The Algebra of DAGs

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Joint work with Marco Devesas Campos
A Question of Robin Milner

a contextual bigraph $H: (3, \{x, x'\}) \rightarrow (2, \emptyset)$

its place graph

its link graph
A Question of Robin Milner

On the generalization from tree structure . . .

. . . to dag structure.
Axioms for DAG structure

Problem:

Give an algebraic characterisation of the symmetric monoidal category \( \text{Dag} \) with

- **objects**: finite ordinals, and
- **morphisms**: finite interfaced dags.
Composition:
The Landscape of Algebraic Structures
The Mathematical Setting

Symmetric Monoidal Equational Presentations

[Lawvere, MacLane]
The Mathematical Setting

Symmetric Monoidal Equational Presentations

Examples:

1. Commutative monoids

   Operators

   \[ \eta : 0 \to 1 , \quad \nabla : 2 \to 1 \]

   Equations

   \[
   \nabla(x_0, \eta) \equiv x_0 , \quad x_0 \equiv \nabla(\eta, x_0) \\
   \nabla(x_0, \nabla(x_1, x_2)) \equiv \nabla(\nabla(x_0, x_1), x_2) , \quad \nabla(x_0, x_1) \equiv \nabla(x_1, x_0)
   \]
2. Commutative comonoids

Operators

\[ \epsilon : 1 \rightarrow 0 \quad \delta : 1 \rightarrow 2 \]

Equations
**PROduct and Permutation categories**

**Definition:** A PROP is a symmetric strict monoidal category with underlying monoid structure on objects given by finite ordinals under addition.

**Examples:**

1. Dag

2. The free PROP $P[E]$ on a symmetric monoidal equational presentation $E$. 
$\mathbf{P}[\mathcal{E}]$ may be constructed syntactically, with morphisms given by equivalence classes of expressions generated by

\[
\begin{align*}
\text{id}_n &: n \to n \\
\frac{f : \ell \to m \ , \ g : m \to n}{f ; g : \ell \to n} \\
\frac{f_1 : m_1 \to n_1 \ , \ f_2 : m_2 \to n_2}{f_1 + f_2 : m_1 + m_2 \to n_1 + n_2} \\
\sigma_{m,n} &: m + n \to n + m \\
o &: n \to m \ 	ext{an operator}
\end{align*}
\]

under the congruence determined by the laws of symmetric strict monoidal categories together with the identities of the equational presentation $\mathcal{E}$. 
Algebraic Characterization of DAG Structure

Theorem: For $\mathcal{D}$ the symmetric monoidal equational presentation of a node together with that of degenerate commutative bialgebras,

$$P[\mathcal{D}] \cong \text{Dag}.$$
1. The free PROP $P^{[\emptyset]}$ on the empty equational presentation is the free symmetric strict monoidal category on an object, viz. the category $\text{Perm}$ of \textit{finite ordinals and permutations}.
2. The free PROP $\mathbb{P}[\bullet]$ on the equational presentation of a node $
abla: 1 \to 1$ is the free symmetric strict monoidal category on the additive monoid of natural numbers.
2. The free PROP $\mathbf{P}[\bullet]$ on the equational presentation of a node $\bullet : 1 \to 1$ is the free symmetric strict monoidal category on the additive monoid of natural numbers, viz. the category $\mathbf{Perm}_\mathbb{N}$ of finite ordinals and $\mathbb{N}$-labelled permutations.
3. The free PROP \( P[ComMon] \) on the equational presentation of commutative monoids is the free cocartesian category on an object, i.e. the category \( \text{Fun} \) of finite ordinals and functions.
3. The free PROP $P[\text{ComMon}]$ on the equational presentation of commutative monoids is the free cocartesian category on an object, i.e. the category $\text{Fun}$ of finite ordinals and functions.
4. The free PROP $\mathbf{P}[\bullet + \text{ComCoMon}]$ on the equational presentation of a node together with that of commutative comonoids is the subcategory $\text{Forest}$ of $\text{Dag}$ consisting of forests.

[Moerdijk, Milner]
a morphism
a morphism

its forest representation

\[
\begin{align*}
4 & \rightarrow & 3 & \rightarrow & 3 & \rightarrow & 3 & \rightarrow & 2 & \rightarrow & 0 \\
3 & \rightarrow & 0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & 3 \\
3 & \rightarrow & 0 & \rightarrow & 1 & \rightarrow & 2 \\
3 & \rightarrow & 0 & \rightarrow & 1 \\
2 & \rightarrow & 0 & \rightarrow & 1 \\
2 & \rightarrow & 0 & \rightarrow & 1
\end{align*}
\]
a forest
a layered normal form

```
4 0 1 2 3
3 0 1 2 3
2 0 1 2 3
2 0 1
```

a forest

```
0 1 2 3
0
1 2 3
0 1
```
5. The free PROP $P[\text{ComBiAlg}]$ on the equational presentation of commutative bialgebras is the free category with biproducts on an object, viz. the category $\text{Mat}_\mathbb{N}$ of finite ordinals and $\mathbb{N}$-valued matrices.

[MacLane, Pirashvili, Lack]
5. The free PROP $\mathbf{P}[\text{ComBiAlg}]$ on the equational presentation of commutative bialgebras is the free category with biproducts on an object, viz. the category $\mathbf{Mat}_\mathbb{N}$ of finite ordinals and $\mathbb{N}$-valued matrices.

[MacLane, Pirashvili, Lack]

The equational presentation of commutative bialgebras is that of commutative monoids and commutative comonoids where the comonoid structure is a monoid homomorphism and the comonoid structure is a monoid homomorphism.
(a) The commutative bialgebra structure turns the symmetric monoidal structure into biproduct structure.
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(b) Every morphism $m \rightarrow n$ has a unique representation as an $m \times n$ matrix with entries in the endo-hom on $1$. 
(a) The commutative bialgebra structure turns the symmetric monoidal structure into biproduct structure.

(b) Every morphism \( m \rightarrow n \) has a unique representation as an \( m \times n \) matrix with entries in the endo-hom on 1.

(c) The endo-hom on 1 is the multiplicative monoid of natural numbers.
6. The free PROP $\mathbf{P}[\text{DegCombAlg}]$ on the equational presentation of degenerate commutative bialgebras is the category $\text{Rel}$ of finite ordinals and relations.

The degeneracy axiom:

\[
\begin{align*}
2 & = 0 \begin{array}{c}
\Downarrow\\Downarrow
\end{array} 0 = 0 - 0 = 1
\end{align*}
\]
7. The free PROP $\mathbf{P}[\bullet + \text{DegComBiAlg}]$ on the equational presentation of a node together with that of degenerate commutative bialgebras is $\text{Dag}$. 
7. The free PROP \( P[\bullet + \text{DegComBiAlg}] \) on the equational presentation of a node together with that of degenerate commutative bialgebras is \( \text{Dag} \).

**Proof:**

(a) Define *universal topological interpretations* \([D]_\tau\) of dags \( D \) according to topological sortings \( \tau \) of \( D \).
7. The free PROP \( P[\bullet + \text{DegComBiAlg}] \) on the equational presentation of a node together with that of degenerate commutative bialgebras is \( \text{Dag} \).

**Proof:**

(a) Define *universal topological interpretations* \([D]_\tau\) of dags \( D \) according to topological sortings \( \tau \) of \( D \).

(b) Prove the *invariance* of topological interpretations, viz. that \([D]_\tau = [D]_{\tau'}\) for all topological sortings \( \tau \) and \( \tau' \) of \( D \).
7. The free PROP $\mathbb{P}[ullet + \text{DegComBiAlg}]$ on the equational presentation of a node together with that of degenerate commutative bialgebras is $\text{Dag}$.

**Proof:**

(a) Define *universal topological interpretations* $[\mathcal{D}]_\tau$ of dags $\mathcal{D}$ according to topological sortings $\tau$ of $\mathcal{D}$.

(b) Prove the *invariance* of topological interpretations, viz. that $[\mathcal{D}]_\tau = [\mathcal{D}]_{\tau'}$ for all topological sortings $\tau$ and $\tau'$ of $\mathcal{D}$.

(c) Establish the *compositionality* of the interpretation function to obtain an *initial-algebra semantics*.
Example:

\[ D = \]

\[
\begin{align*}
0 & \rightarrow \rightarrow 0 \\
1 & \rightarrow \rightarrow 0 \\
2 & \rightarrow \rightarrow 1
\end{align*}
\]
Example:

$$D = \begin{array}{c}
\xrightarrow{0} & \xrightarrow{1} & \xrightarrow{2} \\
\xrightarrow{1} & \xrightarrow{0} & \xrightarrow{1} \\
\xrightarrow{2} & \xrightarrow{1} & \xrightarrow{0} \\
\end{array}$$

$$[D]_{(0,3)} = \begin{array}{ccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 & 4 & 4 & 4 \\
\end{array}$$
Example:

$D = \begin{array}{c}
0 \\
1 \\
2 \\
0 \\
1 \\
1 \\
2 \\
3 \\
3 \\
4 \\
\end{array}$

$\llbracket D \rrbracket_{(0,0)} = \begin{array}{c}
0 - 0 - 0 - 0 - 0 - 0 \\
1 - 1 - 1 - 1 - 1 - 1 \\
2 - 2 - 2 - 2 - 2 - 2 \\
3 - 3 - 3 - 3 - 3 - 3 \\
4 - 4 - 4 - 4 - 4 - 4 \\
\end{array}$

$\llbracket D \rrbracket_{(0,0)} = \begin{array}{c}
0 - 0 - 0 - 0 - 0 - 0 \\
1 - 1 - 1 - 1 - 1 - 1 \\
2 - 2 - 2 - 2 - 2 - 2 \\
3 - 3 - 3 - 3 - 3 - 3 \\
4 - 4 - 4 - 4 - 4 - 4 \\
\end{array}$
\[ D = \begin{array}{c}
0 \\
1 \\
2 \\
\end{array} \rightarrow \begin{array}{c}
0 \\
1 \\
\end{array} \]

\[ D' = \begin{array}{c}
0 \\
1 \\
\end{array} \rightarrow \begin{array}{c}
0 \\
1 \\
\end{array} \]

\[ D; D' = \begin{array}{c}
0 \\
1 \\
2 \\
\end{array} \rightarrow \begin{array}{c}
0 \\
1 \\
\end{array} \]

\[ [D; D']((\cdot < \cdot < \cdot)) = [D]((\cdot < \cdot)) ; [D']((\cdot)) \]