Mathematical Synthesis
of
Equational Deduction Systems

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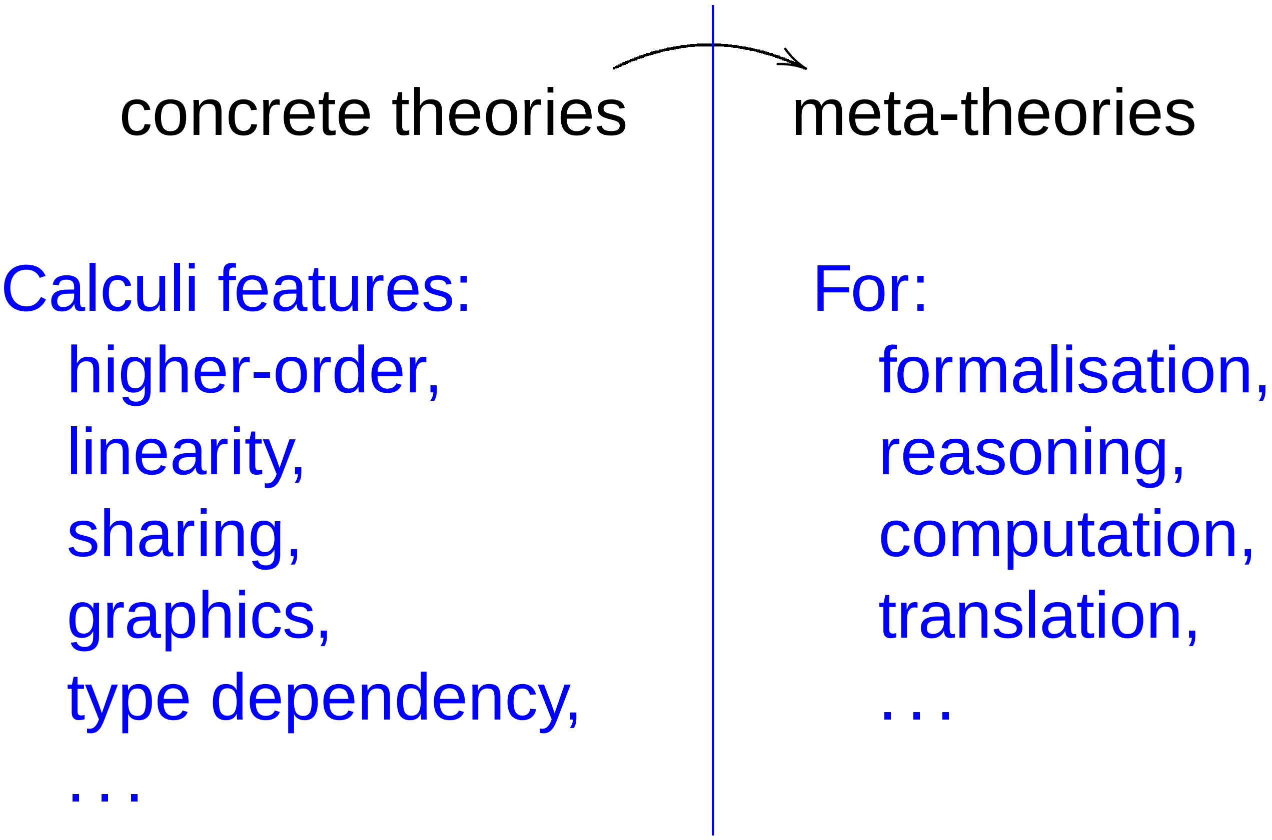
TLCA 2009
3.VII.2009

Context

concrete theories

meta-theories

Context



concrete theories

Calculi features:

- higher-order,
- linearity,
- sharing,
- graphics,
- type dependency,
- . . .

meta-theories

For:

- formalisation,
- reasoning,
- computation,
- translation,
- . . .

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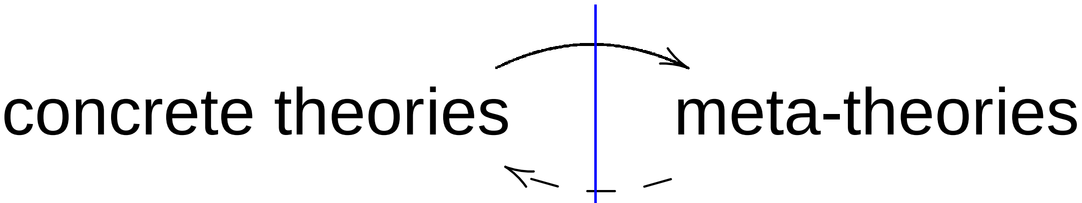
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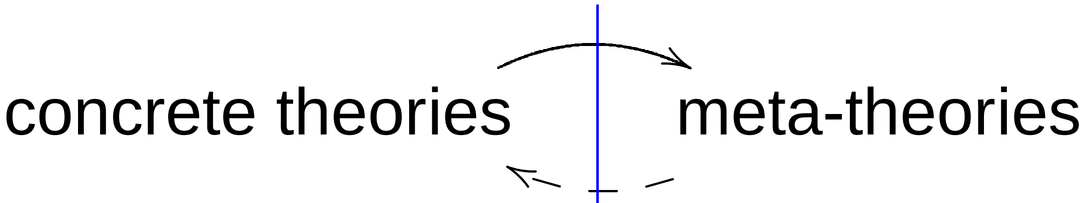
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Programme

[1] mathematical
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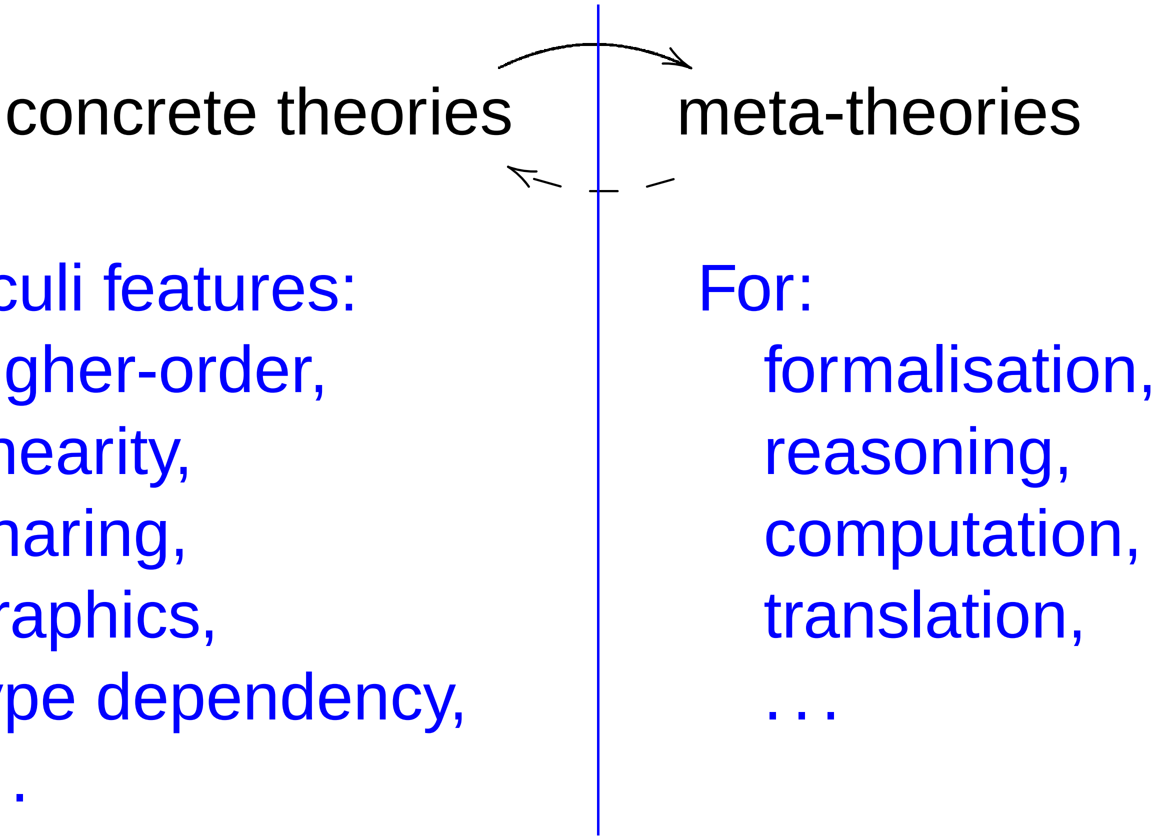

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This Talk
Part I

Mathematical *algebraic* framework and methodology for the synthesis of deduction systems for equational reasoning and computation by rewriting.

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Part I

Mathematical *algebraic* framework and methodology for the synthesis of deduction systems for equational reasoning and computation by rewriting.

- *Semantics*
  - Algebraic model theory
- *Syntax*
  - Initial-algebra semantics (⇒ compositionality)
  - structural recursion
  - induction principle
  - theory of translations

This Talk
Part II

TLCA

Typed Lambda Calculi:
- [simply-typed] λ-calculus,
- Martin Löf type theory,
- . . .

? What are Typed Lambda Calculi?

This Talk
Part II

TLCA

Typed Lambda Calculi:

[simply-typed] λ-calculus,
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$\boxed{?}$ What are Typed Lambda Calculi?

$\boxed{!}$ Simple Type Theories are
Second-Order Equational Presentations!

Part I

Equational Meta-Logic

Universal Algebra

Syntax	Semantics

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signatures: $\Sigma = \{ \Sigma_n \in \mathcal{S}\!\mathit{et} \}_{n \in \mathbb{N}}$	

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Birkhoff's Deduction Problem

Devise a deduction system such that

$t \equiv t'$ is derivable from a set of equations $\mathcal{E}$

soundness $\Big\Downarrow$ $\Big\Uparrow$ completeness

for all $\underline{A} \models \mathcal{E}$, $\underline{A} \models t \equiv t'$

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Birkhoff's Equational Logic of Universal Algebra

$$\frac{(t \equiv t') \in \mathcal{E}}{t \equiv t'}$$

$$\frac{t_i \equiv t_i' \quad (i = 1, \ldots, n)}{f(t_1, \ldots, t_n) \equiv f(t_1', \ldots, t_n')} \quad (f : n)$$

$$\frac{t \equiv t'}{t[\rho] \equiv t'[\rho]} \quad (\rho \text{ a substitution})$$

Analysis of Universal Algebra

Syntax	Semantics
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Monadic Algebra

Syntax	Semantics
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Monadic Algebra

Generalised Syntax	Semantics
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Deduction Problem

Devise a deduction system such that

$\mathrm{t} \equiv \mathrm{t}' : \mathrm{I} \to \mathsf{TV}$ is derivable from a set of generalised equations $\mathcal{E}$

soundness $\Downarrow$ $\Uparrow$ completeness

for all $\underline{A} \models \mathcal{E}$, $\underline{A} \models \mathrm{t} \equiv \mathrm{t}'$

Equational Meta-Logic Rules

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$$(\text{Subst})\ \frac{\begin{array}{c} t_1 \equiv t_1' : U \to TV \\ t_2 \equiv t_2' : V \to TW \end{array}}{t_1[t_2] \equiv t_1'[t_2'] : U \to TW}$$

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$$(\textsf{LocChar})\ \frac{\begin{array}{c} \{\, e_i : U_i \to U \,\}_{i \in I} \text{ a cover} \\ t\, e_i \equiv t'\, e_i : U_i \to TV \quad (i \in I) \end{array}}{t \equiv t' : U \to TV}$$

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$$(\text{Ext})\ \frac{t \equiv t' : U \to TV}{\langle I \rangle t \equiv \langle I \rangle t' : I \otimes U \to T(I \otimes V)}$$

Soundness

If $t \equiv t' : I \to TV$ is derivable from $\mathcal{E}$

then $\underline{A} \models t \equiv t'$, for all $\underline{A} \models \mathcal{E}$.

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Internal Soundness and Completeness

For

$\widetilde{TV}$ the free algebra satisfying $\mathcal{E}$

and

$q : TV \to \widetilde{TV}$ the associated quotient map,

the following are equivalent:

1. $\underline{A} \models t \equiv t' : U \to TV$, for all $\underline{A} \models \mathcal{E}$
2. $\widetilde{TV} \models t \equiv t' : U \to TV$
3. $q\,t = q\,t' : U \to \widetilde{TV}$

Reconstruction of Birkhoff's Equational Logic

$$\mathcal{S}\boldsymbol{et}$$

syntactic structure

= signature: $\Sigma = \{\, \Sigma_n \in \mathcal{S}\boldsymbol{et} \,\}_{n \in \mathbb{N}}$

Reconstruction of Birkhoff's Equational Logic

$$\begin{tikzcd}
\mathbf{\Sigma\text{-}Alg} \arrow[d, shift left=3] \\
\mathbfit{Set} \arrow[u, shift left=3] \arrow[u, phantom, "\dashv"] \arrow[loop left, "\mathbb{T}_\Sigma"']
\end{tikzcd}$$

$$\mathsf{T}_\Sigma(X) \;\cong\; X + \coprod_{n\in\mathbb{N}} \Sigma_n \times (\mathsf{T}_\Sigma X)^n$$

interpretation

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Reconstruction of Birkhoff's Equational Logic

$$\begin{tikzcd}
\mathbf{\Sigma\text{-}Alg} \arrow[d, shift left=2ex] \arrow[d, phantom, "\dashv"] \\
\mathcal{S}et \arrow[u, shift left=2ex] \arrow[loop left, "\mathbb{T}_\Sigma"]
\end{tikzcd}$$

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interpretation

concretion

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Equational Logic

$$\mathsf{V} \vdash \mathsf{t} \equiv \mathsf{t}'$$

Substitution Rule

$$\text{(Subst)}\ \dfrac{V \vdash s \equiv s' \qquad \text{(LocChar)}\ \dfrac{W \vdash t_x \equiv t'_x \quad (x \in V)}{W \vdash \{\, t_x \,\}_{x \in V} \equiv \{\, t'_x \,\}_{x \in V}}}{W \vdash s[t_x/x]_{x \in V} \equiv s'[t'_x/x]_{x \in V}}$$

Reconstruction of Goguen and Meseguer's Multi-Sorted Equational Logic

$\mathcal{S}et^{\mathsf{S}}$

syntactic structure

= signature: $\Sigma = \{\, \Sigma_\sigma \in \mathcal{S}et^{\mathsf{S}} \,\}_{\sigma \in \mathsf{S}^*}$

General method for the extension from the mono-sorted to the multi-sorted case.

Reconstruction of Goguen and Meseguer's Multi-Sorted Equational Logic

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$$(\mathsf{T}_\Sigma \mathsf{X})_{\mathsf{s}} \cong \mathsf{X}_{\mathsf{s}} + \coprod_{\sigma=(\mathsf{s}_1 \ldots \mathsf{s}_n) \in \mathsf{S}^*} \Sigma_{\sigma,\mathsf{s}} \times \prod_{i=1}^{n} (\mathsf{T}_\Sigma \mathsf{X})_{\mathsf{s}_i}$$

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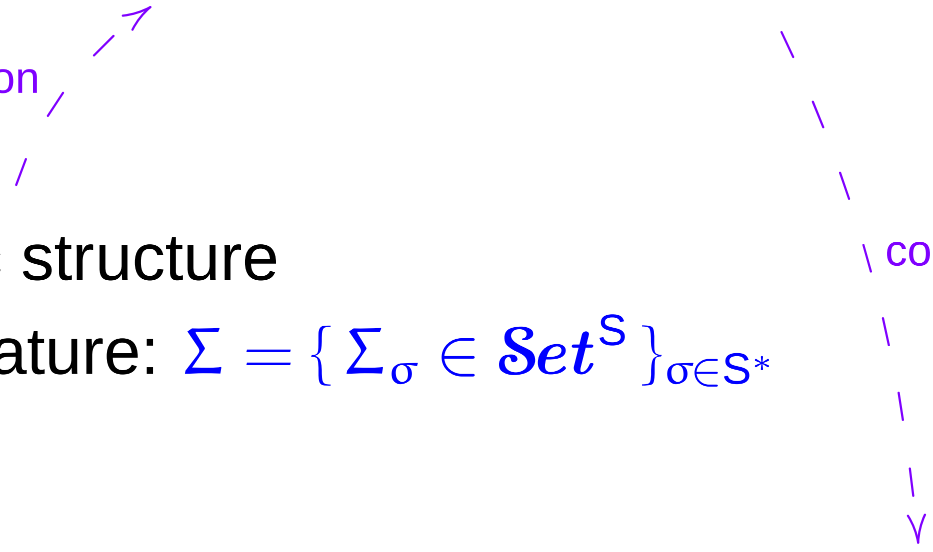
$$(T_\Sigma X)_s \;\overset{\sim}{=}\; X_s + \coprod_{\sigma=(s_1\ldots s_n)\in S^*} \Sigma_{\sigma,s} \times \prod_{i=1}^{n} (T_\Sigma X)_{s_i}$$



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Equational Logic

$$\vec{x} : \vec{\sigma} \vdash t \equiv t' : s$$

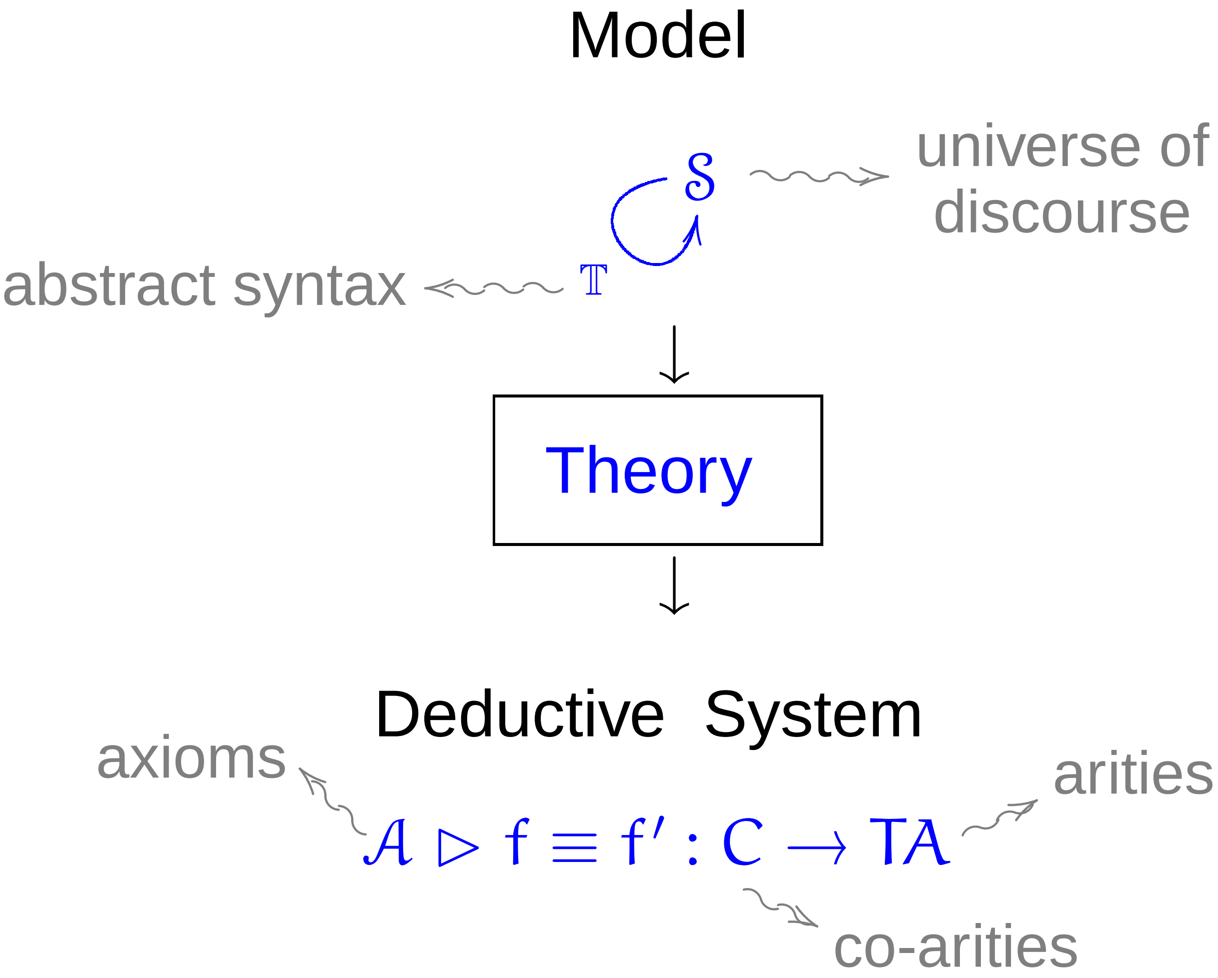
General method for the extension from the mono-sorted to the multi-sorted case.

Methodology

Model

$$\begin{tikzcd}
 & \mathcal{S} \arrow[loop left] \arrow[r, squiggly] & \text{universe of discourse} \\
\text{abstract syntax} & \mathbb{T} \arrow[l, squiggly] &
\end{tikzcd}$$

Methodology



Model

$\mathbb{T}$ $\mathcal{S}$

abstract syntax

universe of discourse

Theory

Deductive System

$$\mathcal{A} \rhd f \equiv f' : C \to TA$$

axioms

arities

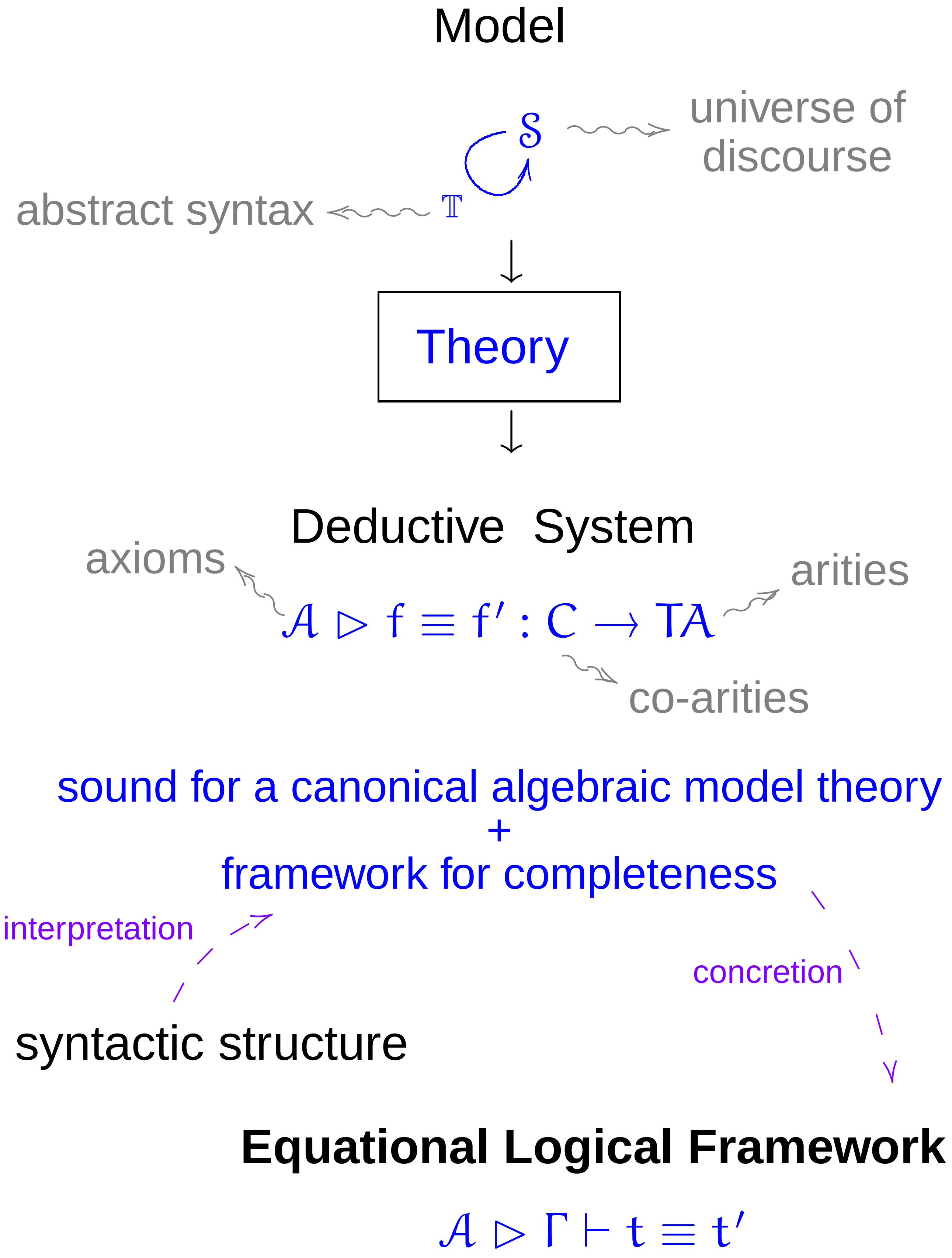
co-arities

sound for a canonical algebraic model theory

+

framework for completeness

Methodology



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Equational Logical Framework

$\mathcal{A} \rhd \Gamma \vdash t \equiv t'$

Part II

The Universal Algebra of Simple Type Theory

Simply Typed Theories

types	algebraic
terms	algebraic with binding

Simply Typed Theories

	algebraic	simply typed	dependently typed
		theories	
types	unstructured	algebraic	algebraic with binding
terms	algebraic	algebraic with binding	algebraic with binding

- The paradigmatic simple type theory:

  λ-calculus

  $$(\beta) \quad (\lambda x.\, M)\, N = M[N/x]$$

  $$(\eta) \quad \lambda x.\, M\, x = M \qquad \big(x \notin \mathrm{FV}(M)\big)$$

- Simply-typed λ-calculus, computational λ-calculus, . . .

- The paradigmatic simple type theory:

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- Simply-typed λ-calculus, computational λ-calculus, . . .

- The syntactic theory should account for:
  - variables and meta-variables
  - variable binding and α-equivalence
  - capture-avoiding and meta substitution
  - mono and multi sorting

Algebraic Model

syntactic structure =

- arities: an operator of arity $\vec{n} = (n_1 \dots n_k)$ takes k arguments, respectively binding n_i variables.

Algebraic Model

finite sets (contexts)
and functions (renamings)

$\mathcal{S}et^{\mathbb{F}}$

$X \in \mathcal{S}et^{\mathbb{F}}$ is a functor $\begin{cases} X\Gamma \ (\Gamma \in \mathbb{F}) \\ \mathbb{F}(\Gamma, \Delta) \to \mathcal{S}et(X\Gamma, X\Delta) \end{cases}$

E.g. the object of variables is $V\Gamma = \Gamma$

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\+

- **substitution**

Algebras with Substitution

(Σ-monoids)

- algebra structure:

$$\Sigma X \xrightarrow{\xi} X$$

- substitution structure:

$$\text{monoid} \quad V \xrightarrow{e} X \xleftarrow{m} X \bullet X$$

$$\left(\equiv \quad \begin{array}{c} \Gamma \longrightarrow X\Gamma \longleftarrow X\Delta \times (X\Gamma)^{\Delta} \\ \text{subject to the laws of substitution} \end{array} \right)$$

subject to the compatibility condition:

$$\begin{tikzcd}
\Sigma(X) \bullet X \arrow[r] \arrow[d, "\xi \bullet X"'] & \Sigma(X \bullet X) \arrow[r, "\Sigma m"] & \Sigma X \arrow[d, "\xi"] \\
X \bullet X \arrow[rr, "m"'] & & X
\end{tikzcd}$$

λ Pre-Models with Substitution

- Signature.

$$\begin{aligned} \Sigma_\lambda \quad &= \quad \mathsf{0} \,@\, \mathsf{0} && \text{(application)} \\ &\mid \quad \lambda(1) && \text{(abstraction)} \end{aligned}$$

- Algebras.

$$\mathsf{app} : \mathrm{L}^2 \to \mathrm{L} \qquad \mathsf{abs} : \mathrm{L}^{\mathrm{V}} \to \mathrm{L}$$

$$\mathsf{var} : \mathrm{V} \longrightarrow \mathrm{L} \longleftarrow \mathrm{L} \bullet \mathrm{L} : \mathsf{subst}$$

$$\begin{tikzcd}
\mathrm{L}^2 \bullet \mathrm{L} \arrow[r, "\cong"] \arrow[d, "\mathsf{app} \bullet \mathrm{L}"'] & (\mathrm{L} \bullet \mathrm{L})^2 \arrow[r, "\mathsf{subst}^2"] & \mathrm{L}^2 \arrow[d, "\mathsf{app}"] \\
\mathrm{L} \bullet \mathrm{L} \arrow[rr, "\mathsf{subst}"'] & & \mathrm{L}
\end{tikzcd}$$

$$\begin{tikzcd}
\mathrm{L}^{\mathrm{V}} \bullet \mathrm{L} \arrow[r] \arrow[d, "\mathsf{abs} \bullet \mathrm{L}"'] & (\mathrm{L} \bullet \mathrm{L})^{\mathrm{V}} \arrow[r, "\mathsf{subst}^{\mathrm{V}}"] & \mathrm{L}^{\mathrm{V}} \arrow[d, "\mathsf{abs}"] \\
\mathrm{L} \bullet \mathrm{L} \arrow[rr, "\mathsf{subst}"'] & & \mathrm{L}
\end{tikzcd}$$

Monadic Model

$$\begin{tikzcd} \Sigma\text{-}\mathbf{Mon} \arrow[d, shift left=2] \arrow[d, phantom, "\dashv"] \\ \mathcal{S}et^{\mathbb{F}} \arrow[u, shift left=2] \arrow[loop left, "\mathcal{M}_\Sigma"] \end{tikzcd}$$

Thm:

1. $\mathcal{M}_\Sigma(X) \cong V + X \bullet \mathcal{M}_\Sigma(X) + \Sigma(\mathcal{M}_\Sigma X)$

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Thm:

1. $$\mathcal{M}_\Sigma(X) \cong V + X \bullet \mathcal{M}_\Sigma(X) + \Sigma(\mathcal{M}_\Sigma X)$$

2. For Σ induced by a binding signature,

   $\mathcal{M}_\Sigma$ is a strong monad .

concretion

Syntactic theory for variables, meta-variables, variable binding, α-equivalence, capture-avoiding substitution, meta-substitution.

Second-Order Syntactic Theory (I)

- Syntax:

  For X an object of meta-variables,

$$
\begin{array}{lll}
t \in \mathcal{M}_\Sigma(X)_\Gamma & & \\
\quad ::= \lfloor x \rfloor & & (x \in \Gamma) \\
\quad\ \ \mid \ \ M[t_1, \ldots, t_\ell] & & \left(\begin{array}{l} M \in X(\ell) \\ t_i \in (M_\Sigma X)_\Gamma \end{array} \right) \\
\quad\ \ \mid \ \ f\big((\vec{x_1})t_1, \ldots, (\vec{x_k})t_k\big) & & \left(\begin{array}{l} f \in \Sigma_{(|\vec{x_1}|\ldots|\vec{x_k}|)} \\ t_i \in (M_\Sigma X)_{\Gamma,\vec{x_i}} \end{array} \right)
\end{array}
$$

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\quad \mid \; \mathsf{f}\big((\vec{x_1})t_1, \ldots, (\vec{x_k})t_k\big) & & \left(\begin{array}{l} \mathsf{f} \in \Sigma_{(|\vec{x_1}| \ldots |\vec{x_k}|)} \\ t_i \in (\mathcal{M}_\Sigma X)_{\Gamma, \vec{x_i}} \end{array} \right)
\end{array}
$$

- Capture-avoiding substitution:

$$
\mathcal{M}_\Sigma(X) \bullet \mathcal{M}_\Sigma(X) \longrightarrow \mathcal{M}_\Sigma(X)
$$

$$
\left(\equiv \mathcal{M}_\Sigma(X)_\Delta \times \big(\mathcal{M}_\Sigma(X)_\Gamma\big)^\Delta \longrightarrow \mathcal{M}_\Sigma(X)_\Gamma \right)
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- Meta-substitution:

$$\mathcal{M}_\Sigma(X) \times \big(\mathcal{M}_\Sigma(Y)\big)^X \longrightarrow \mathcal{M}_\Sigma(Y)$$

$$\left(\equiv \mathcal{M}_\Sigma(X)_\Gamma \times \prod_{\ell \in \mathbb{N}} X(\ell) \Rightarrow \left((\mathcal{M}_\Sigma Y)^{V^\ell} \right)_\Gamma \to \mathcal{M}_\Sigma(Y)_\Gamma \right)$$

Second-Order Syntactic Theory (II)

- *Canonical specification* and derived *correct definition* of
  - variable renaming,
  - capture-avoiding simultaneous substitution,
  - meta-variable renaming,
  - meta-substitution.
- Canonical *algebraic model theory*.

Second-Order Equational Presentations

$$x_1, \ldots, x_n \,;\, M_1[m_1], \ldots, M_k[m_k] \vdash t \equiv t'$$

Second-Order Algebraic Models

$$[\![t]\!] = [\![t']\!] : V^n \times A^{V^{m_1}} \times \ldots \times A^{V^{m_k}} \to A$$

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λ-calculus

$$(\beta) \quad \mathsf{M}[1], \mathsf{N}[0] \vdash \lambda\big((x)\mathsf{M}[\lfloor x \rfloor]\big) \,@\, \mathsf{N}[\,] \equiv \mathsf{M}\big[\mathsf{N}[\,]\big]$$

$$(\eta) \quad \mathsf{M}[\,] \vdash \lambda\big((x)\ \mathsf{M}[\,] \,@\, \lfloor x \rfloor\big) \equiv \mathsf{M}[\,]$$

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$$x_1, \ldots, x_n \,;\, M_1[m_1], \ldots, M_k[m_k] \vdash t \equiv t'$$

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$$(\eta) \quad M[\,] \vdash \lambda\big((x)\; M[\,] \mathbin{@} \lfloor x \rfloor\big) \equiv M[\,]$$

Second-Order Algebraic Models

$$[\![t]\!] = [\![t']\!] : V^n \times A^{V^{m_1}} \times \ldots \times A^{V^{m_k}} \to A$$

λ-models with substitution

$$\begin{tikzcd}
L^V \times L \arrow[r] \arrow[d, "\mathsf{abs}\times\mathrm{id}"'] \arrow[dr, phantom, "(\beta)"] & L \bullet L \arrow[d, "\mathsf{subst}"] \\
L \times L \arrow[r, "\mathsf{app}"'] & L
\end{tikzcd}$$

$$\begin{tikzcd}
 & & L \arrow[dll] \arrow[d, "\mathrm{id}"] \\
(V \times L)^V \arrow[d, "(\mathsf{var}\times L)^V"'] & (\eta) & L \\
(L \times L)^V \arrow[rr, "\mathsf{app}^V"'] & & L^V \arrow[u, "\mathsf{abs}"']
\end{tikzcd}$$

Second-Order Equational Logic (I)

$$\text{(Subst)}\ \frac{\begin{array}{c} t_1 \equiv t_1' : U \to TV \\ t_2 \equiv t_2' : V \to TW \end{array}}{t_1[t_2] \equiv t_1'[t_2'] : U \to TW}$$

Second-Order Equational Logic (I)

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Substitution Rule

$$\frac{\begin{array}{l} \vec{x}\,;\, M_1[m_1], \ldots, M_k[m_k] \vdash t \equiv u \\ \begin{array}{l} y_1^{(i)}, \ldots, y_{m_i}^{(i)}\,;\, N_1[n_1], \ldots, N_\ell[n_\ell] \\ \qquad \vdash t_i \equiv u_i \end{array} \qquad (i = 1, \ldots, k) \end{array}}{\begin{array}{l} \vec{x}\,;\, N_1[n_1], \ldots, N_\ell[n_\ell] \\ \quad \vdash\ t\{\, M_i := (y_1^{(i)}, \ldots, y^{(i)m_i}\,)\, t_i \,\} \\ \qquad\quad \equiv\ u\{\, M_i := (y_1^{(i)}, \ldots, y^{(i)m_i}\,)\, u_i \,\} \end{array}}$$

Second-Order Equational Logic (II)

$$(\mathsf{LocChar})\ \dfrac{\begin{array}{c}\{\, e_i : U_i \to U \,\}_{i\in I} \text{ a cover} \\ t\, e_i \equiv t'\, e_i : U_i \to \mathsf{TV} \quad (i \in I)\end{array}}{t \equiv t' : U \to \mathsf{TV}}$$

Local Character Rule

$$\imath : \{\, x_1, \ldots, x_k \,\} \rightarrowtail \{\, y_1, \ldots, y_\ell \,\}$$

$$\dfrac{y_1, \ldots, y_\ell \,;\, \ldots, M_i[m_i], \ldots \vdash t[\imath] \equiv u[\imath]}{x_1, \ldots, x_k \,;\, \ldots, M_i[m_i], \ldots \vdash t \equiv u}$$

Second-Order Equational Logic (III)

$$(\mathsf{Ext})\ \frac{t \equiv t' : U \to TV}{\langle I \rangle t \equiv \langle I \rangle t' : I \otimes U \to T(I \otimes V)}$$

Extension Rule

$$\frac{\vec{x}\,;\ \ldots, M_i[m_i], \ldots\ \vdash\ t_1 \equiv t_2}{\vec{x}, y_1, \ldots, y_\ell\,;\ \ldots, M_i[m_i + \ell], \ldots \vdash t_1{}^{\#} \equiv t_2{}^{\#}}$$

$$t^{\#} = t\{\, M_i := (z_1^{(i)}, \ldots, z_{m_i}^{(i)})\, M_i[z_1^{(i)}, \ldots, z_{m_i}^{(i)}, y_1, \ldots, y_\ell]\,\}$$

Algebraic Simple Type Theory

Algebraic model theory

&

Second-order equational logic

for second-order equational presentations

+

Theory of translations