\big(\ell.\,\mathbf{yield}(\ell, \mathsf{N}[\,])\,,\ \mathbf{thunk}(\,\mathsf{k}.\,\mathsf{M}[\mathsf{k}]\,)\big) \ \equiv\ \mathbf{sub}\big(\,\mathsf{k}.\,\mathsf{M}[\mathsf{k}]\,,\ \mathsf{N}[\,]\big)$$
  - $$\mathbf{sub}\big(\,\ell.\,\mathbf{thunk}\big(x.\,\mathsf{M}[\ell, x]\big)\,,\ \mathsf{N}[\,]\,\big)$$
    $$\equiv\ \ \mathbf{thunk}\big(\,x.\,\mathbf{sub}\big(\,\ell.\,\mathsf{M}[\ell, x]\,,\ \mathsf{N}[\,]\,\big)\,\big)$$
  - $$\mathbf{sub}\big(\,e.\,\mathbf{yield}\big(\mathsf{L}[e], \mathsf{M}[e]\big)\,,\ \mathsf{N}[\,]\,\big)$$
    $$\equiv\ \ \mathbf{sub}\big(\ \ \ell.\,\mathbf{yield}(\,\ell\,,\ \mathbf{sub}\big(e.\,\mathsf{M}[e], \mathsf{N}[\,]\big)\,)\ \ ,\ \ \mathbf{sub}\big(\,e.\,\mathbf{var}(\mathsf{L}[e]), \mathsf{N}[\,]\,\big)\ \ \big)$$

- Fixpoints
  - For a computation $t(\cdot)$,

    $$Y_t = \mathbf{sub}\big(\, x.\, t\big(\mathbf{yield}(x, \mathbf{var}\; x)\big)\;,\; \mathbf{thunk}\big(\, x.\, t\big(\mathbf{yield}(x, \mathbf{var}\; x)\big)\,\big)\big)$$

    we have

    $$Y_t \equiv t(Y_t)$$

  - For a parameterised computation $t[x](c.\mathsf{M}[c])$, let

    $$\begin{aligned} \mathsf{T} = \;& \mathbf{thunk}(f.\, \mathbf{thunk}(x. \\ & \quad t[x]\big(\, c.\, \mathbf{sub}(p.\, \mathbf{yield}(p, \mathbf{var}\; c), \mathbf{yield}(f, \mathbf{var}\; f))\,\big))) \end{aligned}$$

    For

    $$Y_t[x] = \mathbf{sub}\big(\, f.\, t[x](c.\, \mathbf{sub}(p.\mathbf{yield}(p, \mathbf{var}\; c), \mathbf{yield}(f, \mathsf{T})))\;,\; \mathsf{T}\,\big)$$

    we have

    $$Y_t[x] \;\equiv\; t[x]\big(\, c.\, Y_t[c]\,\big)$$

- Producer-Consumer

Let

$$\mathbf{In}[a] \equiv \mathbf{in}(\mathbf{yield}(a, \mathbf{thunk}(x.\mathbf{In}[x])))$$

$$\mathbf{Out} \equiv \mathbf{thunk}(y.\mathbf{out}(\mathbf{out}(\mathbf{yield}(y, \mathbf{Out}))))$$

Then

$$\mathbf{sub}(\, a.\,\mathbf{In}[a]\,,\ \mathbf{Out}\,) \equiv \mathbf{in}(\mathbf{out}(\mathbf{out}(\mathbf{sub}(x.\mathbf{In}[x], \mathbf{Out}))))$$

▸ Left Lambda Algebras