Algebraic Foundations for Type Theories

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The known

$$\begin{tikzcd}
& \text{Cat. Alg.} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
\text{Univ. Alg.} \arrow[rr, leftrightarrow] & 1^{\text{st}}\text{-order} & \text{Eq. Log.}
\end{tikzcd}$$

Alg. Theories

(mon, grp,
ring, mod, . . .)

► [1, 5, 6]

The Unknown

$$\begin{tikzcd}[column sep=small] & \text{Cat. Alg.} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ \text{Univ. Alg.} \arrow[rr, leftrightarrow, "1^{\text{st}}\text{-order}"] & & \text{Eq. Log.} \end{tikzcd}$$

$$\frac{\text{(diagram above)}}{\text{Alg. Theories}} = \frac{?}{\text{Type Theories}}$$

Alg. Theories

(mon, grp,
ring, mod, . . .)

Type Theories

(simply typed,
dep. typed,
polymorphic,
linear, . . .)

- [1, 5, 6], [29].

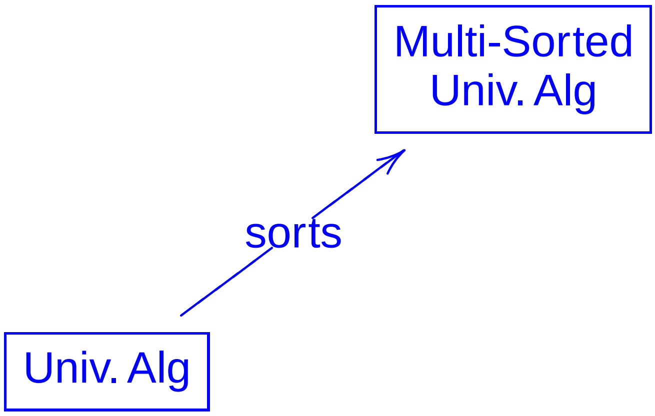
Programme

mathematical models $\rightsquigarrow$ meta-theories

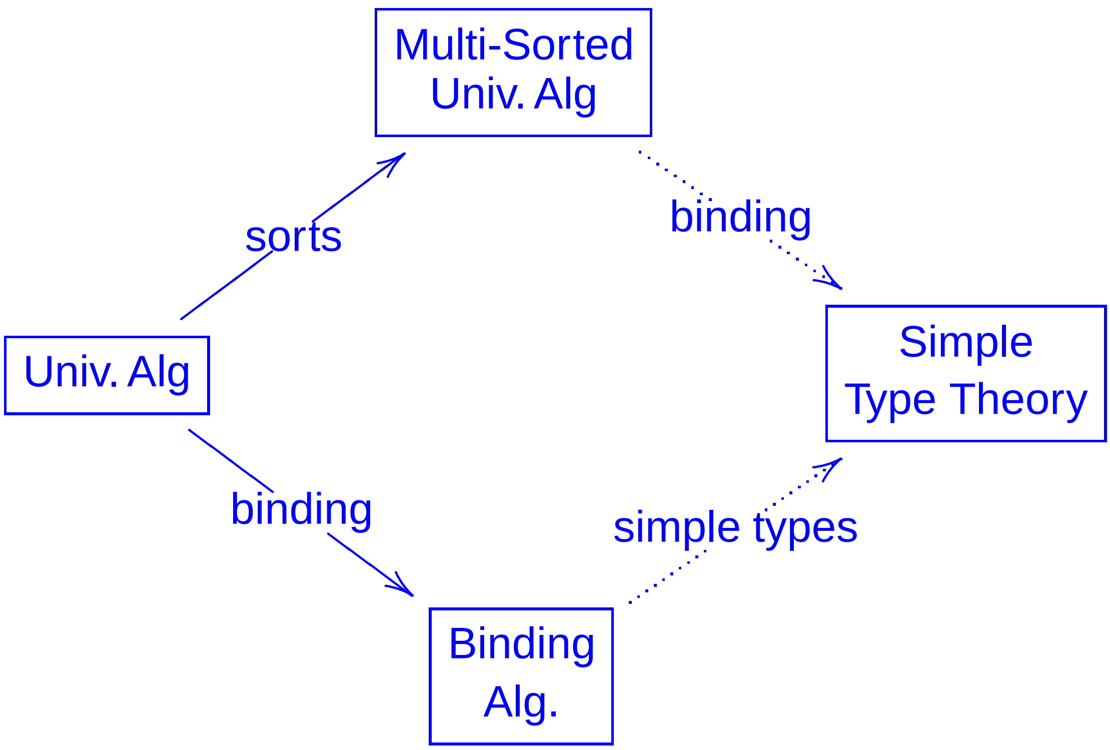
- Development of *algebraic meta-theories* for formal languages.
  - *Semantics*
    - Model theory.
  - *Syntax*
    - Initial-algebra semantics.
    - Structural induction and recursion.
    - Substitution.
- Synthesis of *deduction systems* for equational reasoning and computation by rewriting.

- [29], [26].

The Space

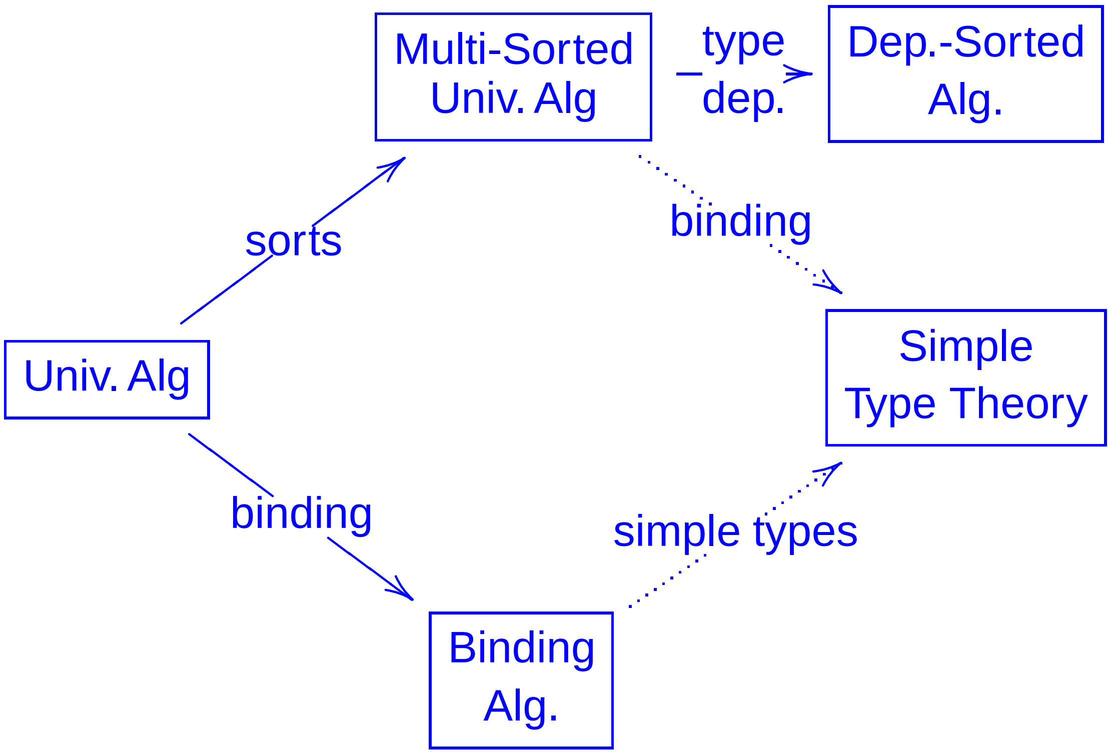


The Space


Multi-Sorted
Univ. Alg
sorts
binding
Univ. Alg
Simple
Type Theory
binding
simple types
Binding
Alg.

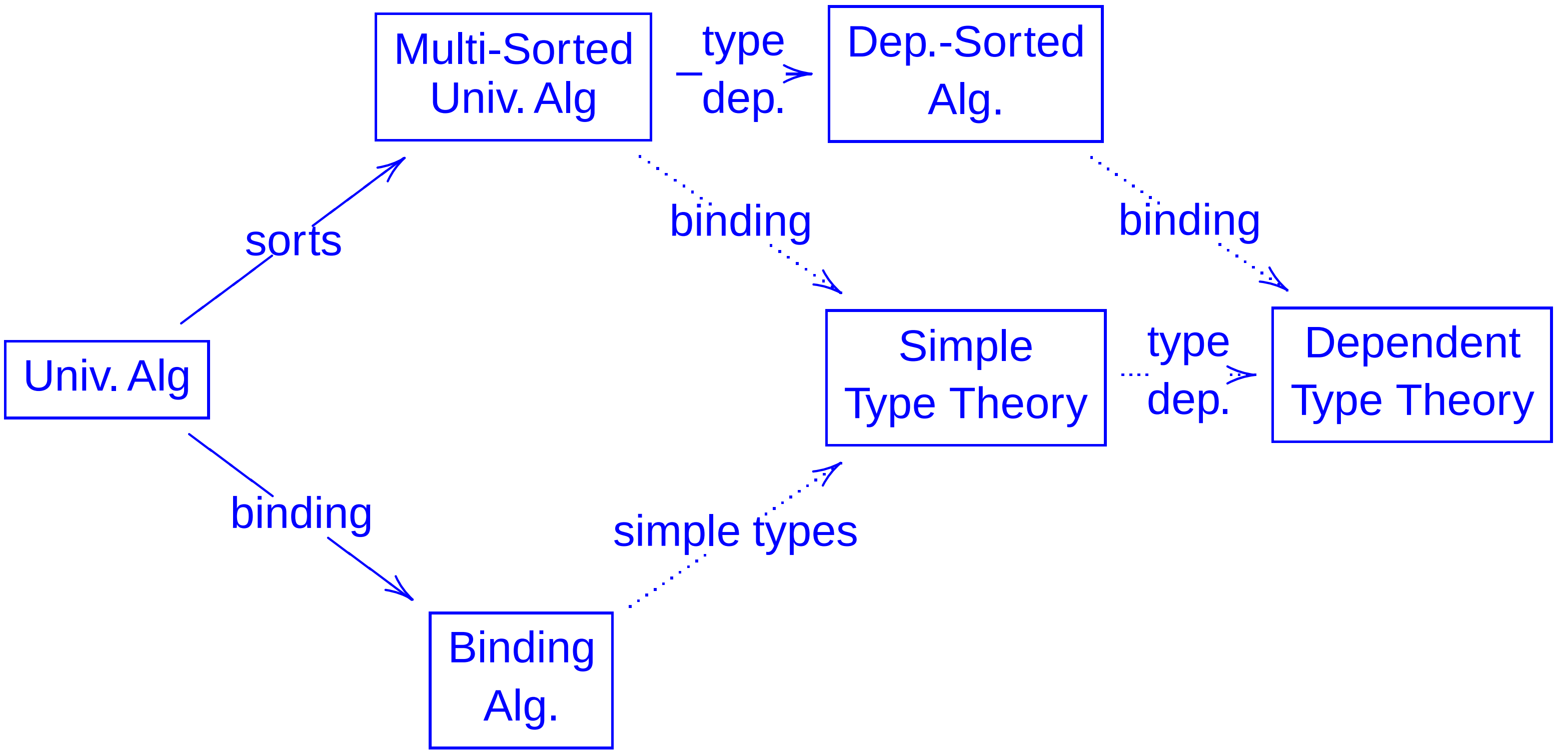
► [1], [2]

The Space



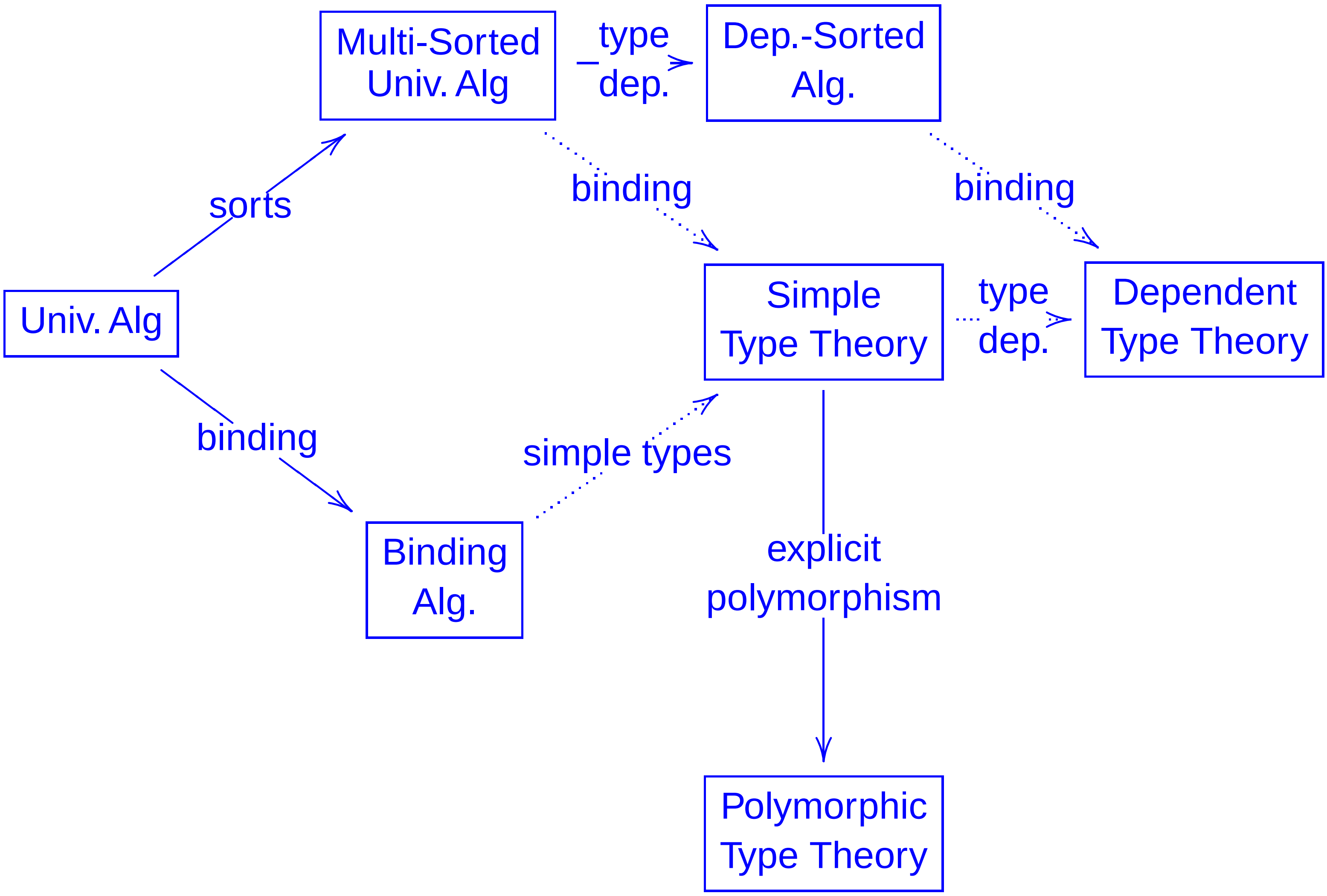
► [1], [2], [14]

The Space



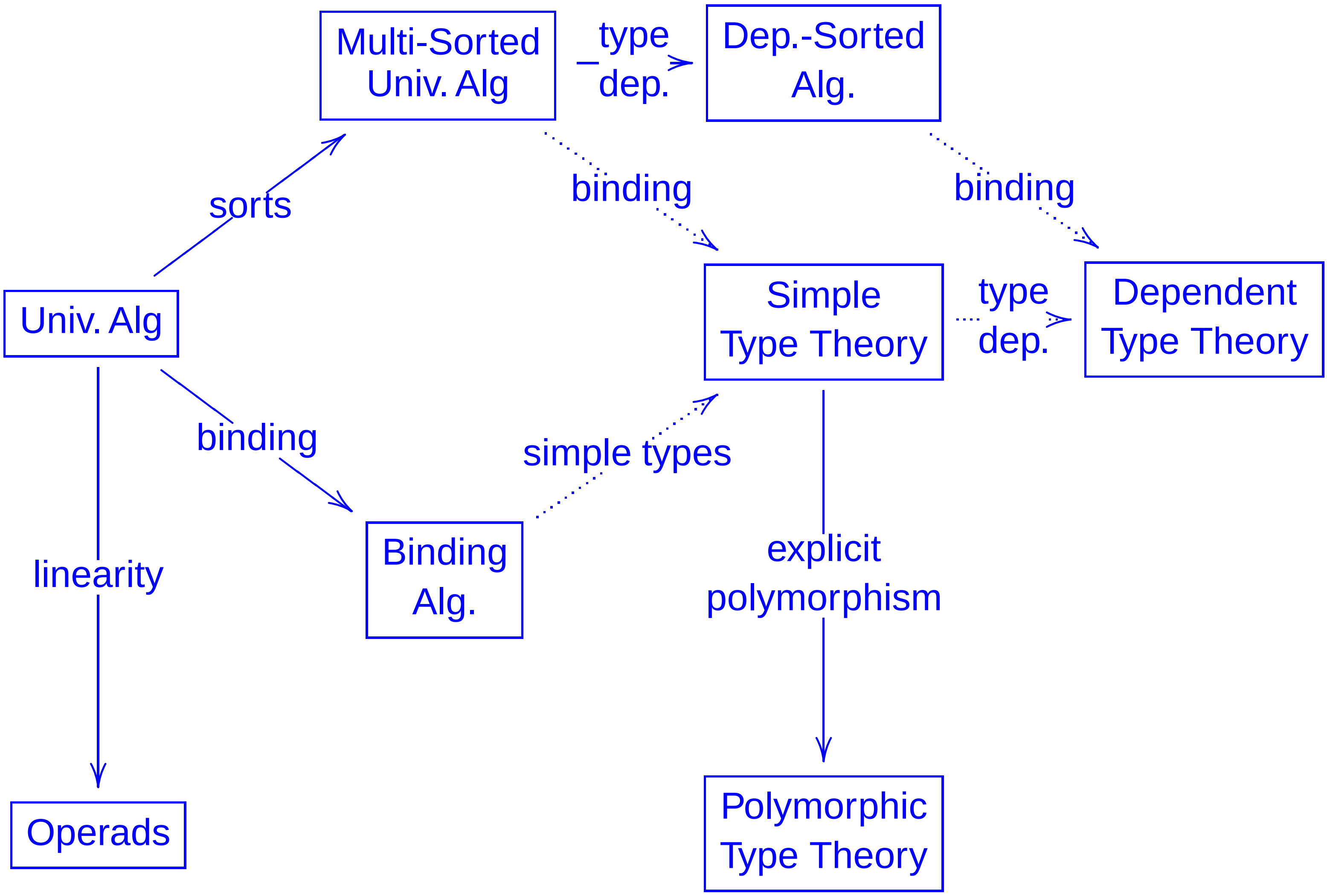
- [1], [2], [14], [10]

The Space



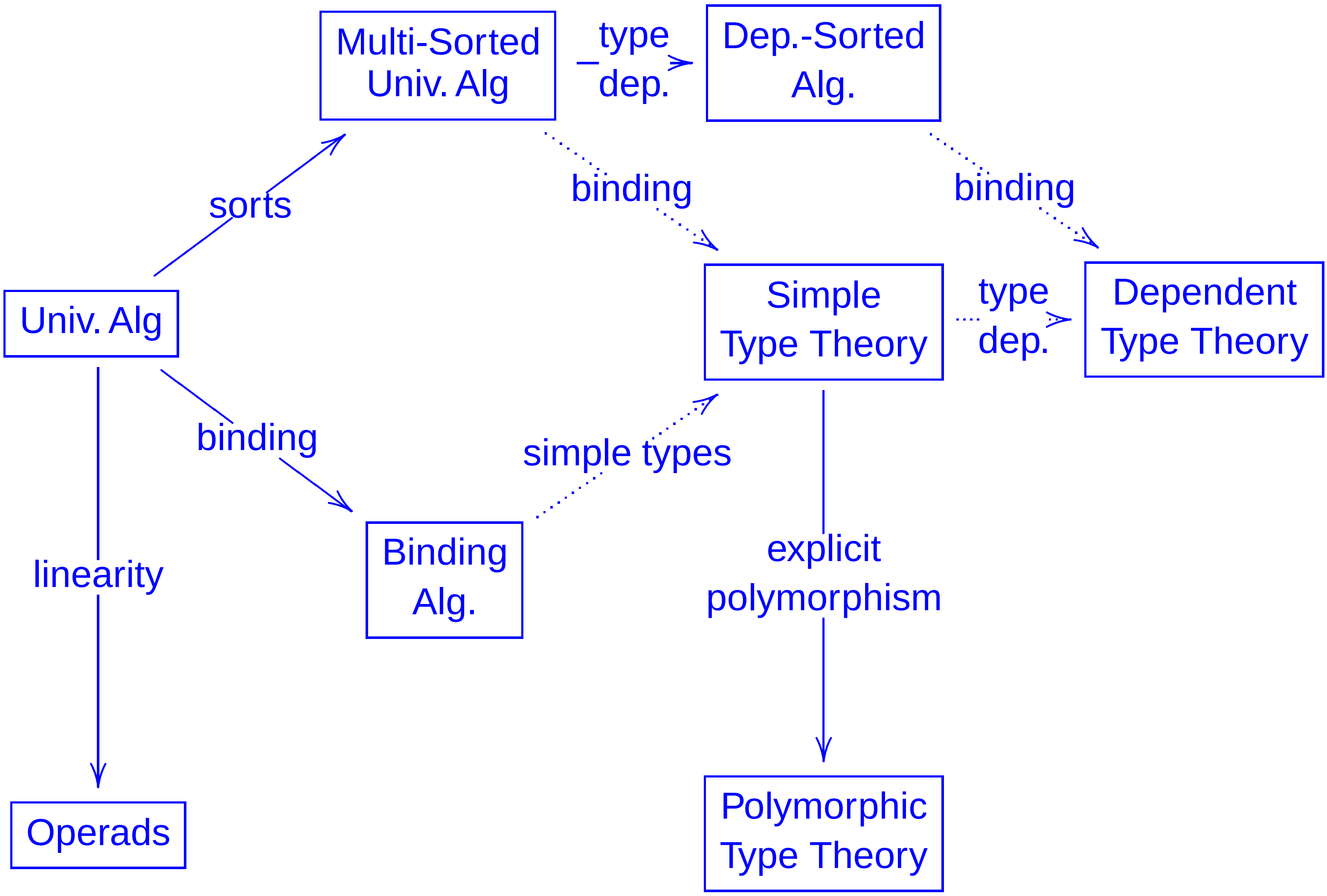
► [1], [2], [14], [10], [8, 12]

The Space



▶ [1], [2], [14], [10], [8, 12], [9, 11].

The Space



The Talk

I Modelling of simple type theories.

II Modelling of dependent type theories.

III Foundations.

▶ [1], [2], [14], [10], [8, 12], [9, 11].

I

Algebraic Modelling of Simple Type Theories

Simple Type Theory

	algebraic theories	simply-typed theories
types	unstructured	algebraic
terms	algebraic	algebraic with binding

The syntactic theory should account for:

- variables and meta-variables
- variable binding and α-equivalence
- capture-avoiding and meta substitution
- mono and multi sorting

- [19, 20, 22, 25].

Categories of Contexts

Def: An S-sorted *context structure* is given by

- a small category $\mathbb{C}$ with terminal object,
- objects $\langle\sigma\rangle \in \mathbb{C}$ for all $\sigma \in S$,
- product diagrams

$$\begin{tikzcd} & \Gamma, \sigma \arrow[dl] \arrow[dr] & \\ \Gamma & & \langle\sigma\rangle \end{tikzcd}$$

  for all $\Gamma \in \mathbb{C}$ and $\sigma \in S$.

Example: $\mathrm{FinSet}^{\mathrm{op}}$ is the initial *mono-sorted* context structure.

Algebraic Models of Variable Binding in $\widehat{\mathbb{C}}^{\mathbb{S}}$

Example: Untyped λ-calculus.

- Syntax:

$$t \;::=\; x \mid t'(t'') \mid \lambda x.\, t'$$

- [3]

Algebraic Models of Variable Binding in $\widehat{\mathbb{C}}^{S}$

Example: Untyped λ-calculus.

- Syntax:

$$t \quad ::= \quad x \mid t'(t'') \mid \lambda x.\, t'$$

- Algebras:

$$\begin{aligned} \mathsf{var} : &\quad 1 \to A^{\mathbf{y}\langle s \rangle} \\ \mathsf{app} : &\quad A^2 \to A \qquad\qquad \text{in } \widehat{\mathbb{C}} \\ \mathsf{abs} : &\quad A^{\mathbf{y}\langle s \rangle} \to A \end{aligned}$$

$\widehat{\mathbb{C}} =^{\mathrm{def}} \mathcal{S}et^{\mathbb{C}^{\mathrm{op}}}$ and $\mathbf{y} : \mathbb{C} \hookrightarrow \widehat{\mathbb{C}}$ is the Yoneda embedding

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- [3], [20, 22].

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NB:

$$P^{\mathbf{y}\langle s\rangle}(\Gamma) \;=\; P(\Gamma, s)$$

as

$$\begin{tikzcd} \mathbb{C} \arrow[r, hook] \arrow[d, "{-,s}"'] & \widehat{\mathbb{C}} \arrow[d, shift right=2, "(-)\times\mathbf{y}\langle s\rangle"'] \arrow[d, phantom, "\dashv"] \\ \mathbb{C} \arrow[r, hook] & \widehat{\mathbb{C}} \arrow[u, shift right=2, "(-)^{\mathbf{y}\langle s\rangle}"'] \arrow[u, phantom, shift left=6, "\dashv"] \end{tikzcd}$$

- [3], [20, 22].

Algebraic Models of Variable Binding in $\widehat{\mathbb{C}}^{S}$

Example: Untyped λ-calculus.

- Syntax:

$$t \quad ::= \quad x \mid t'(t'') \mid \lambda x.\, t'$$

- Algebras:

$$\begin{aligned} \mathsf{var}: &\quad \mathbf{y}\langle s\rangle \longrightarrow A \\ \mathsf{app}: &\quad A^2 \longrightarrow A \\ \mathsf{abs}: &\quad A^{\mathbf{y}\langle s\rangle} \longrightarrow A \end{aligned} \qquad \text{in } \widehat{\mathbb{C}}$$

- Initial model:

$$\Lambda \in \mathcal{S}et^{\mathsf{FinSet}}$$

with $\Lambda(n)$ the set of α-equivalence classes of λ-terms with free variables amongst $x_1, \ldots, x_n$.

- [20, 22].

Single-Variable Substitution

- Substitution algebras:

$$\mathsf{subst} : A^{\mathbf{y}\langle s\rangle} \times A \longrightarrow A$$

satisfying

... natural axioms ...

- [20, 22].

Single-Variable Substitution

- Substitution algebras:

$$\mathsf{subst} : A^{\mathbf{y}\langle s \rangle} \times A \to A$$

  satisfying

  . . . natural axioms . . .

- Initial model:

$$\Lambda \in \mathcal{S}et^{\mathsf{FinSet}}$$

  with capture-avoiding single-variable substitution.

- [20, 22].

Substitution Algebras

1. $u : A \vdash \mathsf{var}(x)\left[{}^{u}/_{x}\right] = u$

2. $t : A, u : A \vdash t\left[{}^{u}/_{x}\right] = t$

3. $t : A^{\mathbf{y}\langle s\rangle \times \mathbf{y}\langle s\rangle}, x : \mathbf{y}\langle s\rangle$
   $\vdash t(x,y)\left[{}^{\mathsf{var}(x)}/_{y}\right] = t(x,x)$

4. $t : A^{\mathbf{y}\langle s\rangle \times \mathbf{y}\langle s\rangle}, u : A^{\mathbf{y}\langle s\rangle}, v : A$
   $\vdash \left(t(y,x)\left[{}^{u(x)}/_{y}\right]\right)\left[{}^{v}/_{x}\right]$
   $= \left(t(y,x)\left[{}^{v}/_{x}\right]\right)\left[{}^{u(x)\left[{}^{v}/_{x}\right]}/_{y}\right]$

► [20, 22].

Substitution Algebras

1. $u : A \vdash \mathsf{var}(x)[{}^{u}/_{x}] = u$

2. $t : A, u : A \vdash t[{}^{u}/_{x}] = t$

3. $t : A^{\mathbf{y}\langle s\rangle \times \mathbf{y}\langle s\rangle}, x : \mathbf{y}\langle s\rangle$
$$\vdash t(x,y)[{}^{\mathsf{var}(x)}/_{y}] = t(x,x)$$

4. $t : A^{\mathbf{y}\langle s\rangle \times \mathbf{y}\langle s\rangle}, u : A^{\mathbf{y}\langle s\rangle}, v : A$
$$\vdash (t(y,x)[{}^{u(x)}/_{y}])[{}^{v}/_{x}]$$
$$= (t(y,x)[{}^{v}/_{x}])[{}^{u(x)[{}^{v}/_{x}]}/_{y}]$$

5. $t, t' : A^{\mathbf{y}\langle s\rangle}, u : A$
$$\vdash \mathsf{app}(t(x), t'(x))[{}^{u}/_{x}]$$
$$= \mathsf{app}(t(x)[{}^{u}/_{x}], t'(x)[{}^{u}/_{x}])$$

6. $t : A^{\mathbf{y}\langle s\rangle \times \mathbf{y}\langle s\rangle}, u : A$
$$\vdash \mathsf{abs}(\lambda\!\!\lambda y.\, t(y,x))[{}^{u}/_{x}]$$
$$= \mathsf{abs}(\lambda\!\!\lambda y.\, t(y,x)[{}^{u}/_{x}])$$

▶ [20, 22].

Free Constructions

$$
\begin{tikzcd}
\mathsf{SubstAlg} \arrow[d, shift left] \\
\mathcal{S}et^{\mathsf{FinSet}} \arrow[u, shift left] \arrow[u, phantom, "\dashv"]
\end{tikzcd}
$$

- Explicit description of syntax with variable binding.
- Induction principle for syntax with variable biding.
- Definition of capture-avoiding substitution by structural recursion.

- [23, 25].

Free Constructions

$$\begin{tikzcd} \mathrm{SubstAlg} \arrow[d, shift left, ""] \arrow[d, phantom, "\dashv"] \\ \mathcal{S}et^{\mathrm{FinSet}} \arrow[u, shift left] \end{tikzcd}$$

- Explicit description of syntax with variable binding.
- Induction principle for syntax with variable biding.
- Definition of capture-avoiding substitution by structural recursion.
- Mathematical foundations for *metavariables*.

- [23, 25].

Metavariables

$$\begin{tikzcd} \mathsf{SubstAlg} \arrow[d, shift left] \\ \mathcal{S}et^{\mathsf{FinSet}} \arrow[u, shift left] \arrow[u, phantom, "\dashv"] \arrow[loop below, "\mathcal{M}"] \end{tikzcd}$$

- Kleisli maps

$$\mathbf{y}(n) \to \mathcal{M}\left(\coprod_i \mathbf{y}(m_i)\right)$$

are in bijective correspondence with terms

$$\begin{aligned} t \quad ::= \quad & x \mid t'(t'') \mid \lambda x.\, t' \\ \mid \quad & M_i[t_1, \ldots, t_{m_i}] \end{aligned}$$

with free variables amongst $x_1, \ldots, x_n$.

- Definition of *meta-substitution* by structural recursion:

$$\mathcal{M}(X) \times Y^X \to \mathcal{M}(Y)$$

- [13, 23, 25].

Second-Order Equational Presentations

Example: Untyped λ-calculus.

$$(\beta)\ \mathrm{M} : 1 \rhd x \vdash \big(\lambda x.\, \mathrm{M}[x]\big)\, x = \mathrm{M}\big[x\big]$$

$$(\eta)\ \mathrm{M} : 0 \rhd \cdot \vdash \lambda x.\, \mathrm{M}[\,]\,(x) = \mathrm{M}[\,]$$

Second-Order Equational Presentations

Example: Untyped λ-calculus.

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$$(\eta)\ \mathrm{M} : 0 \rhd \cdot \vdash \lambda x.\, \mathrm{M}[\,]\,(x) = \mathrm{M}[\,]$$

Second-Order Equational Logic

(Extended Metasubstitution Rule)

$$\frac{\begin{array}{c} \mathrm{M}_1 : m_1, \ldots, \mathrm{M}_k : m_k \rhd \Gamma \vdash s \equiv t \\ \Theta \rhd \Delta\,,\, x_{i,1}\,, \ldots, x_{i,m_i} \vdash s_i \equiv t_i \quad (1 \leq i \leq k) \end{array}}{\Theta \rhd \Gamma, \Delta \quad \vdash s\{\mathrm{M}_i := (\vec{x_i})s_i\}_{1\leq i\leq k} \equiv t\{\mathrm{M}_i := (\vec{x_i})t_i\}_{1\leq i\leq k}}$$

► [13], [31].

Results

Extension of the mathematical theory of (first-order) algebraic structure to simple type theory:

$$\begin{tikzcd} & \text{Cat. Alg.} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] \arrow[d, phantom, "2^{\text{nd}}\text{-order}"] & \\ \text{Univ. Alg.} \arrow[rr, leftrightarrow] & & \text{Eq. Log.} \end{tikzcd}$$

- Conservativity of Second-Order Equational Logic over Birkhoff's (first-order) Equational Logic.
- Soundness and completeness of Second-Order Equational Logic.
- Soundness and completeness of (bidirectional) Second-Order Term Rewriting.
- Presentation/theory correspondence via classifying categories and internal languages.
- Universal-algebra/categorical-algebra correspondence.
- Theory of syntactic algebraic translations.

► [31, 32].

II

Algebraic Modelling of Dependent Type Theories

Dependent Type Theory

	simply-typed theories	dependently-typed theories
types	algebraic	algebraic with binding
terms	algebraic with binding	algebraic with binding

The syntactic theory should account for:

- type dependency
- variable binding and α-equivalence
- term and type substitution

- [10].

- Dependency.

$$\begin{array}{rcl} & \vdash & C_0 \\ x, y : C_0 & \vdash & C_1(x, y) \end{array}$$

- Binding.

$$\frac{\Gamma, x : \sigma \vdash \tau}{\Gamma \vdash \Pi x : \sigma . \tau}$$

$$\frac{\Gamma, x : \sigma \vdash t : \tau}{\Gamma \vdash \lambda x : \sigma . t \; : \; \Pi x : \sigma . \tau}$$

- Substitution.

$$\frac{\Gamma \vdash t : \Pi x : \sigma . \tau \qquad \Gamma \vdash u : \sigma}{\Gamma \vdash t(u) : \tau\left[{}^{u}/_{x}\right]}$$

$$(\lambda x : \sigma . t)(u) = t\left[{}^{u}/_{x}\right]$$

Categories of Dependent Contexts

Def: A *dependently-typed context structure* is given by

- a small category $\mathbb{C}$ with terminal object,
- a presheaf $S \in \widehat{\mathbb{C}}$,
- a functorial assignment of pullbacks

$$\begin{tikzcd}
 & \Delta, \sigma[f] \arrow[dl, "\pi_{\Delta \vdash \sigma[f]}"'] \arrow[dr, "f^*"] & \\
\Delta \arrow[dr, "f"'] & & \Gamma, \sigma \arrow[dl, "\pi_{\Gamma \vdash \sigma}"] \\
 & \Gamma &
\end{tikzcd}$$

for all $\Gamma \vdash \sigma$ and $f : \Delta \to \Gamma$ in $\mathbb{C}$.

► [14].

Type-Dependent Binding

- **Local context-extension lemma.**

  For $\Gamma \vdash \sigma$ and $P \in \widehat{\mathbb{C}_{/\Gamma}}$,

$$P^{\mathbf{y}(\pi_{\Gamma \vdash \sigma})}(\Delta \xrightarrow{f} \Gamma) \;\cong\; P(\Delta, \sigma[f] \to \Gamma) \quad .$$

Type-Dependent Binding

- **Local context-extension lemma.**

  For $\Gamma \vdash \sigma$ and $P \in \widehat{\mathbb{C}_{/\Gamma}}$,

  $$P^{\mathbf{y}(\pi_{\Gamma\vdash\sigma})}(\Delta \xrightarrow{f} \Gamma) \;\cong\; P(\Delta, \sigma[f] \to \Gamma) \;\; .$$

- Type-dependent binding operators.

  $$\Pi_\sigma : {S_\Gamma}^{\mathbf{y}(\pi_{\Gamma\vdash\sigma})} \to S_\Gamma \text{ in } \widehat{\mathbb{C}_{/\Gamma}}$$

Decomposition of Binding Arities

- For $\Gamma \vdash \sigma$, consider the adjunction

$$\begin{tikzcd} \mathbb{C}_{/\Gamma} \arrow[r, shift left=2, "\pi^*"] \arrow[r, phantom, "\top"] & \mathbb{C}_{/\Gamma,\sigma} \arrow[l, shift left=2, "\mathbb{C}_{/\pi}"] \end{tikzcd}$$

Decomposition of Binding Arities

- For $\Gamma \vdash \sigma$, consider the adjunction

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- It induces the adjoint situation:

$$\begin{tikzcd} \widehat{\mathbb{C}_{/\Gamma}} \arrow[r, shift left=6, "(\pi^*)_*"] \arrow[r, phantom, shift left=4, "\top"] \arrow[r, phantom, "\top"] \arrow[r, phantom, shift right=4, "\top"] \arrow[r, shift right=2, "\epsilon_\sigma"] & \widehat{\mathbb{C}_{/\Gamma,\sigma}} \arrow[l, shift right=2, "\delta_\sigma"'] \arrow[l, shift left=6, "(\mathbb{C}_{/\pi})_!"] \end{tikzcd}$$

- **Binding-arity decomposition lemma.**

  For $\Gamma \vdash \sigma$, the monad $(-)^{\mathbf{y}(\pi_{\Gamma\vdash\sigma})}$ on $\widehat{\mathbb{C}_{/\Gamma}}$ is induced by the adjunction $\varepsilon_\sigma \dashv \delta_\sigma$.

Binding Arities

- We thus obtain the following situation

$$
\begin{tikzcd}[column sep=large]
\widehat{\mathbb{C}_{/\Gamma}} \arrow[loop left, "(-)^{\mathbf{y}(\pi_{\Gamma \vdash \sigma})}"] \arrow[r, phantom, "\cong"] & \widehat{\mathbb{C}}_{/\mathbf{y}(\Gamma)} \arrow[rr, shift left=9] \arrow[rr, shift left=6, phantom, "\top"] \arrow[rr, shift left=3, leftarrow, "\prod_\sigma" description] \arrow[rr, phantom, "\top"] \arrow[rr, shift right=3, "(\mathbf{y}\pi)^*" description] \arrow[rr, shift right=6, phantom, "\top"] \arrow[rr, shift right=9, leftarrow, "\sum_\sigma" description] & & \widehat{\mathbb{C}}_{/\mathbf{y}(\Gamma,\sigma)} \arrow[r, phantom, "\cong"] & \widehat{\mathbb{C}_{/\Gamma,\sigma}}
\end{tikzcd}
$$

Binding Arities

- We thus obtain the following situation

$$
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\widehat{\mathbb{C}_{/\Gamma}} \arrow[loop left, "(-)^{\mathbf{y}(\pi_{\Gamma\vdash\sigma})}"] \arrow[r, phantom, "\cong"] & \widehat{\mathbb{C}}_{/\mathbf{y}(\Gamma)} \arrow[r, shift left=9] \arrow[r, shift left=4.5, phantom, "\top"] \arrow[r, phantom, "\top"] \arrow[r, shift right=3, "(\mathbf{y}\pi)^*"] \arrow[r, shift right=7.5, phantom, "\top"] & \widehat{\mathbb{C}}_{/\mathbf{y}(\Gamma,\sigma)} \arrow[l, shift right=3, "\prod_\sigma"'] \arrow[l, shift left=9, "\sum_\sigma"] \arrow[r, phantom, "\cong"] & \widehat{\mathbb{C}_{/\Gamma,\sigma}}
\end{tikzcd}
$$

Type-Dependent Binding

- Type-dependent binding operators.

$$\Pi_\sigma : \prod_\sigma \big(\mathbf{y}(\Gamma,\sigma)^* S\big) \to S \ \text{ in } \widehat{\mathbb{C}}$$

Binding Arities

- We thus obtain the following situation

$$\begin{tikzcd}[column sep=huge]
\widehat{\mathbb{C}_{/\Gamma}} \arrow[loop left, "(-)^{\mathbf{y}(\pi_{\Gamma\vdash\sigma})}"] \cong \widehat{\mathbb{C}}_{/\mathbf{y}(\Gamma)}
\arrow[r, shift left=12]
\arrow[r, phantom, shift left=8, "\top"]
\arrow[r, leftarrow, shift left=4, "\prod_\sigma" description]
\arrow[r, phantom, "\top"]
\arrow[r, shift right=4, "(\mathbf{y}\pi)^*" description]
\arrow[r, phantom, shift right=8, "\top"]
\arrow[r, leftarrow, shift right=12, "\sum_\sigma" description]
& \widehat{\mathbb{C}}_{/\mathbf{y}(\Gamma,\sigma)} \cong \widehat{\mathbb{C}_{/\Gamma,\sigma}}
\end{tikzcd}$$

Type-Dependent Binding

- Type-dependent binding operators.

$$\Pi_\sigma : \prod_\sigma \big(\mathbf{y}(\Gamma,\sigma)^* S\big) \to S \quad \text{in } \widehat{\mathbb{C}}$$

Term Binding

- Term binding operators.

  For $\Gamma \vdash \sigma$ and $\Gamma, \sigma \vdash \tau$,

$$\begin{tikzcd}
\prod_\sigma(T_\tau) \arrow[r, "\lambda"] \arrow[d] & T \arrow[d] \\
\mathbf{y}(\Gamma) \arrow[r, "\mathbf{y}\big(\Pi_\sigma(\tau)\big)"'] & S
\end{tikzcd} \quad \text{in } \widehat{\mathbb{C}}$$

Variables

$$\begin{tikzcd}
\mathbf{y}(\Gamma, \sigma) \arrow[rr] \arrow[dr, "\mathbf{y}(\pi_{\Gamma \vdash \sigma})"'] & & \mathsf{T}_\sigma \arrow[dl] \\
& \mathbf{y}(\Gamma) &
\end{tikzcd}$$

- This is to give a term $\Gamma, \sigma \vdash p : \sigma[\pi_{\Gamma \vdash \sigma}]$.

Variables

$$\begin{tikzcd} \mathbf{y}(\Gamma, \sigma) \arrow[rr] \arrow[dr, "\mathbf{y}(\pi_{\Gamma \vdash \sigma})"'] & & T_\sigma \arrow[dl] \\ & \mathbf{y}(\Gamma) & \end{tikzcd}$$

- This is to give a term $\Gamma, \sigma \vdash p : \sigma[\pi_{\Gamma \vdash \sigma}]$.

NB: Categories with attributes/families

The condition

$$\begin{tikzcd} \mathbf{y}(\Gamma, \sigma) \arrow[rr, "\cong"] \arrow[dr, "\mathbf{y}(\pi_{\Gamma \vdash \sigma})"'] & & T_\sigma \arrow[dl] \\ & \mathbf{y}(\Gamma) & \end{tikzcd}$$

is equivalent to *Dybjer's context comprehension* property: For all maps $f : \Delta \to \Gamma$ in $\mathbb{C}$ and terms $\Delta \vdash t : \sigma[f]$ there exists a unique map $\langle f, t \rangle : \Delta \to (\Gamma, \sigma)$ in $\mathbb{C}$ such that $p\big[\langle f, t \rangle\big] = t$ and $\pi_{\Gamma \vdash \sigma} \circ \langle f, t \rangle = f$.

- The initial model is the *classifying category*.

- [14, 18].

Substitution Structure

For $\Gamma \vdash \sigma$,

$$\begin{tikzcd} {T_\Gamma}^{\mathbf{y}(\pi_{\Gamma\vdash\sigma})} \times T_{\Gamma\vdash\sigma} \arrow[r] \arrow[d] & T_\Gamma \arrow[d] \\ {S_\Gamma}^{\mathbf{y}(\pi_{\Gamma\vdash\sigma})} \times T_{\Gamma\vdash\sigma} \arrow[r] & S_\Gamma \end{tikzcd} \quad \text{in } \widehat{\mathbb{C}}_{/\mathbf{y}(\Gamma)}$$

subject to axioms.

II

Algebraic Foundations

Kan Extensions

Every

$$f : \mathbb{X} \to \mathbb{Y}$$

induces

$$\begin{tikzcd} \mathcal{P}\mathbb{X} \arrow[r, bend left=50, "f_*"] \arrow[r, phantom, shift left=3, "\top"] \arrow[r, phantom, shift right=3, "\top"] \arrow[r, bend right=50, "f_!"'] & \mathcal{P}\mathbb{Y} \arrow[l, "f^*"'] \end{tikzcd}$$

where

$$\mathcal{P}\mathbb{C} =^{\text{def}} \boldsymbol{\mathcal{S}et}^{\mathbb{C}}$$

and

$$\begin{aligned} f_* P\, y &= \mathrm{Ran}_f P\, y &= \int_{x\in\mathbb{X}} [\,\mathbb{Y}(y, fx) \Rightarrow Px\,] \\ f^* Q\, x &= Q(fx) & \\ f_! P\, y &= \mathrm{Lan}_f P\, y &= \int^{x\in\mathbb{X}} \mathbb{Y}(fx, y) \times Px \end{aligned}$$

► [4].

Generalised Dependent Polynomial Functors

The class of

generalised dependent polynomial functors

is the closure under natural isomorphism of the functors

$$\mathcal{P}\mathbb{A} \longrightarrow \mathcal{P}\mathbb{B}$$

arising as composites

$$\mathcal{P}\mathbb{A} \xrightarrow{s^*} \mathcal{P}\mathbb{I} \xrightarrow{f_*} \mathcal{P}\mathbb{J} \xrightarrow{t_!} \mathcal{P}\mathbb{B}$$

from diagrams

$$\mathbb{A} \xleftarrow{s} \mathbb{I} \xrightarrow{f} \mathbb{J} \xrightarrow{t} \mathbb{B}$$

in $\mathcal{C}at$.

$$t_! f_* s^* A\, b = \int^{j \in \mathbb{J}} \mathbb{B}(tj, b) \times \int_{i \in \mathbb{I}} \left[\mathbb{J}(j, fi) \Rightarrow A(si) \right]$$

Examples:

- *Dependent polynomial functors* (aka *indexed containers*) between slices of $\mathcal{S}et$ are [isomorphic to] generalised dependent polynomial functors.

- [21, 24].

- **Untyped abstract syntax**
  1. The rule

$$\frac{\Gamma \vdash t \qquad \Gamma \vdash t'}{\Gamma \vdash t(t')}$$

has associated the generalised dependent polynomial endofunctor represented by

$$\mathsf{FinSet} \xleftarrow{\nabla_2} 2 \cdot \mathsf{FinSet} \xrightarrow{\nabla_2} \mathsf{FinSet} \xrightarrow{\mathrm{id}} \mathsf{FinSet}$$

- **Untyped abstract syntax**
  1. The rule
$$\frac{\Gamma \vdash t \qquad \Gamma \vdash t'}{\Gamma \vdash t(t')}$$
     has associated the generalised dependent polynomial endofunctor represented by
$$\mathsf{FinSet} \xleftarrow{\nabla_2} 2 \cdot \mathsf{FinSet} \xrightarrow{\nabla_2} \mathsf{FinSet} \xrightarrow{\mathrm{id}} \mathsf{FinSet}$$
  2. The rule
$$\frac{\Gamma, x \vdash t}{\Gamma \vdash \lambda x.\, t}$$
     has associated the generalised dependent polynomial endofunctor represented by
$$\mathsf{FinSet} \xleftarrow{+1} \mathsf{FinSet} \xrightarrow{\mathrm{id}} \mathsf{FinSet} \xrightarrow{\mathrm{id}} \mathsf{FinSet}$$

- **Simply typed abstract syntax**

  Let S be the set of simple types and write C for the category $\mathsf{FinSet}_{/\mathsf{S}}$ of S-sorted contexts.

  1. The rule

  $$\frac{\Gamma \vdash t : \tau' \Rightarrow \tau \qquad \Gamma \vdash t' : \tau'}{\Gamma \vdash t(t') : \tau}$$

     has associated the generalised dependent polynomial endofunctor represented by

$$\mathsf{C} \times \mathsf{S} \xleftarrow{[\mathrm{id}\times\Rightarrow,\mathrm{id}\times\pi_1]} 2 \cdot (\mathsf{C} \times \mathsf{S}^2) \xrightarrow{\nabla_2} \mathsf{C} \times \mathsf{S}^2 \xrightarrow{\mathrm{id}\times\pi_2} \mathsf{C} \times \mathsf{S}$$

- **Simply typed abstract syntax**

  Let S be the set of simple types and write C for the category $\mathsf{FinSet}_{/\mathsf{S}}$ of S-sorted contexts.

  1. The rule

     $$\frac{\Gamma \vdash t : \tau' \Rightarrow \tau \qquad \Gamma \vdash t' : \tau'}{\Gamma \vdash t(t') : \tau}$$

     has associated the generalised dependent polynomial endofunctor represented by

$$\mathsf{C} \times \mathsf{S} \xleftarrow{[\mathrm{id} \times \Rightarrow, \mathrm{id} \times \pi_1]} 2 \cdot (\mathsf{C} \times \mathsf{S}^2) \xrightarrow{\nabla_2} \mathsf{C} \times \mathsf{S}^2 \xrightarrow{\mathrm{id} \times \pi_2} \mathsf{C} \times \mathsf{S}$$

  2. The rule

     $$\frac{\Gamma, x : \sigma \vdash t : \tau}{\Gamma \vdash \lambda x.\, t : \sigma \Rightarrow \tau}$$

     has associated the generalised dependent polynomial endofunctor represented by

$$\mathsf{C} \times \mathsf{S} \xleftarrow{+ \times \mathrm{id}} \mathsf{C} \times \mathsf{S} \times \mathsf{S} \xrightarrow{\mathrm{id}} \mathsf{C} \times \mathsf{S} \times \mathsf{S} \xrightarrow{\mathrm{id} \times \Rightarrow} \mathsf{C} \times \mathsf{S}$$

NB: The association of generalised dependent polynomial functors to rules extends to *polymorphic languages*.
In this context, the last component of the representation plays a crucial role as a *pattern-matching* constructor.

► [34].

- **Convolution monoidal closed structure**
  1. *Day's convolution tensor product*
     is [isomorphic to] a generalised dependent polynomial functor.
  2. *Exponentiation to a representable*
     with respect to the closed structure associated to the convolution monoidal structure is a generalised polynomial functor.

- [7].

Generalised Inductive Dependent Polynomial Functors

The class of generalised dependent polynomial functors represented by diagrams of the form

$$\mathbb{A} \longleftarrow \coprod_{k\in K} L_k \cdot \mathbb{J}_k \xrightarrow{\coprod_{k\in K} \nabla_{L_k}} \coprod_{k\in K} \mathbb{J}_k \longrightarrow \mathbb{B}$$

where L_k is finite for all $k \in K$,

- is closed under constants, identities, coproducts, finite products, and composition; and
- admits a (cartesian) differential calculus.

These functors

- are inductive (*viz.* finitary and preserve epis); and
- admit inductively-defined free algebras for equational systems.

Application Areas

- Data types.
  (*e.g.* reasoning)
- Type theory.
  (*e.g.* formalisation)
- Logical frameworks.
  (*e.g.* synthesis)
- Dependently-typed programming.
  (*e.g.* zippers)
- Concurrency theory.
  (*e.g.* models)

Pointers

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