The Algebra of DAGs

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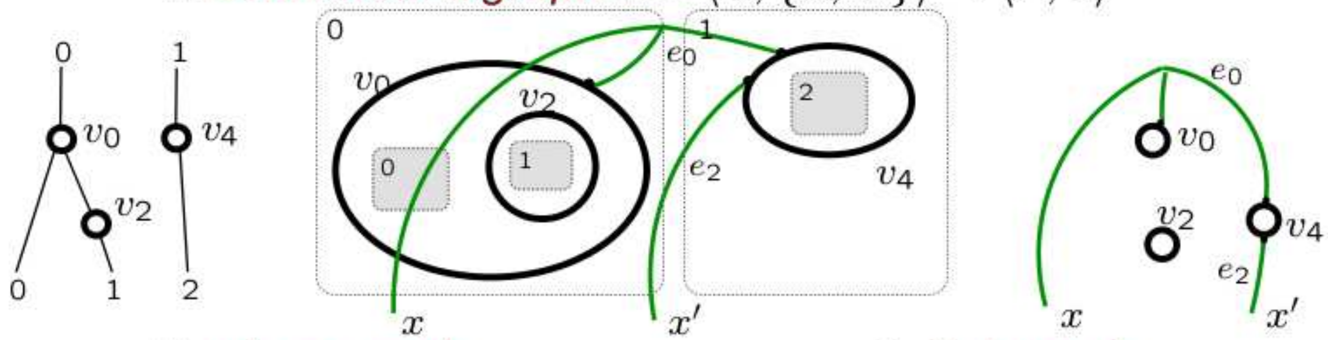
Joint work with Marco Devesas Campos

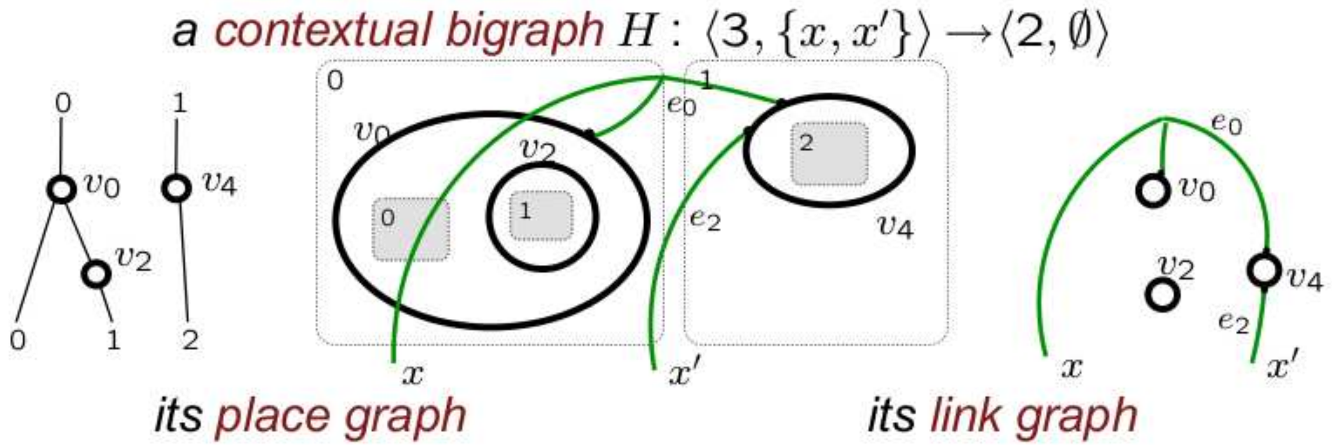
A Question of Robin Milner

a *contextual bigraph* $H : \langle 3, \{x, x'\}\rangle \rightarrow \langle 2, \emptyset\rangle$

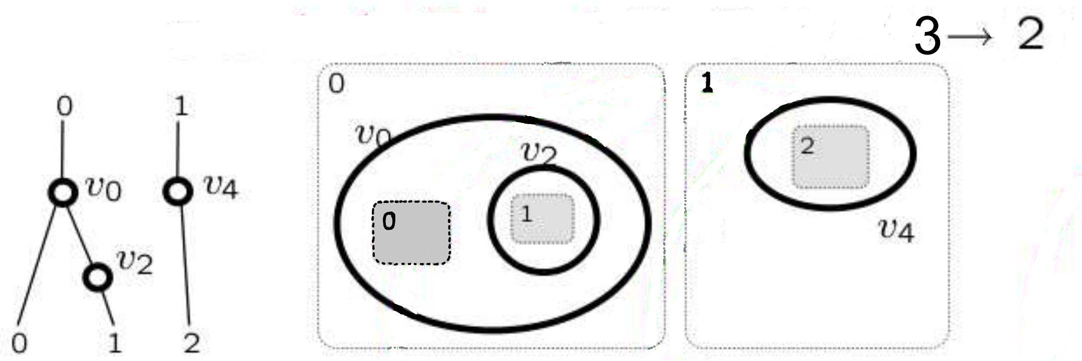


its *place graph* its *link graph*

A Question of Robin Milner



On the generalization from tree structure ...



... to dag structure.

Axioms for DAG structure

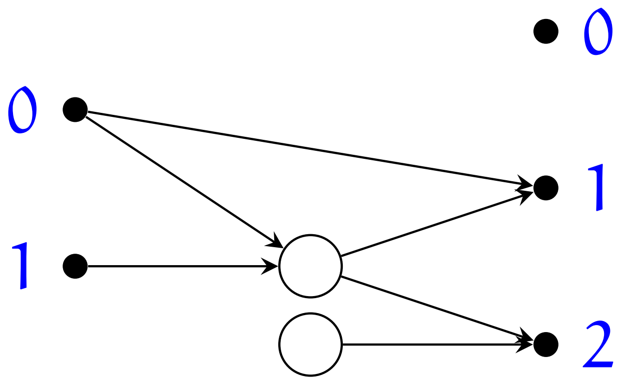
[Gibbons]

Problem:

Give an algebraic characterisation of the symmetric monoidal category $\mathbf{Dag}$ with

objects: finite ordinals, and

morphisms: finite interfaced dags.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1
0
1
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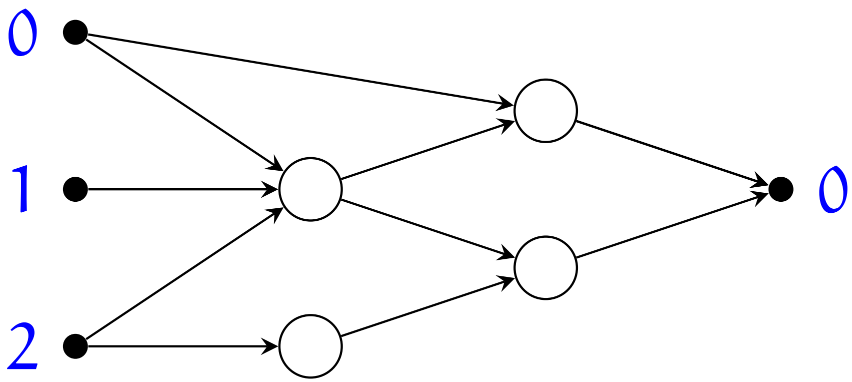
0
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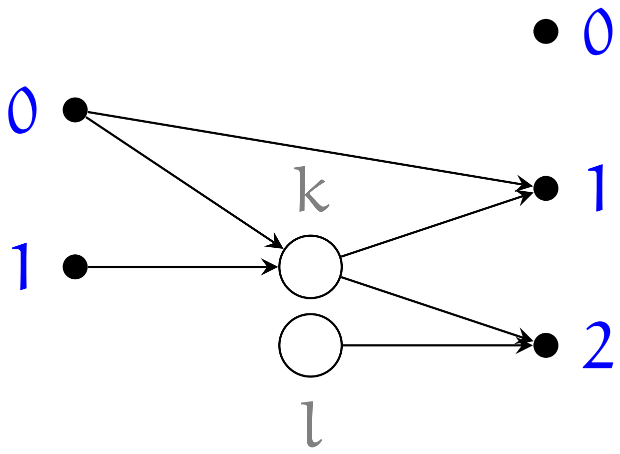
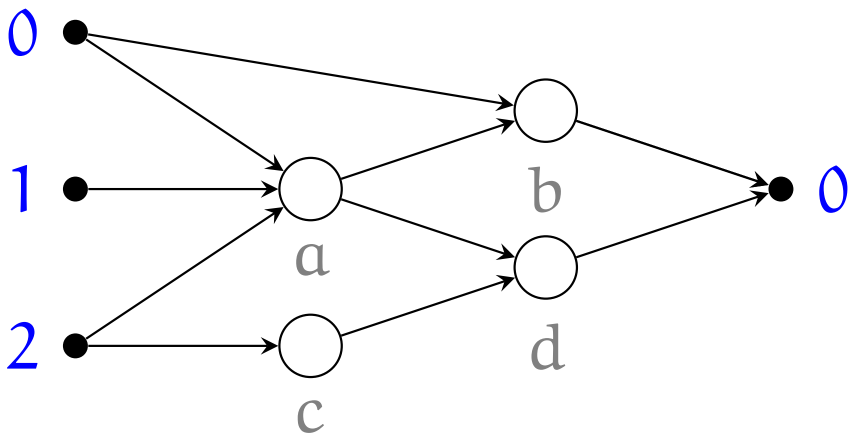
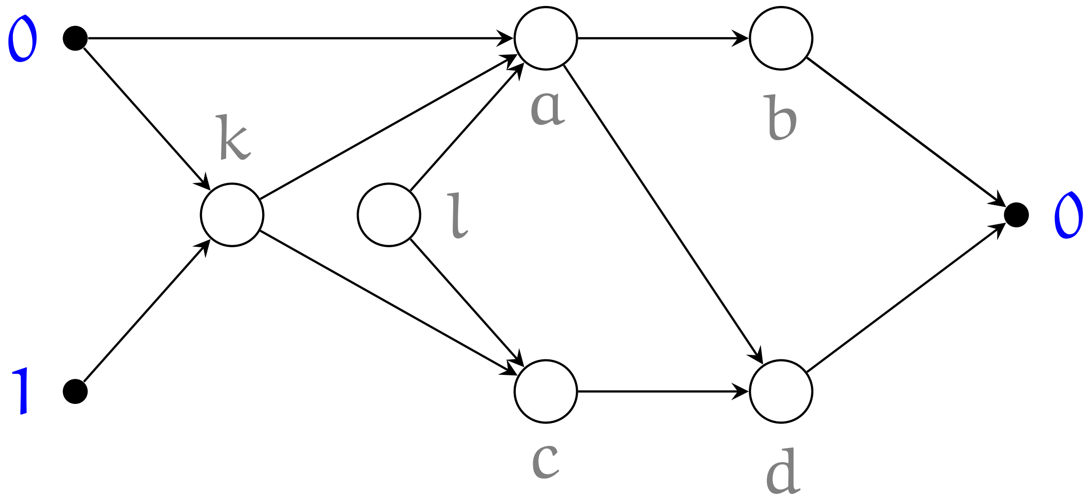
Composition:







The Landscape of Algebraic Structures

$$
\begin{tikzcd}
 & & & \mathbf{Rel} \arrow[rr] \arrow[dr] & & \mathbf{Mat}_{\mathbb{N}} \arrow[r] \arrow[dr] & \mathbf{Mat}_{\mathbb{Z}} \arrow[dr] & \\
\mathbf{Perm} \arrow[rr] \arrow[dr] & & \mathbf{Fun} \arrow[ur] \arrow[dr] & & \mathbf{POrd} \arrow[r] & \mathbf{Dag} \arrow[r] & \mathbf{Dag}_{\mathbb{N}} \arrow[r] & \mathbf{Dag}_{\mathbb{Z}} \\
\mathbb{N} \arrow[r] & \mathbf{Perm}_{\mathbb{N}} \arrow[rr] & & \mathbf{Forest} \arrow[ur] & & & &
\end{tikzcd}
$$

The Mathematical Setting

[Lawvere, MacLane]

Symmetric Monoidal Equational Presentations

The Mathematical Setting

[Lawvere, MacLane]

Symmetric Monoidal Equational Presentations

Examples:

1. Commutative monoids

Operators

$$\eta : 0 \to 1 \quad , \quad \nabla : 2 \to 1$$

Equations

$$\nabla(x_0, \eta) \equiv x_0 \quad , \quad x_0 \equiv \nabla(\eta, x_0)$$

$$\nabla(x_0, \nabla(x_1, x_2)) \equiv \nabla(\nabla(x_0, x_1), x_2) \quad , \quad \nabla(x_0, x_1) \equiv \nabla(x_1, x_0)$$

2. Commutative comonoids

Operators

$$\epsilon : 1 \to 0 \quad , \quad \Delta : 1 \to 2$$

Equations

$$\begin{tikzcd} & 1 \arrow[dl, "\mathrm{id}_1"'] \arrow[d, "\Delta"'] \arrow[dr, "\mathrm{id}_1"] & \\ 1 & 2 \arrow[l, "1+\epsilon"] \arrow[r, "\epsilon+1"'] & 1 \end{tikzcd} \qquad \begin{tikzcd} 1 \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & 2 \arrow[d, "1+\Delta"] \\ 2 \arrow[r, "\Delta+1"'] & 3 \end{tikzcd} \qquad \begin{tikzcd} & 2 \arrow[dd, "\sigma_{1,1}"] \\ 1 \arrow[ur, "\Delta"] \arrow[dr, "\Delta"'] & \\ & 2 \end{tikzcd}$$

PROduct and Permutation categories

Definition: A PROP is a symmetric strict monoidal category with underlying monoid structure on objects given by finite ordinals under addition.

Examples:

1. $\mathbf{Dag}$

2. The free PROP $\mathbf{P}[\mathcal{E}]$ on a symmetric monoidal equational presentation $\mathcal{E}$.

$\mathbf{P}[\mathcal{E}]$ may be constructed syntactically, with morphisms given by equivalence classes of expressions generated by

$$\mathrm{id}_n : n \to n \qquad \frac{f : \ell \to m\ ,\quad g : m \to n}{f\,;g : \ell \to n}$$

$$\frac{f_1 : m_1 \to n_1\ ,\quad f_2 : m_2 \to n_2}{f_1 + f_2 : m_1 + m_2 \to n_1 + n_2} \qquad \sigma_{m,n} : m + n \to n + m$$

$$\frac{\mathsf{o} : n \to m \ \text{ an operator}}{\mathsf{o} : n \to m}$$

under the congruence determined by the laws of symmetric strict monoidal categories together with the identities of the equational presentation $\mathcal{E}$.

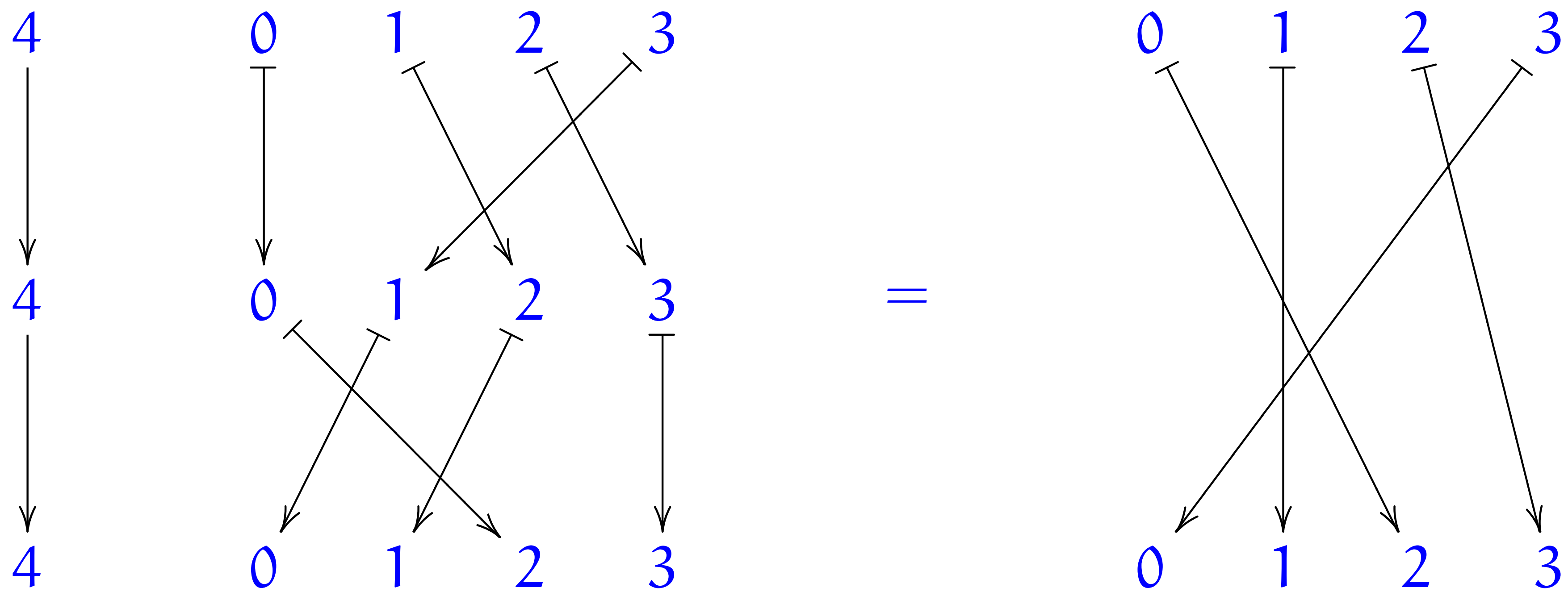
Algebraic Characterization of DAG Structure

Theorem: For $\mathfrak{D}$ the symmetric monoidal equational presentation of a node together with that of degenerate commutative bialgebras,

$$\mathbf{P}[\mathfrak{D}] \;\cong\; \mathbf{Dag} \;.$$

Free PROPs

1. The free PROP $\mathbf{P}[\emptyset]$ on the <u>empty equational presentation</u> is the free symmetric strict monoidal category on an object, viz. the category $\mathbf{Perm}$ of *finite ordinals and permutations*.



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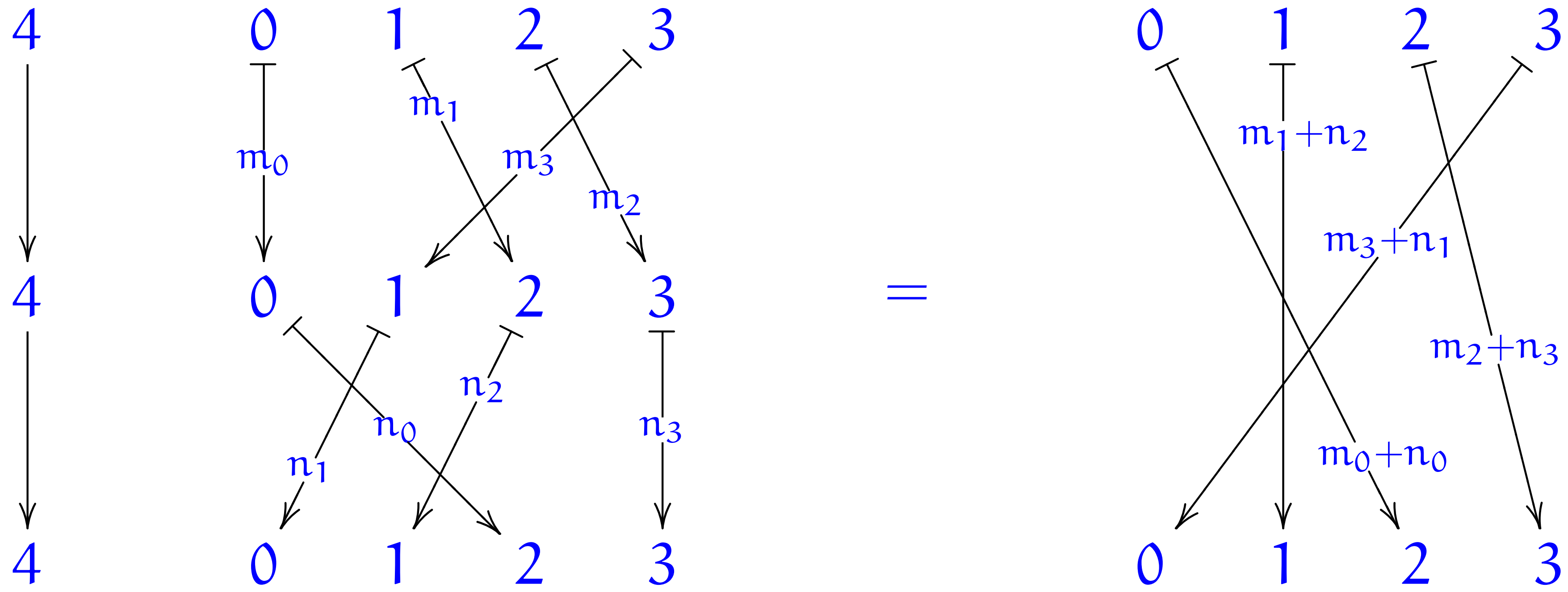

3. The free PROP $\mathbf{P}[\mathcal{C}om\mathcal{M}on]$ on the <u>equational presentation of commutative monoids</u> is the free cocartesian category on an object, i.e. the category $\mathbf{Fun}$ of *finite ordinals and functions*.

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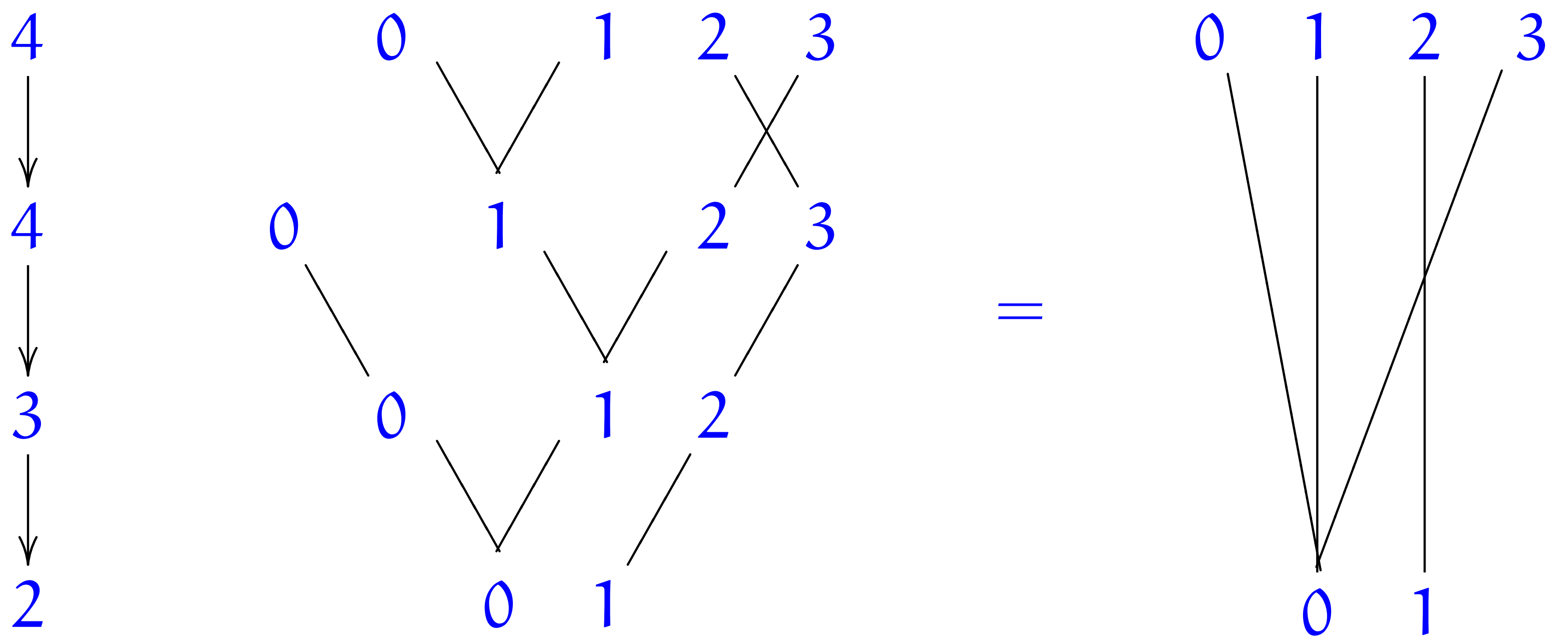
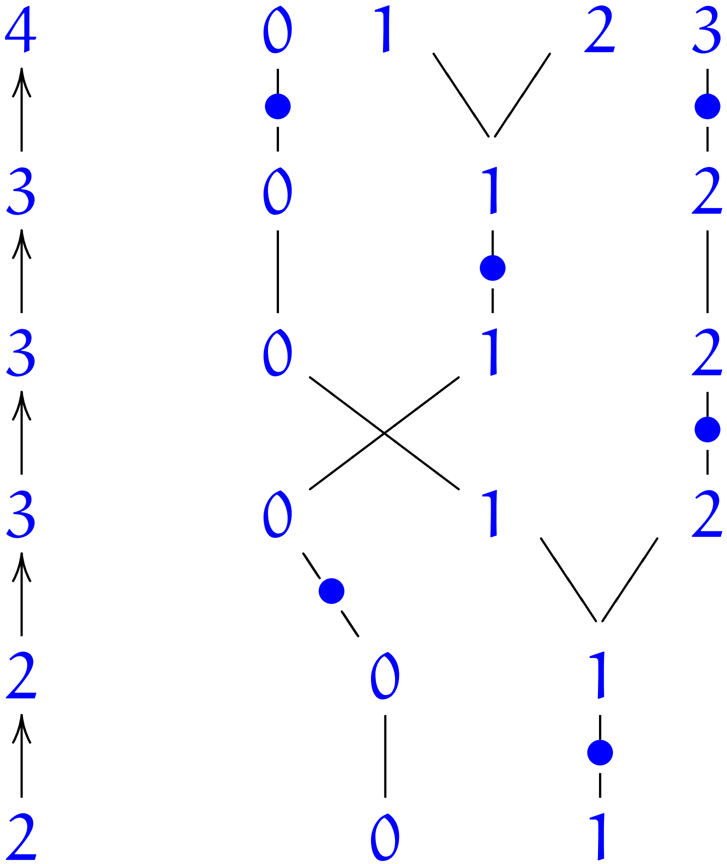


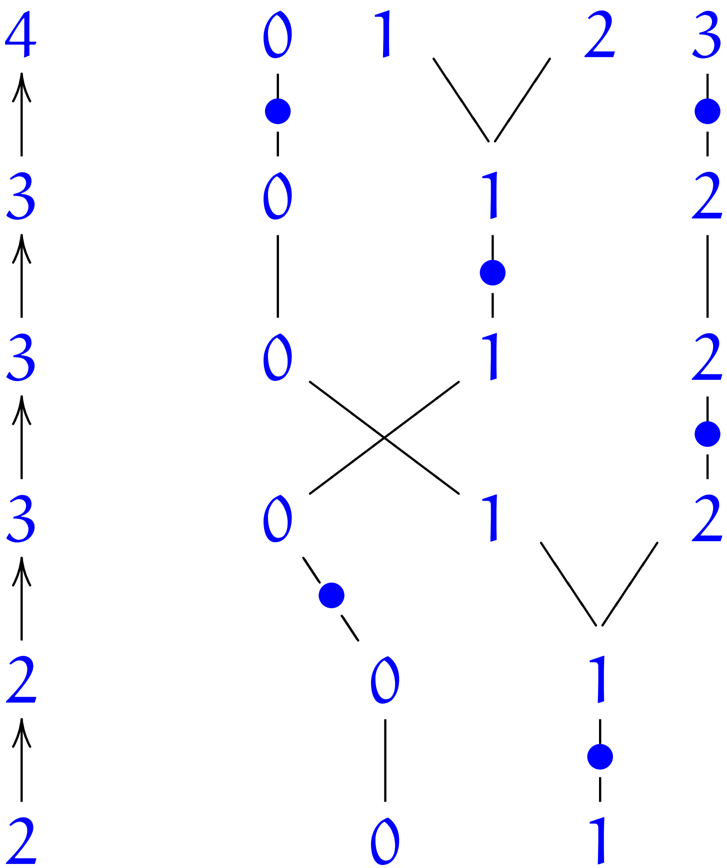
4. The free PROP $\mathbf{P}[\bullet + \mathcal{C}om\mathcal{C}o\mathcal{M}on]$ on the <u>equational presentation of a node together with that of commutative comonoids</u> is the subcategory $\mathbf{Forest}$ of $\mathbf{Dag}$ consisting of forests.

[Moerdijk, Milner]

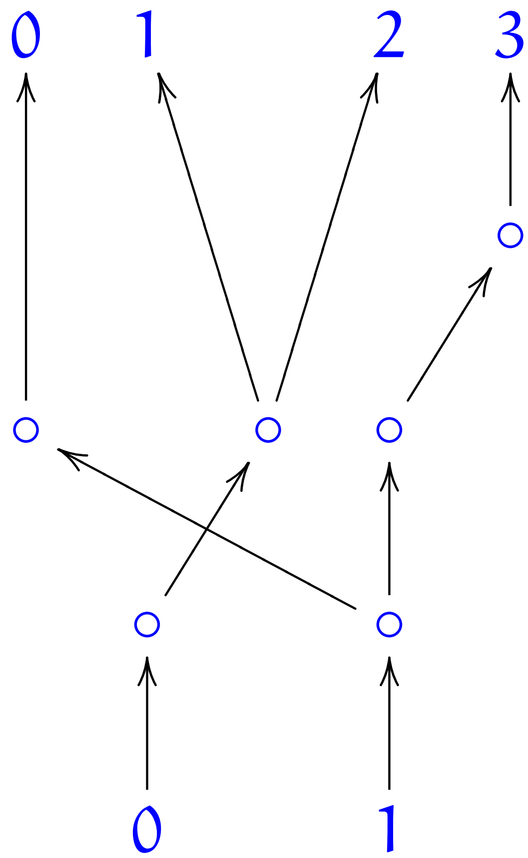
a morphism



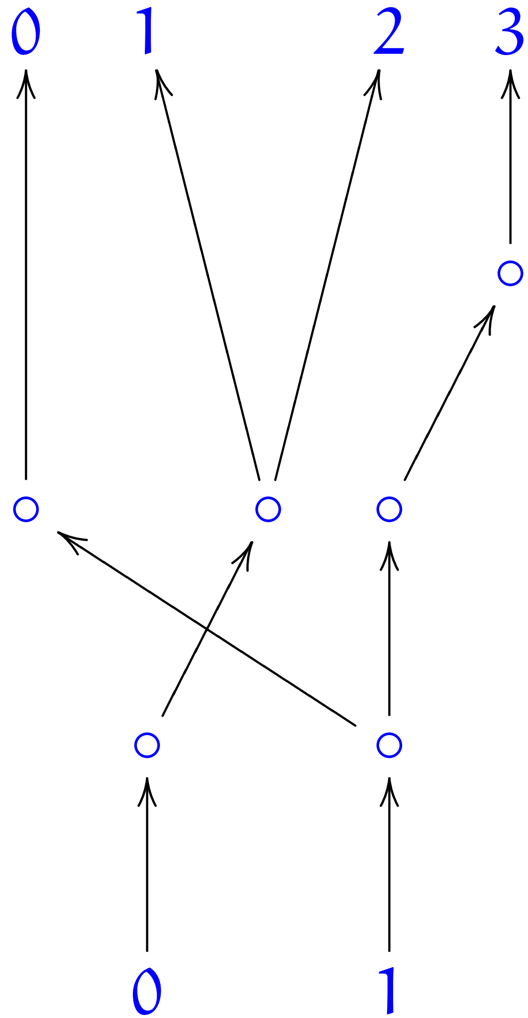
a morphism

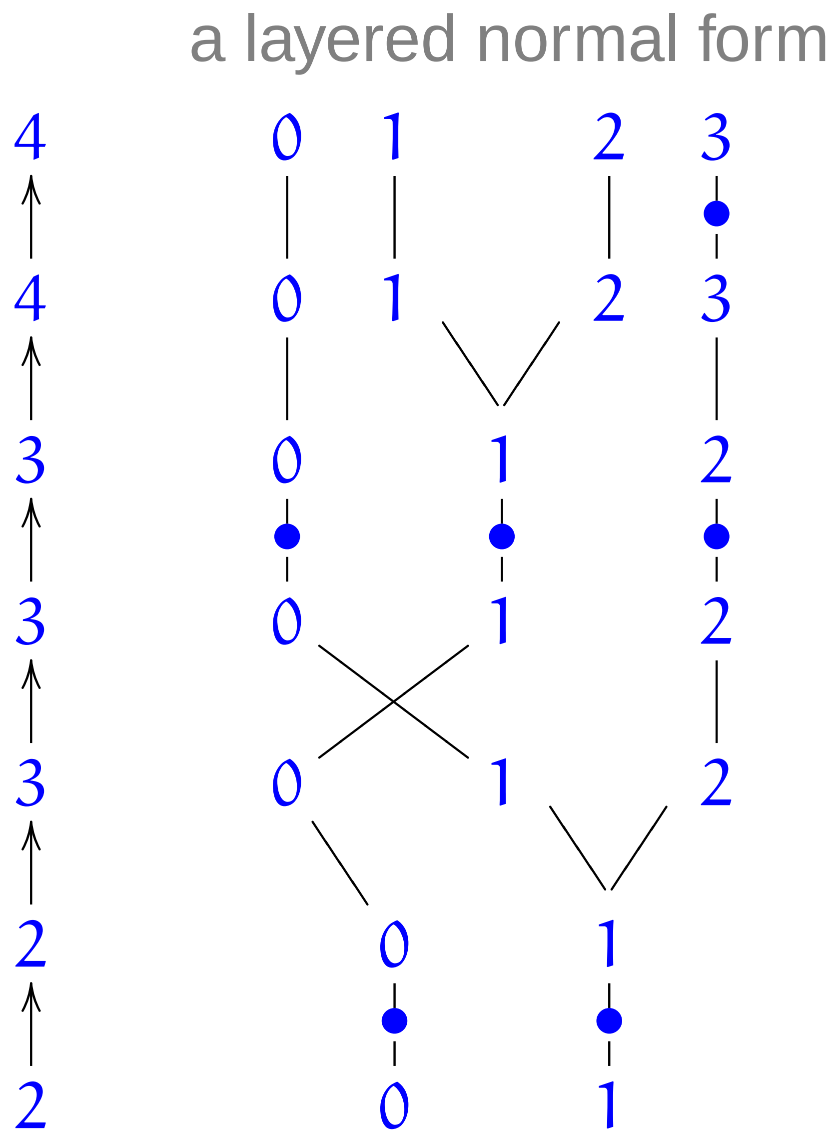


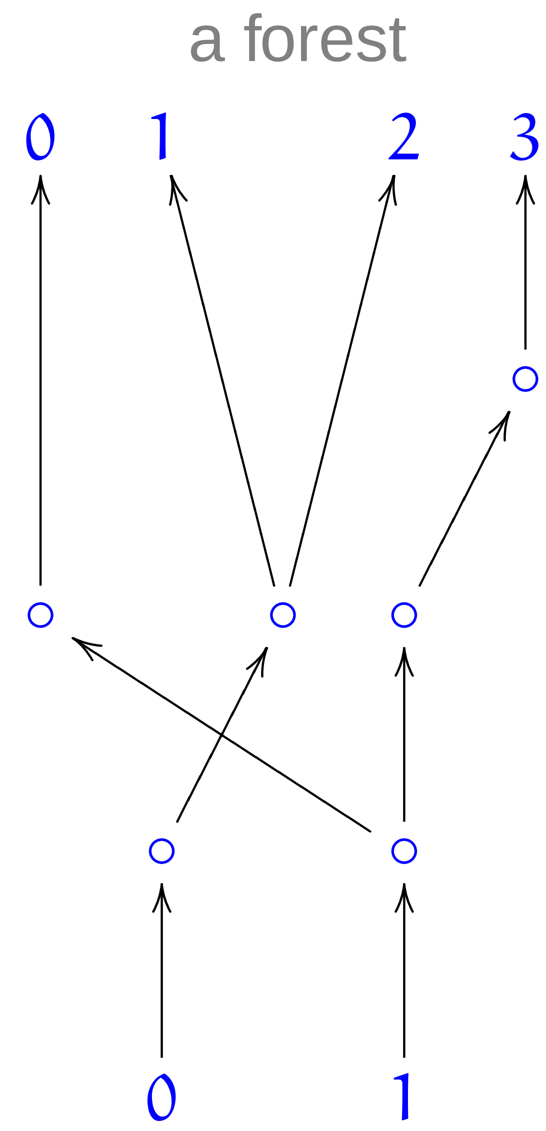
its forest representation



a forest




a layered normal form
4 0 1 2 3
4 0 1 2 3
3 0 1 2
3 0 1 2
3 0 1 2
2 0 1
2 0 1


a forest
0 1 2 3
0 1

5. The free PROP $\mathbf{P}[\,\mathcal{C}om\mathcal{B}i\mathcal{A}lg\,]$ on the equational presentation of commutative bialgebras is the free category with biproducts on an object, viz. the category $\mathbf{Mat}_{\mathbb{N}}$ of *finite ordinals and* $\mathbb{N}$*-valued matrices*.

[MacLane, Pirashvili, Lack]

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[MacLane, Pirashvili, Lack]

The equational presentation of commutative bialgebras is that of commutative monoids and commutative comonoids where the comonoid structure is a monoid homomorphism and the comonoid structure is a monoid homomorphism.

(a) The commutative bialgebra structure turns the symmetric monoidal structure into biproduct structure.

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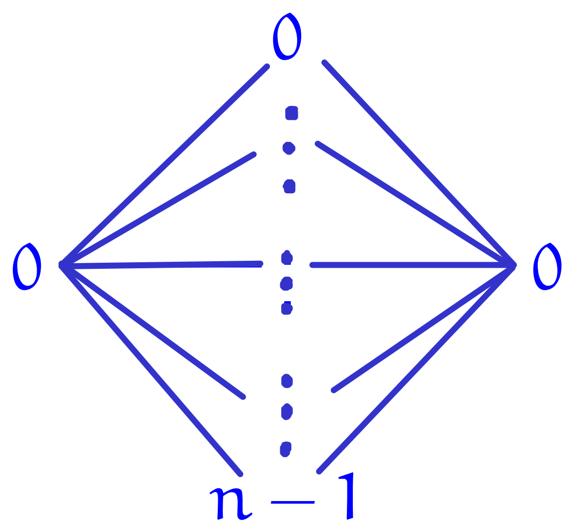
(b) Every morphism $\mathfrak{m} \to \mathfrak{n}$ has a unique representation as an $\mathfrak{m} \times \mathfrak{n}$ matrix with entries in the endo-hom on $\mathbb{1}$.

(a) The commutative bialgebra structure turns the symmetric monoidal structure into biproduct structure.

(b) Every morphism $m \to n$ has a unique representation as an $m \times n$ matrix with entries in the endo-hom on 1.

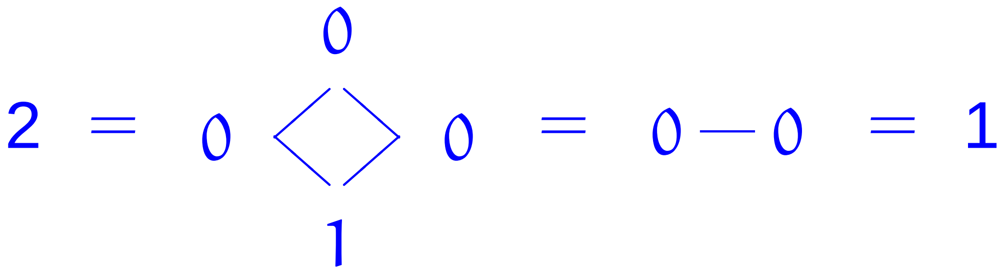
(c) The endo-hom on 1 is the multiplicative monoid of natural numbers.

$n \;=$



6. The free PROP $\mathbf{P}[\,\mathfrak{DegComBiAlg}\,]$ on the equational presentation of degenerate commutative bialgebras is the category $\mathbf{Rel}$ of *finite ordinals and relations*.

The degeneracy axiom:



7. The free PROP $\mathbf{P}[\bullet + \mathcal{DegComBiAlg}]$ on the <u>equational presentation of a node together with that of degenerate commutative bialgebras</u> is $\mathbf{Dag}$.

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Proof:

(a) Define *universal topological interpretations* $[\![\mathrm{D}]\!]_\tau$ of dags D according to topological sortings τ of D.

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Proof:

(a) Define *universal topological interpretations* $[\![\mathrm{D}]\!]_\tau$ of dags D according to topological sortings τ of D.

(b) Prove the *invariance* of topological interpretations, viz. that $[\![\mathrm{D}]\!]_\tau = [\![\mathrm{D}]\!]_{\tau'}$ for all topological sortings τ and τ' of D.

7. The free PROP $\mathbf{P}[\bullet + \mathcal{D}eg\mathcal{C}om\mathcal{B}i\mathcal{A}lg]$ on the equational presentation of a node together with that of degenerate commutative bialgebras is $\mathbf{Dag}$.

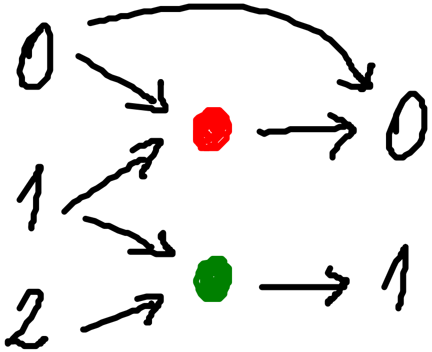
Proof:

(a) Define *universal topological interpretations* $[\![\mathrm{D}]\!]_\tau$ of dags D according to topological sortings τ of D.

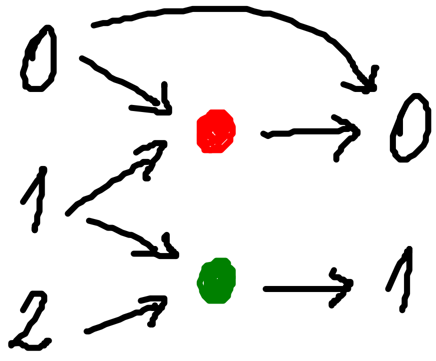
(b) Prove the *invariance* of topological interpretations, viz. that $[\![\mathrm{D}]\!]_\tau = [\![\mathrm{D}]\!]_{\tau'}$ for all topological sortings τ and τ' of D.

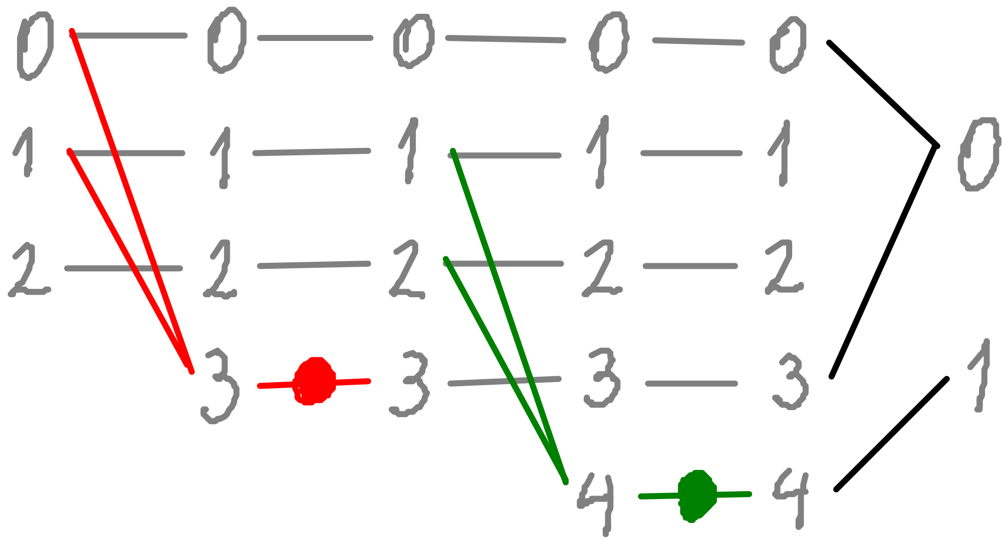
(c) Establish the *compositionality* of the interpretation function to obtain an *initial-algebra semantics*.

Example:

$D =$ 

Example:

$D =$ 

$[\![D]\!]_{(\bullet < \bullet)} =$ 

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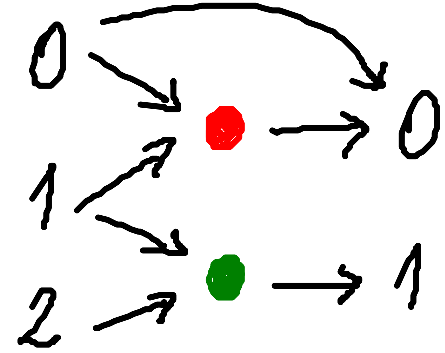
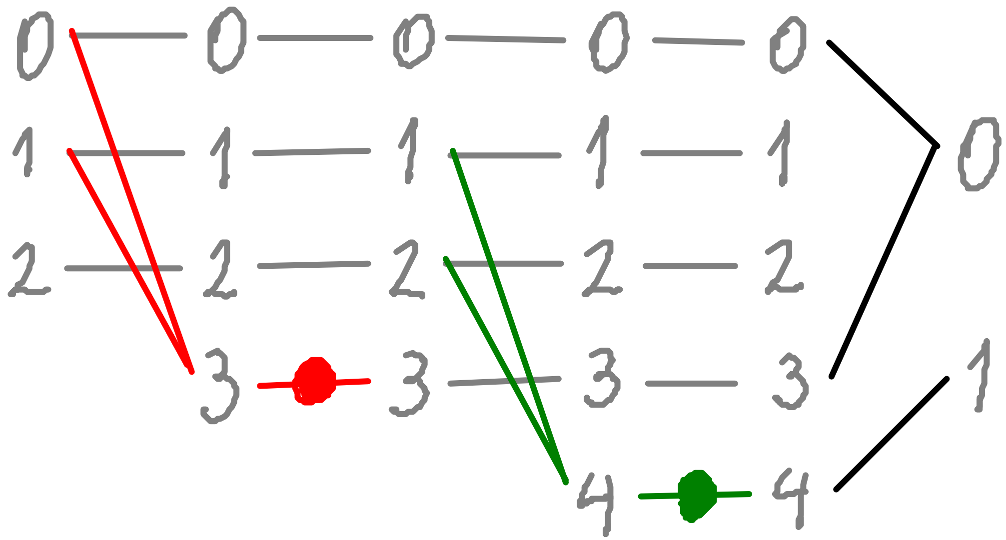
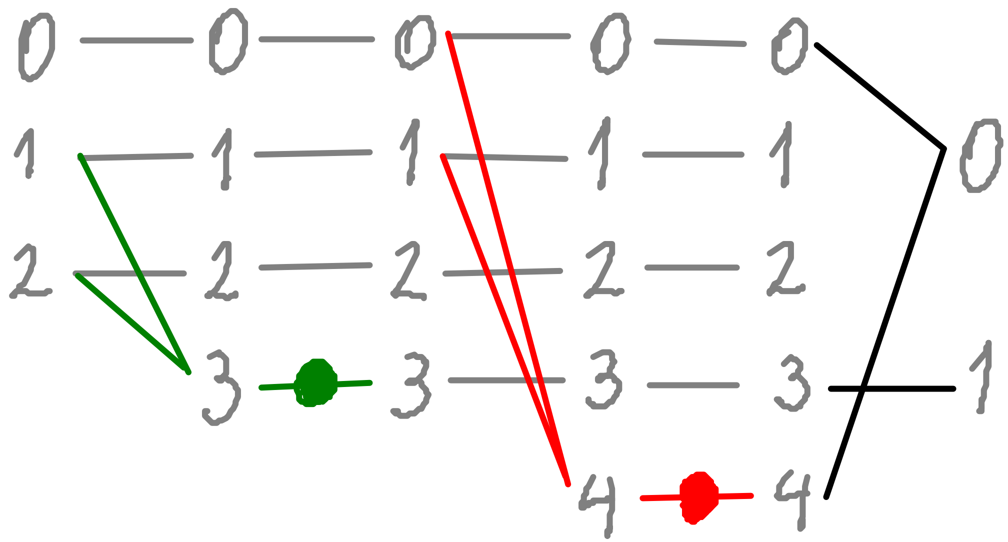
$D =$



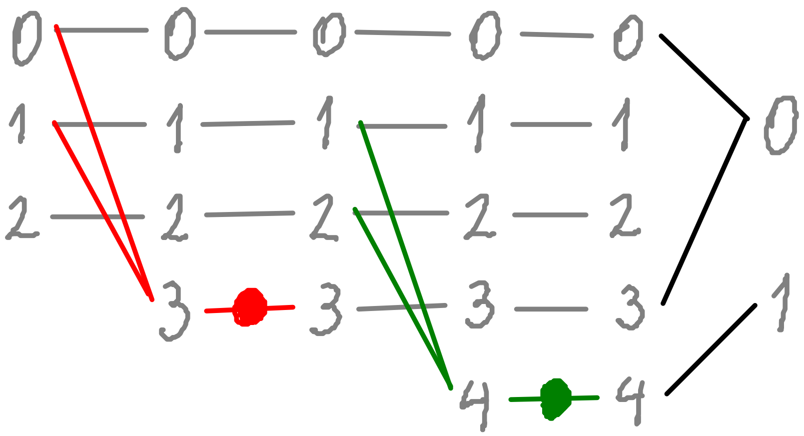
$[D]_{(\bullet < \bullet)} =$

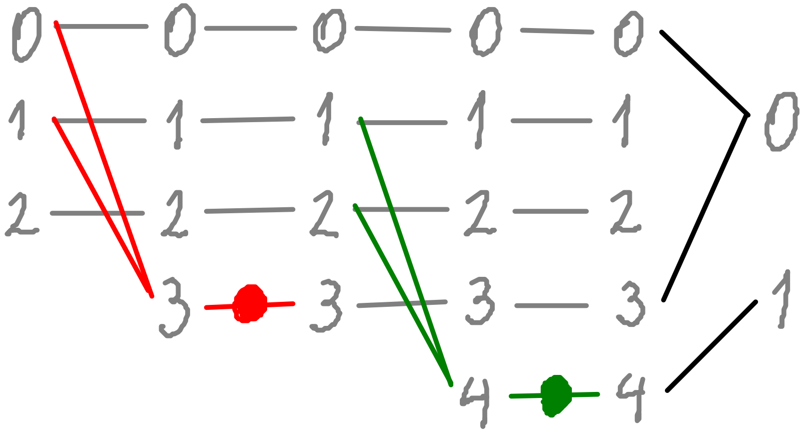
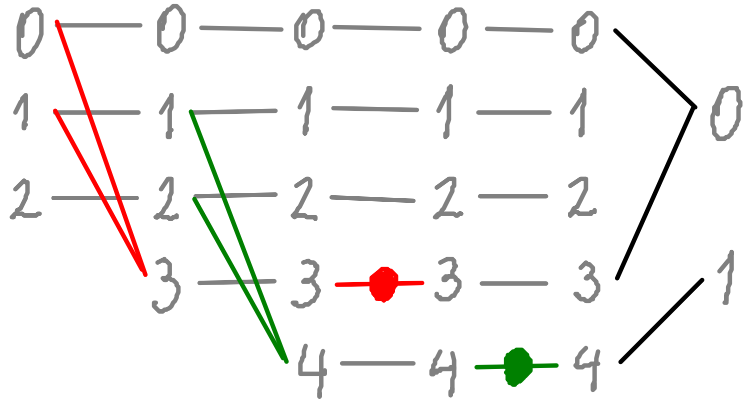


$[D]_{(\bullet < \bullet)} =$



$$[D]_{(\bullet < \bullet)} =$$



$[D]_{(\bullet < \bullet)} =$  $=$ 

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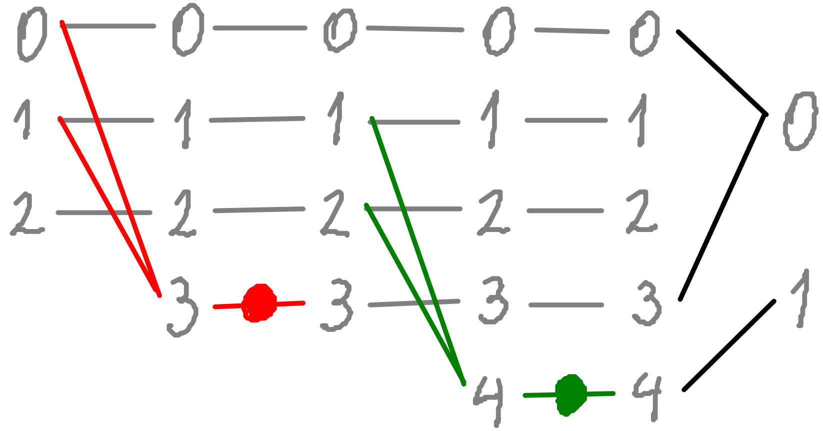

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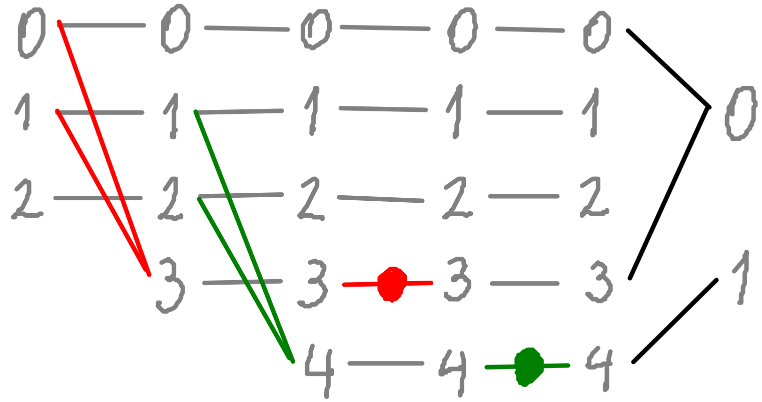

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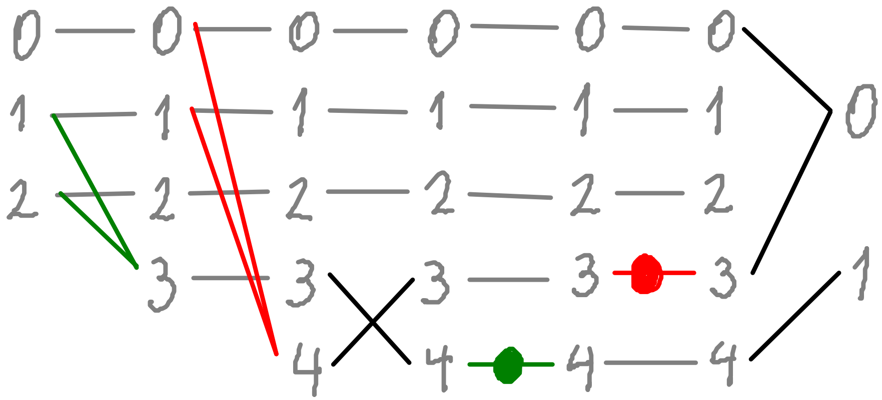
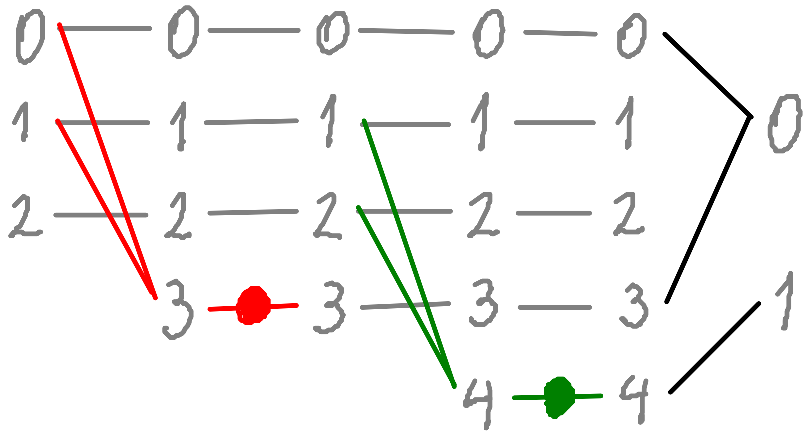
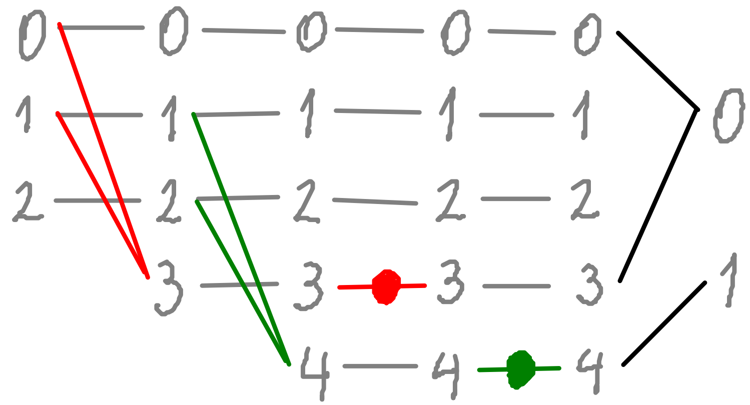
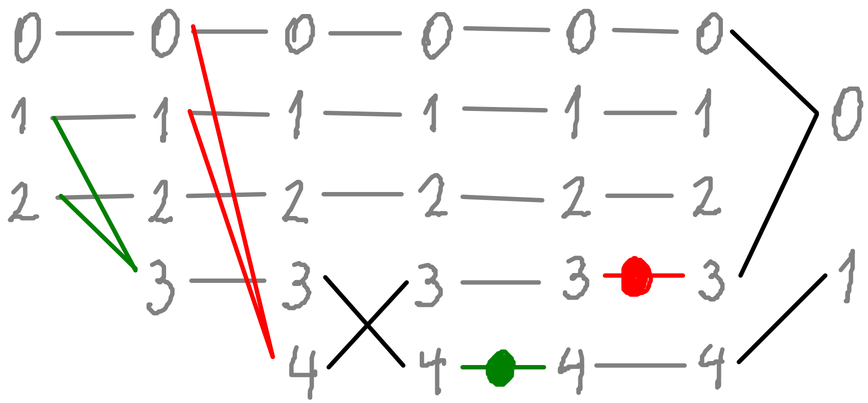
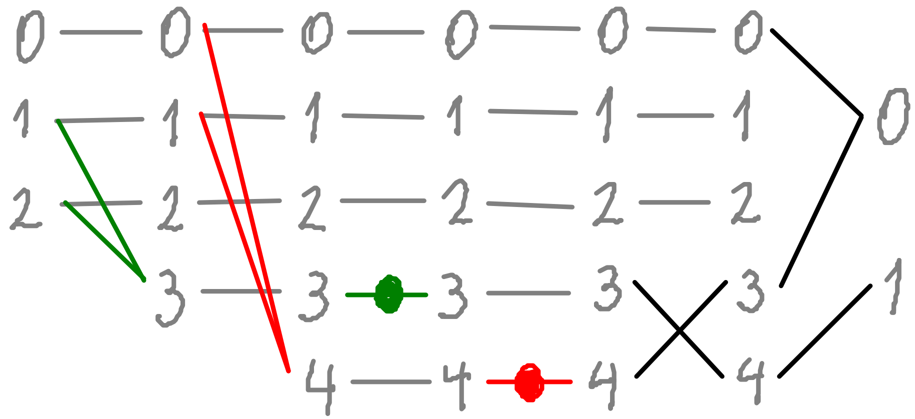
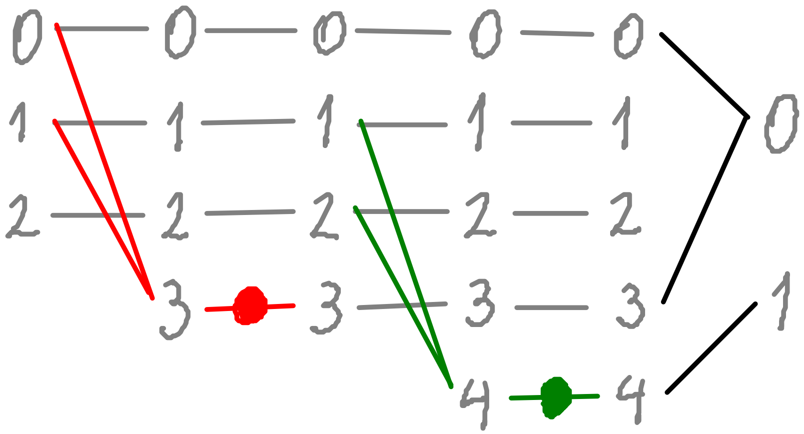
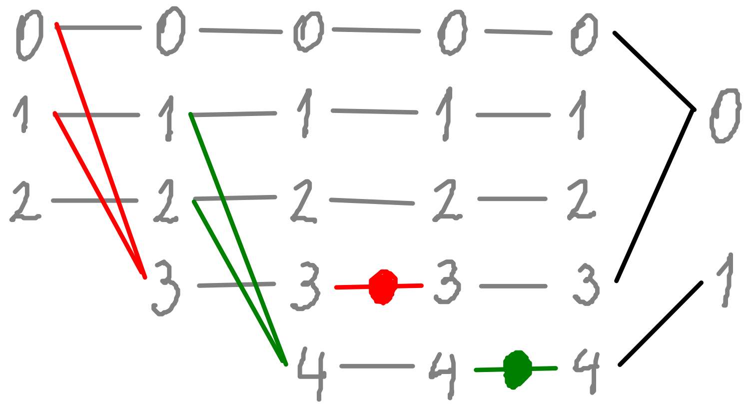
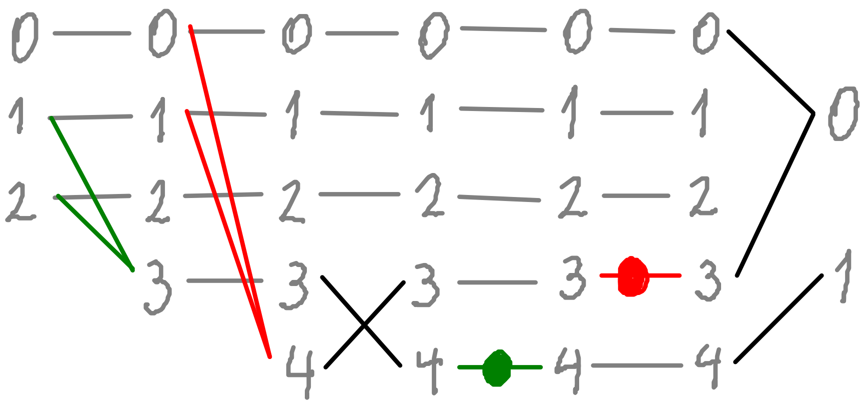
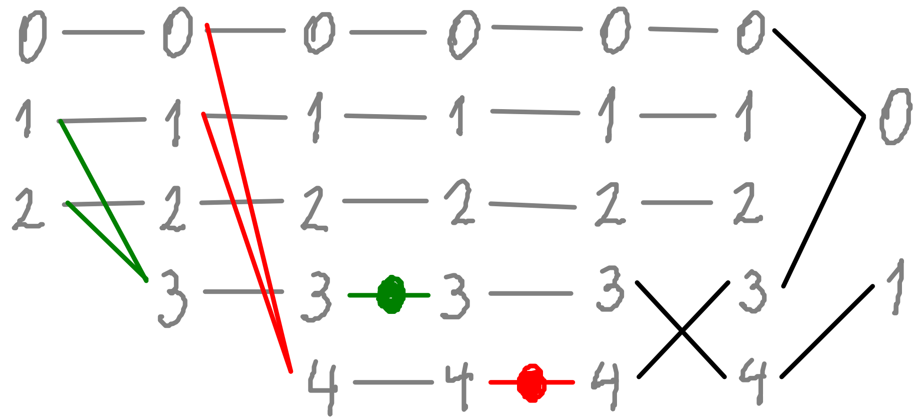
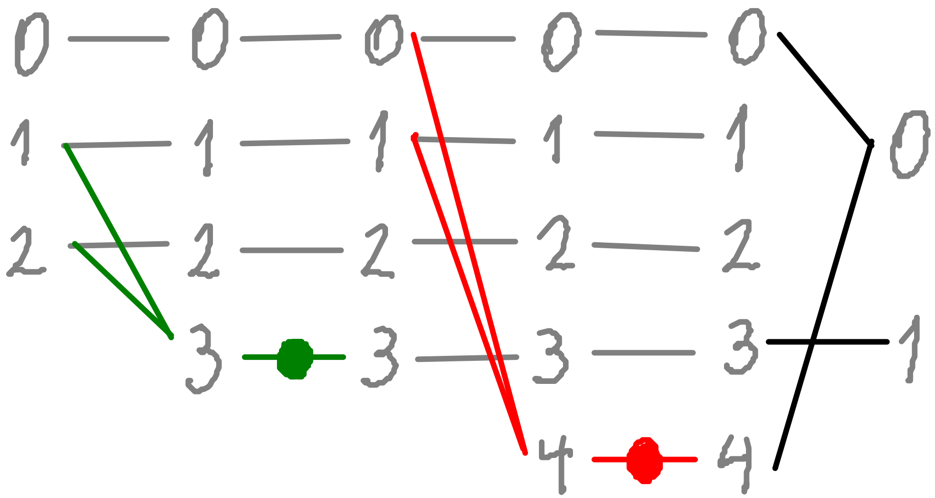


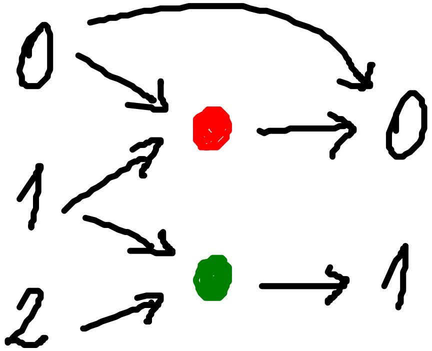
$[D]_{(\bullet < \bullet)} =$  $=$ 

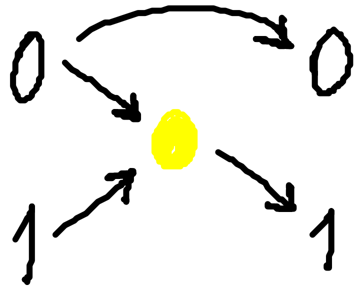
$=$  $=$ 

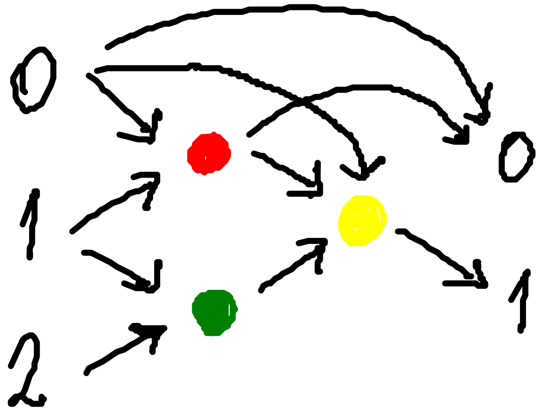
$[D]_{(\bullet < \bullet)} =$  $=$ 

$=$  $=$ 

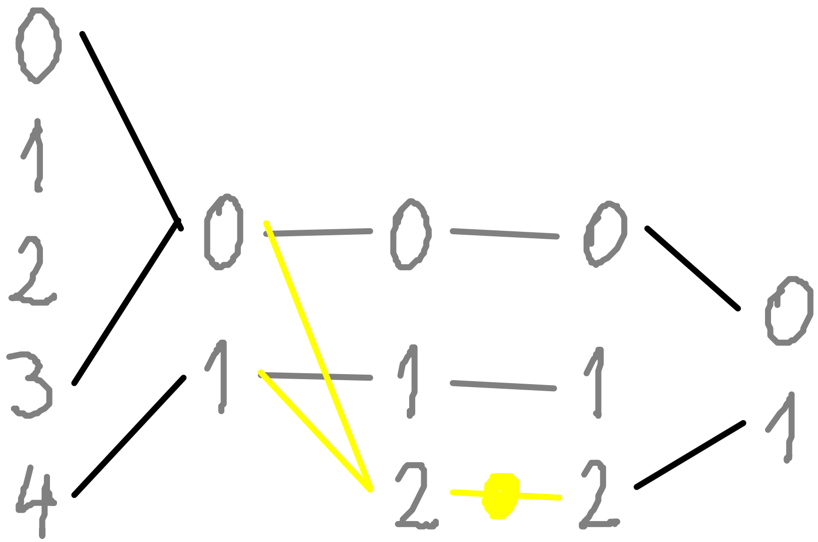
$=$  $= [D]_{(\bullet < \bullet)}$

$D =$ 

$D' =$ 

$D ; D' =$ 

$$[\![D ; D']\!]_{(\bullet < \bullet < \bullet)} = [\![D]\!]_{(\bullet < \bullet)} ; [\![D']\!]_{(\bullet)}$$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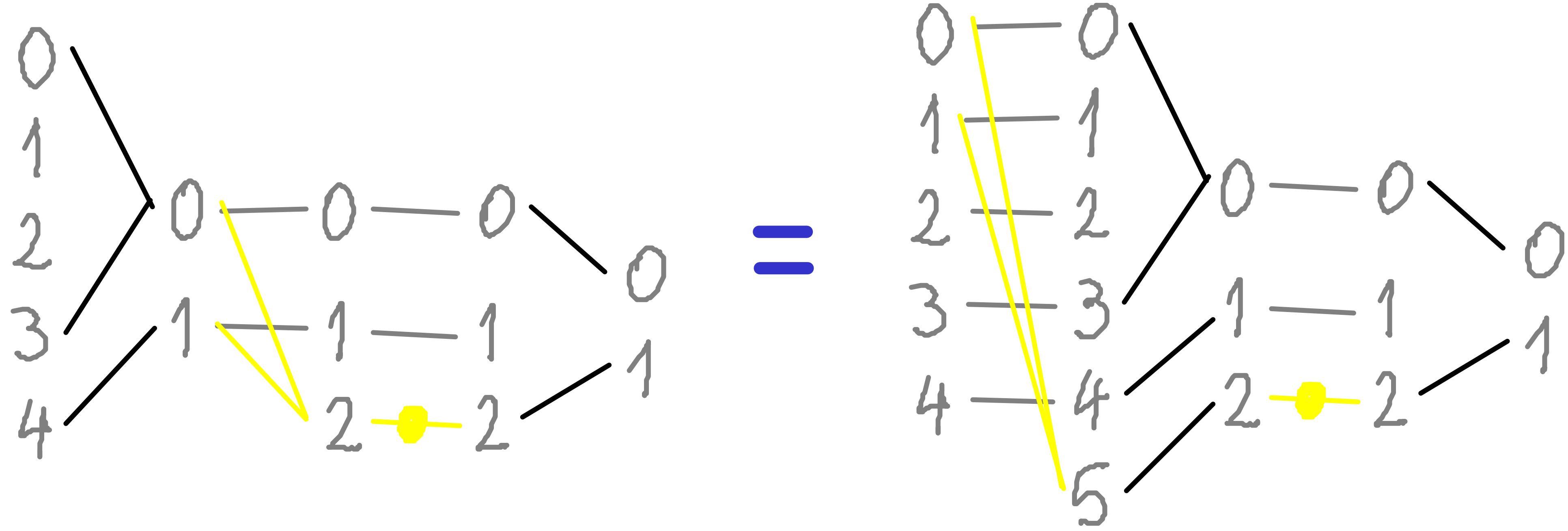

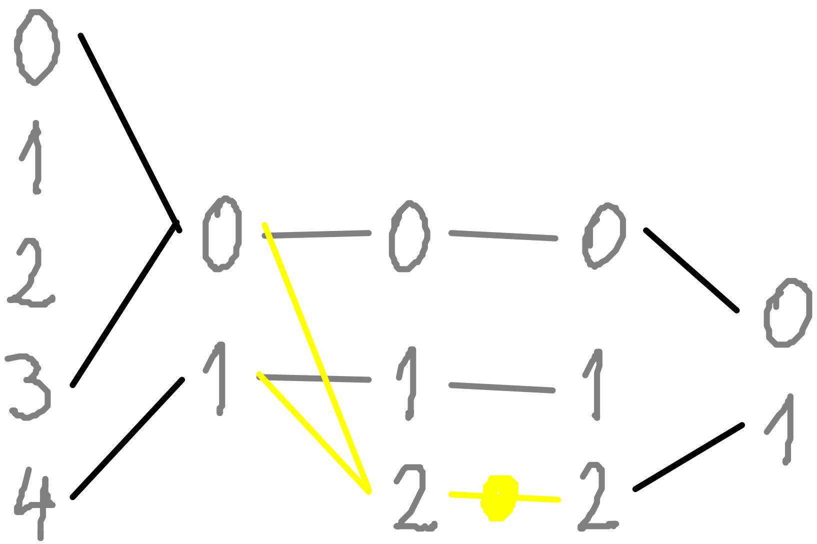

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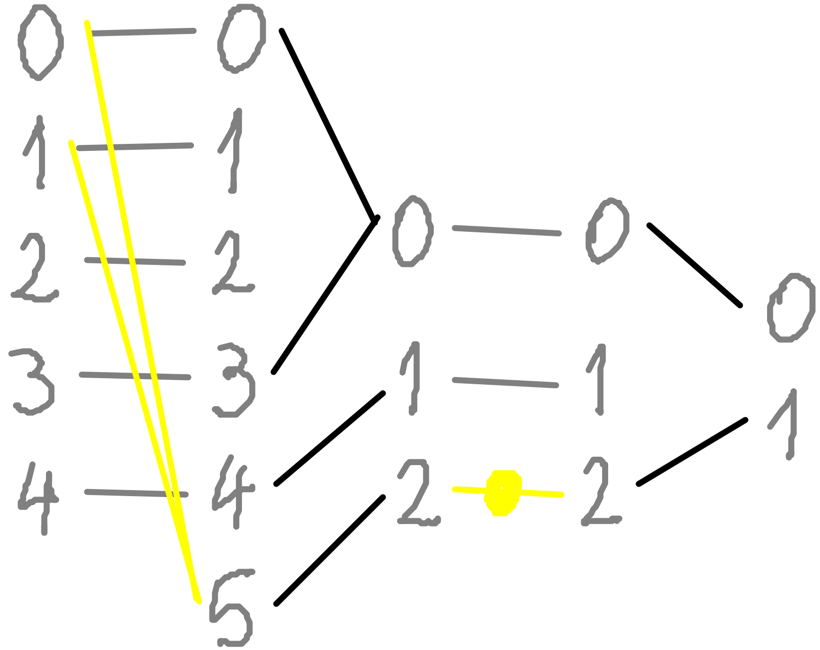

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