An Algebraic Combinatorial Approach to the Abstract Syntax of Opetopic Structures

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Abstract

The starting point of the talk will be the identification of structure common to tree-like combinatorial objects, exemplifying the situation with abstract syntax trees (as used in formal languages) and with opetopes (as used in higher-dimensional algebra). The emerging mathematical structure will be then formalized in a categorical setting, unifying the algebraic aspects of the theory of abstract syntax of [2, 3] and the theory of opetopes of [6]. This realization conceptually allows one to transport viewpoints between these, now bridged, mathematical theories and I will explore it here in the direction of higher-dimensional algebra, giving an algebraic combinatorial framework for a generalisation of the slice construction of [1] for generating opetopes. The technical work will involve setting up a microcosm principle for near-semirings [5] and subsequently exploiting it in the cartesian closed bicategory of generalised species of structures of [4]. Connections to Homotopy Type Theory, (cartesian and symmetric monoidal) equational theories, lambda calculus, and algebraic combinatorics will be mentioned in passing.

References

[1] J. Baez and J. Dolan. Higher-Dimensional Algebra III. n-Categories and the Algebra of Opetopes. *Advances in Mathematics* 135(2):145–206, 1998.

[2] M. Fiore, G. Plotkin and D. Turi. Abstract syntax and variable binding. In *Proceedings of the 14th Annual IEEE Symposium on Logic in Computer Science (LICS'99)*, pages 193–202. IEEE, Computer Society Press, 1999.

[3] M. Fiore. Second-order and dependently-sorted abstract syntax. In *Proceedings of the 23rd Annual IEEE Symposium on Logic in Computer Science (LICS'08)*, pages 57–68. IEEE, Computer Society Press, 2008.

[4] M. Fiore, N. Gambino, M. Hyland, and G. Winskel. The cartesian closed bicategory of generalised species of structures. *J. London Math. Soc.*, 77:203-220, 2008.

[5] M. Fiore and P. Saville. List objects with algebraic structure. In *Proceedings of the 2nd International Conference on Formal Structures for Computation and Deduction (FSCD 2017)*, No. 16, pages 1–18, 2017.

[6] S. Szawiel and M. Zawadowski. The web monoid and opetopic sets. *Journal of Pure and Applied Algebra*, 217:11051140, 2013.

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HDRA 2017
8.IX

GENERAL TOPIC

connection between

type theory
and
higher-dimensional category theory

THIS TALK

GENERAL TOPIC

New connection between abstract syntactic structures in

type theory

and

higher-dimensional category theory

Two aspects of higher-dimensional category theory

(1) Higher-dimensional shapes

simplicial, cubical, globular, opetopic, ...

(2) Higher-dimensional structure

sets, categories, algebras, ...

Two aspects of higher-dimensional category theory

(1) Higher-dimensional shapes

simplicial, cubical, globular, $\boxed{\text{opetopic}}$, ...

(2) Higher-dimensional structure

$\boxed{\text{sets}}$, categories, algebras, ...

THIS TALK

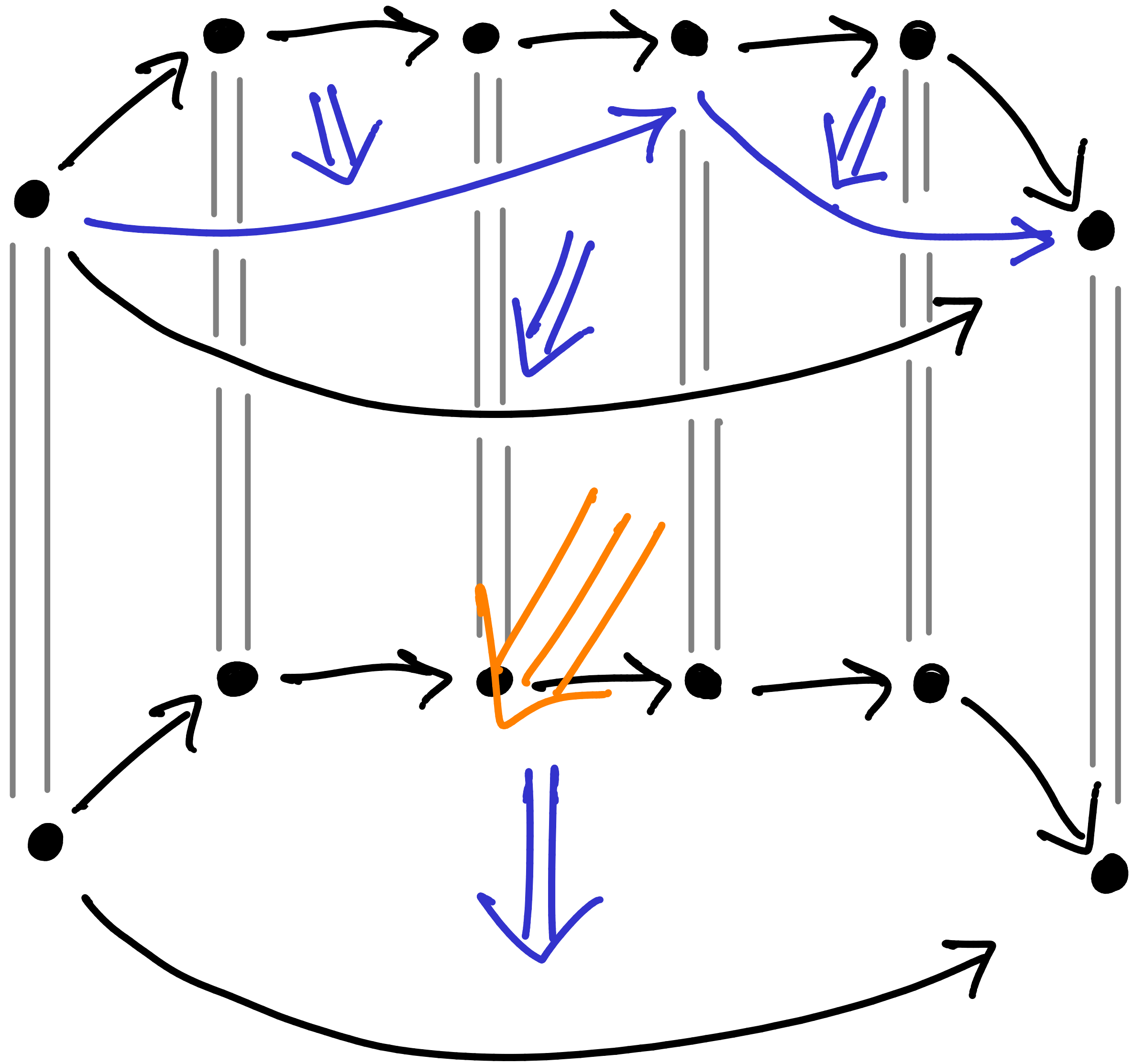
Algebraic Combinatorial Theory for Generalizations of

Opetopes

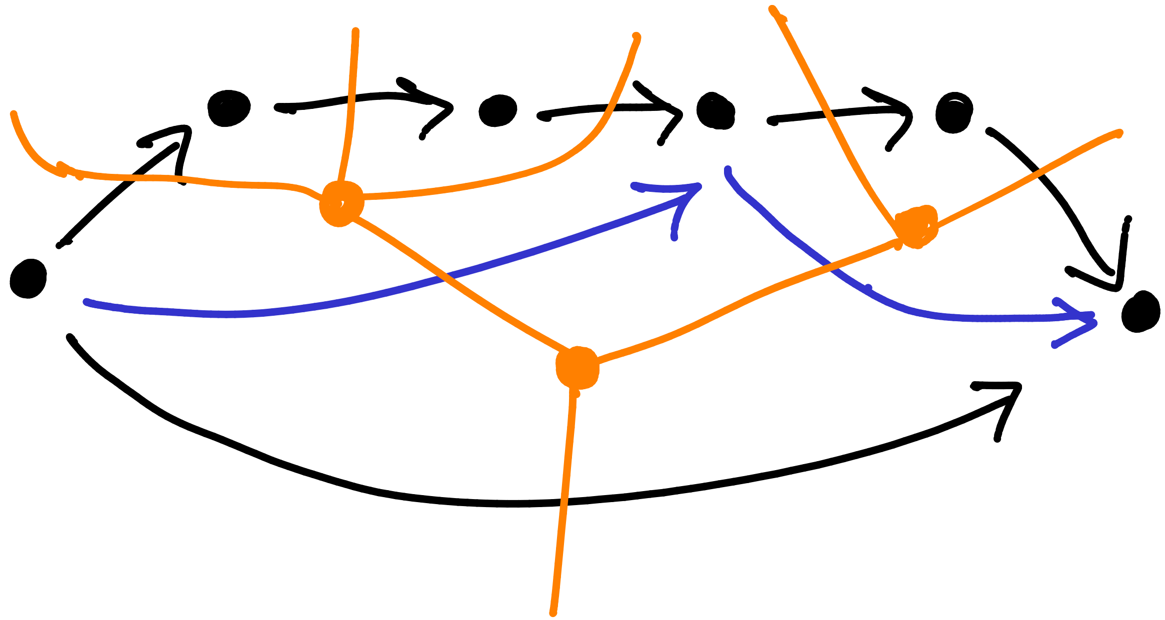
Higher-dimensional Multiarrows

[Baez & Dolan, Hermida & Makkai & Power, Leinster, Cheng, Zawadowski, Kock & Joyal & Batanin & Mascari, Szawiel & Zawadowski, ...]

NB: List structure on arrows.

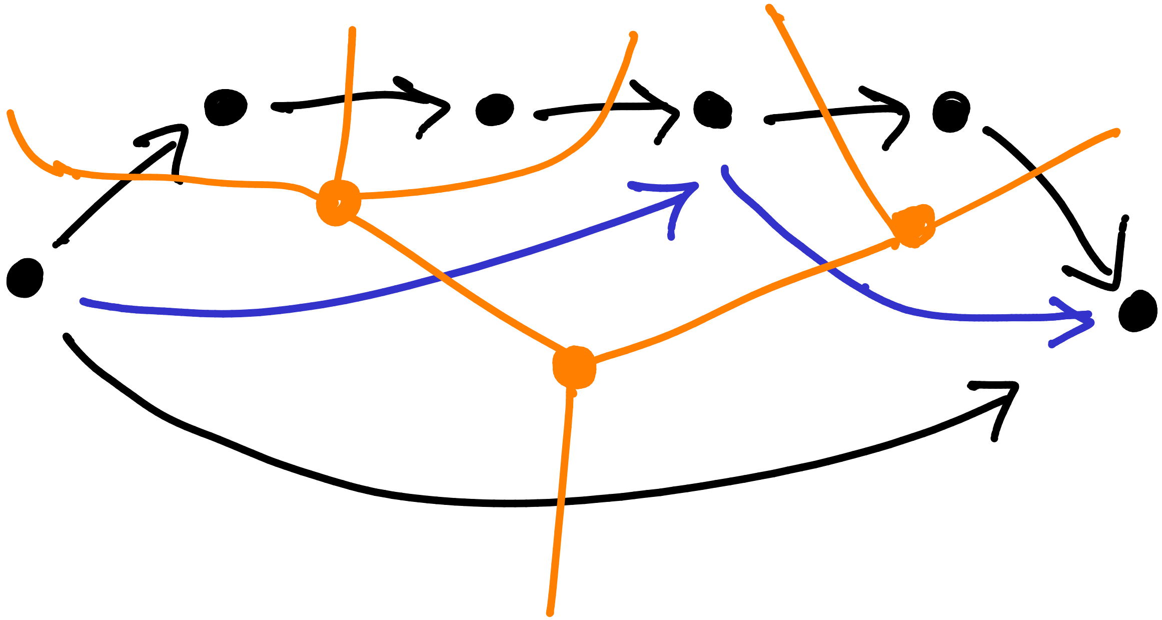


Combinatorial Structure



trees

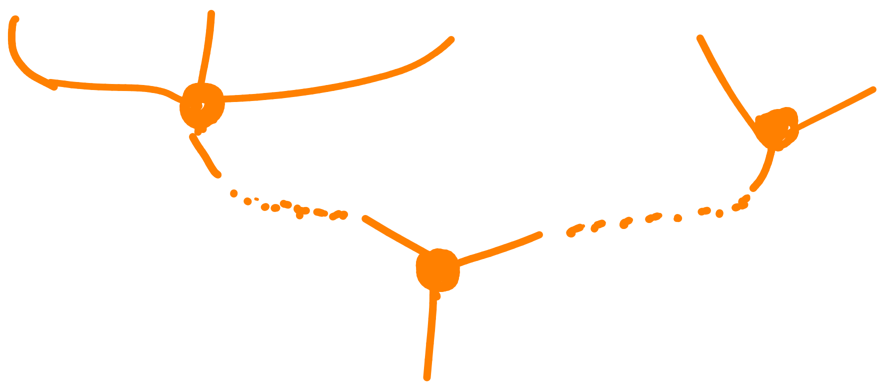
Combinatorial Structure



trees

Algebraic Structure

horizontal composition



leaf grafting

Tree/Grafting structure is List/Monoid structure

Tree/Grafting structure is List/Monoid structure

E.g. the list object

$$F^* = \mu X.\, \mathrm{Id} + F \circ X$$

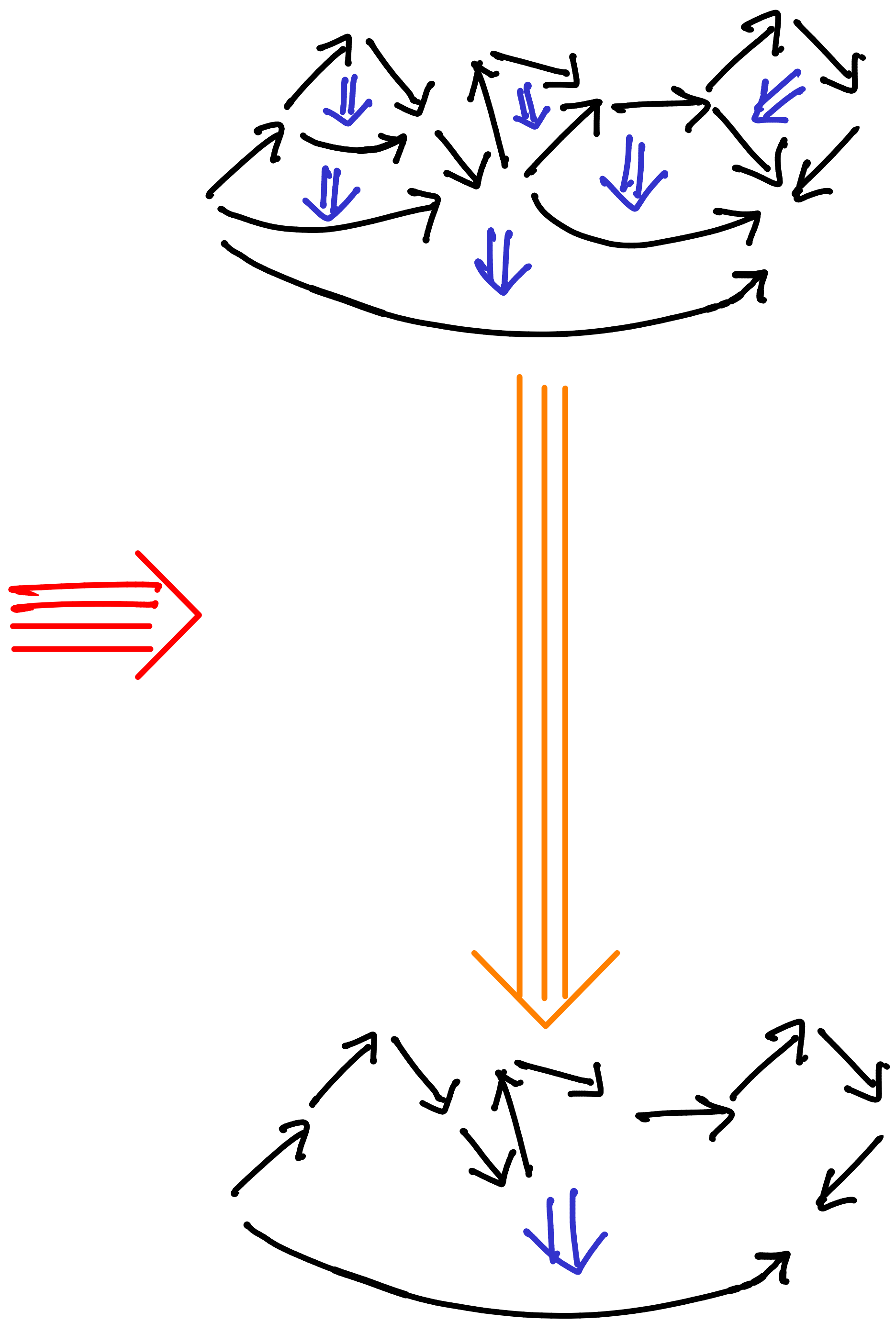
on a signature endofunctor F consists of

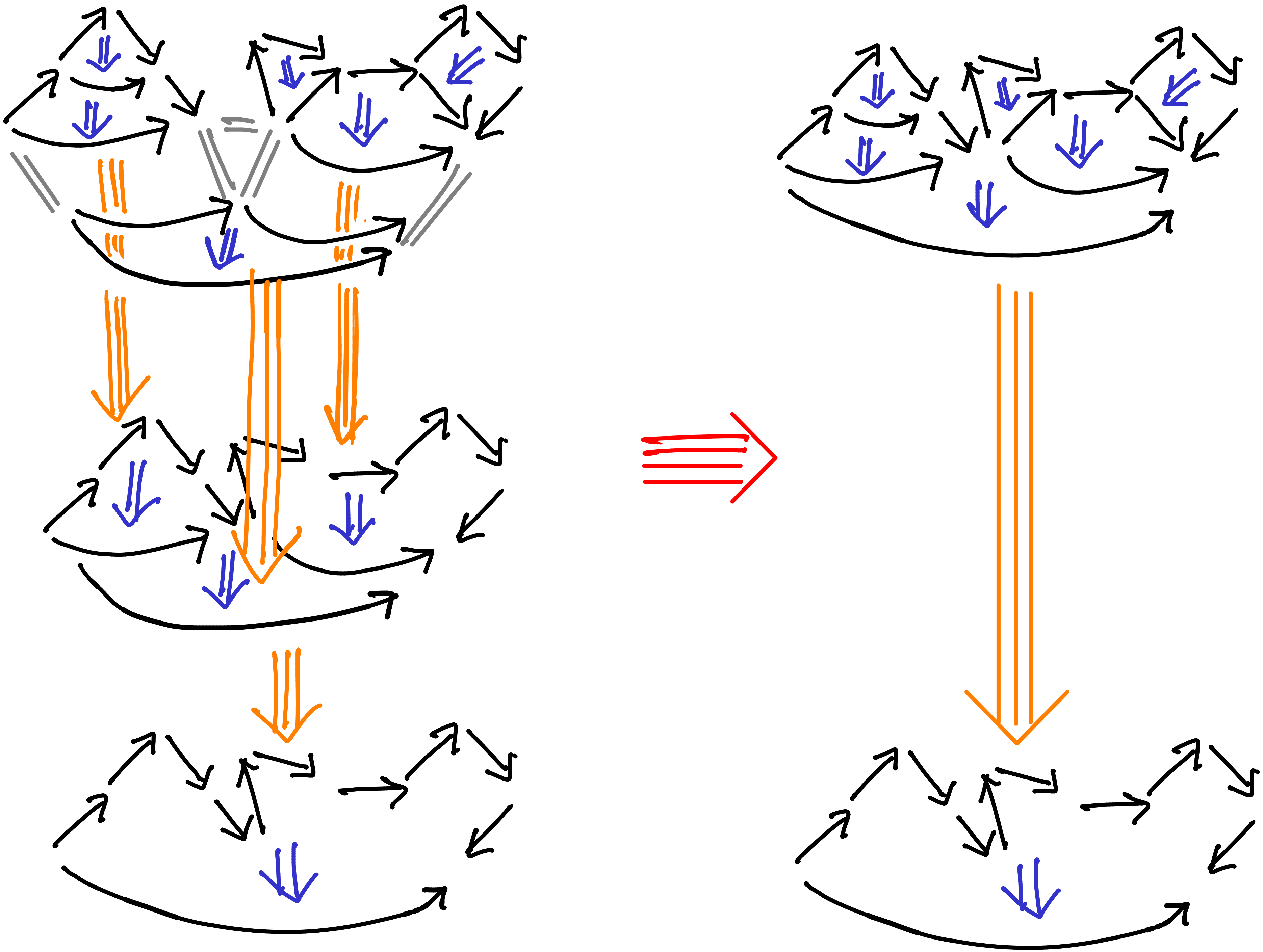
- tree structures

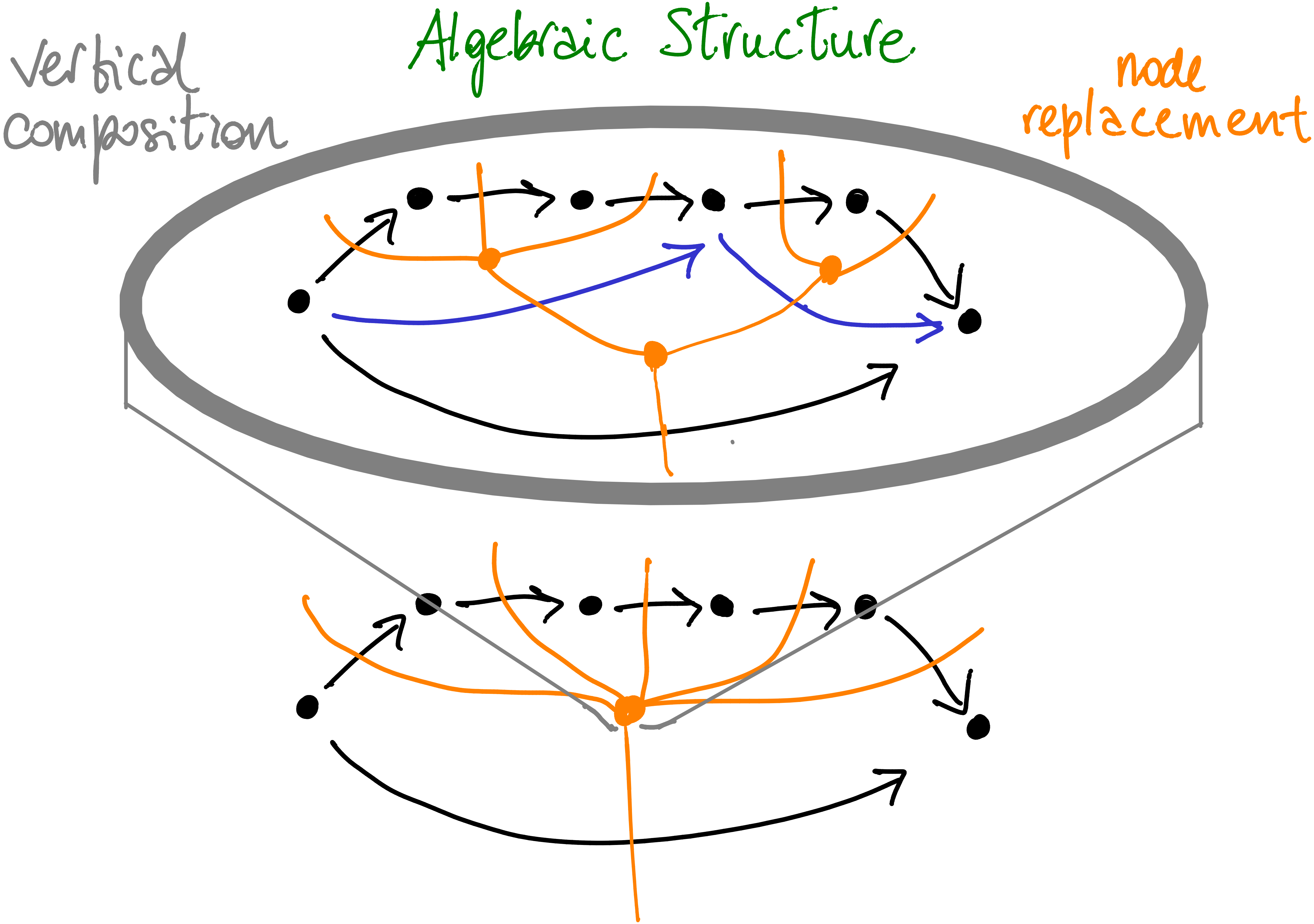
$$F^*(A) \cong A + F(F^* A)$$

$$t \ ::= a \mid f(\ldots, t', \ldots)$$

- with grafting (= substitution) free monoid (= monad) structure.







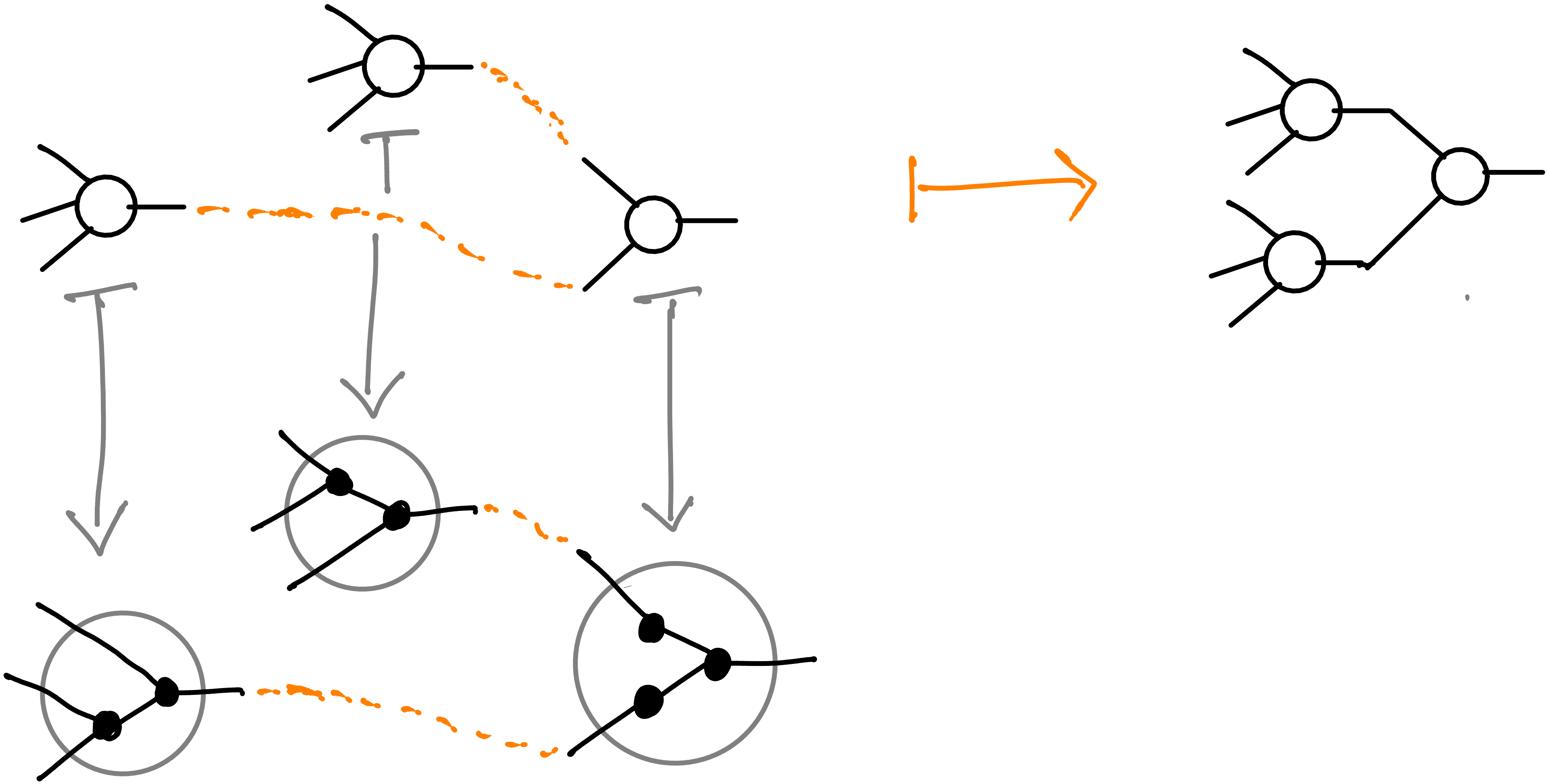
Algebraic Structure
vertical composition
node replacement

Compatibility Law:

$$\begin{tikzcd} {} \arrow[r, "\text{graft}"] \arrow[d, "\text{replace}"'] & {} \arrow[d, "\text{replace}"] \\ {} \arrow[r, "\text{graft}"'] & {} \end{tikzcd}$$

Compatibility Law:

$$\begin{tikzcd} {} \arrow[r, "\text{graft}"] \arrow[d, "\text{replace}"'] & {} \arrow[d, "\text{replace}"] \\ {} \arrow[r, "\text{graft}"'] & {} \end{tikzcd}$$



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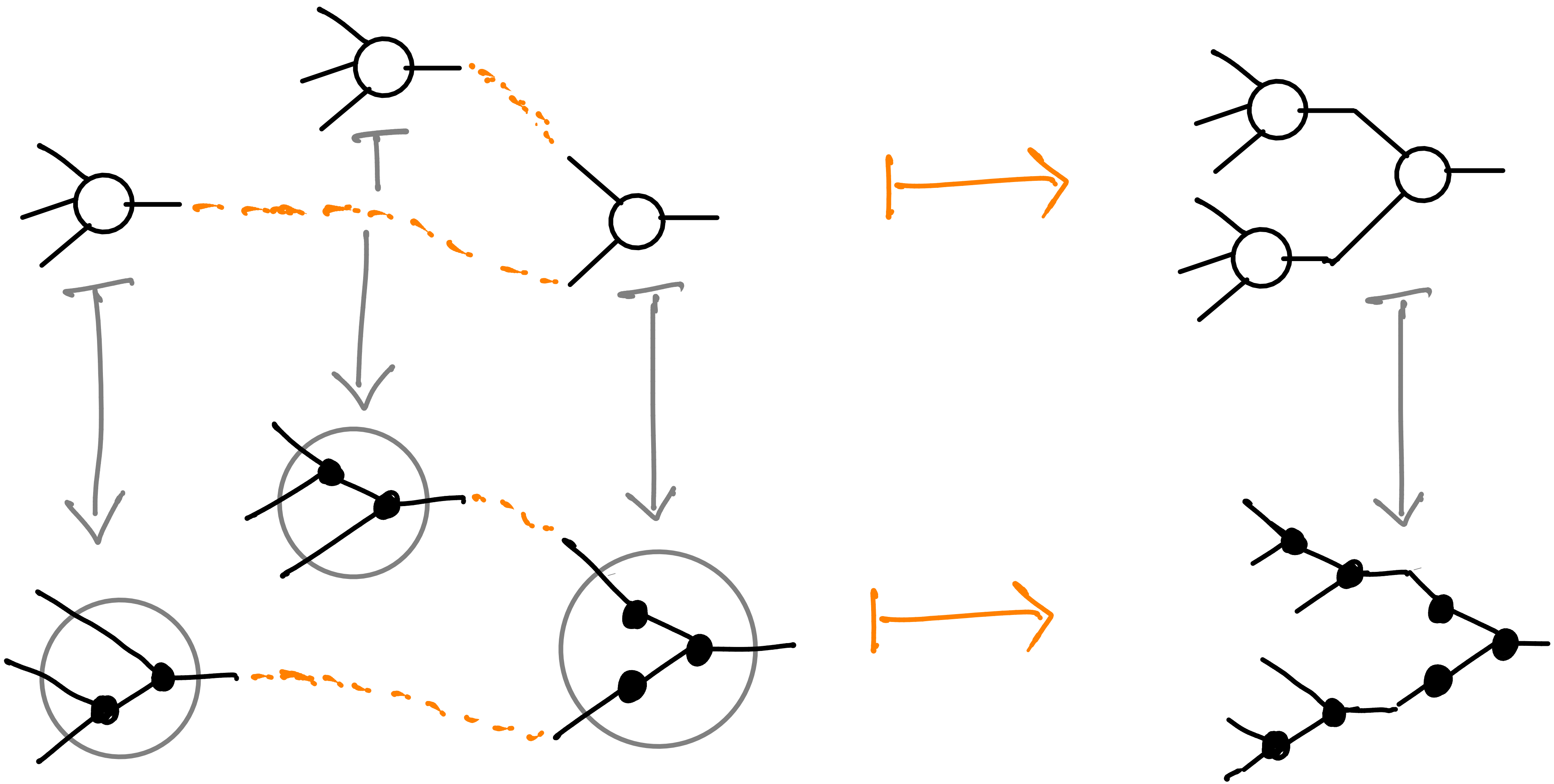

Analysis of horizontal and vertical compositions

(1) composition is monoid structure

(2) horizontal and vertical structures are compatible

Analysis of horizontal and vertical compositions

(1) composition is monoid structure

$$
\begin{tikzcd}
 & C \bullet C \arrow[dr, "h"] & & C * C \arrow[dl, "v"'] & \\
(I, \bullet)\text{-monoid} \arrow[r, squiggly, no head] & & C & & (J, *)\text{-monoid} \arrow[l, squiggly, no head] \\
 & I \arrow[ur] & & J \arrow[ul] &
\end{tikzcd}
$$

(2) horizontal and vertical structures are compatible

- the horizontal and vertical tensor products are endowed with an interchange law
- the horizontal and vertical compositions satisfy a compatibility law relative to the interchange law.

Examples:

- Tensorial strength interchange

(1) Second-order Abstract Syntax [F. 2008]

with parameterised metavariables

(2) Opetopic Structure [F. 2016]

- Monoidal interchange

(3) Internal Strict Higher Category Structure

[F. & Guiraud]

The new connection with type theory

Abstract syntactic character of an inductive <u>universal construction</u> of opetopic structures akin to the structure of algebraic languages with parameterised metavariables

Second-Order Abstract Syntax [Hamana, F.]

t (terms)

$::= x_i$ (variables)

$\mid M[t_1, \ldots, t_n]$ (parameterized metavariables)

$\mid f(\vec{x_1}.t_1, \ldots, \vec{x_n}.t_n)$ (binding operators)

Examples: $@(\lambda x.M[x], N[\,])$ $\qquad M[N[\,]]$

$\forall(x.P[x])$ $\qquad \wedge(P[y], \forall(x.P[x]))$

Two Substitution Operations

(2-leveled
2-dimensional)

- Capture-avoiding substitution

$$t[{}^{t_1}/x_1, \ldots, {}^{t_n}/x_n]$$

- Meta-substitution

$$t\{M_1 := (\vec{x}_1)t_1, \ldots, M_n := (\vec{x}_n)t_n\}$$

Two Substitution Operations

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- Capture-avoiding substitution

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- Meta-substitution

$$t\{M_1 := (\vec{x}_1)\, t_1, \ldots, M_n := (\vec{x}_n)\, t_n\}$$

$$(M_k[u_1, \ldots, u_k])\{M_i := (x_1, - x_{m_i})\, t_i\}_i$$

$$= t_k[{}^{u_1'}/x_1, \ldots, {}^{u_{m_k}'}/x_{m_k}]$$

where $u_j' = u_j\{M_i := (x_1 - x_{m_i})\, t_i\}_i$

Compatibility Law:

$$
\begin{tikzcd}
{} \arrow[r, "\text{subst}"] \arrow[d, "\text{meta subst}"'] & {} \arrow[d, "\text{meta subst}"] \\
{} \arrow[r, "\text{subst}"'] & {}
\end{tikzcd}
$$

$$\left(t\left[{}^{t_i}/_{x_i}\right]_i\right)\{M_j := (\vec{y_j})\,u_j\}_j$$

$$=$$

$$\left(t\{M_j := (\vec{y_j})\,u_j\}_j\right)\left[{}^{t_i\{M_j := (\vec{y_j})\,u_j\}_j}/_{x_i}\right]$$

What is the algebraic structure that axiomatizes two compatible composition/substitution structures for a tensorial strength interchange?

What is the algebraic structure that axiomatizes two compatible composition/substitution structures for a tensorial strength interchange?

- substitution structure = monoid structure

- $$\frac{\text{monoid structure}}{\text{monoidal category}} = \frac{\text{Two compatible monoid structures for a tensorial strength interchange}}{?}$$

Def: A near-semiring category is a category $\mathscr{C}$ with two [skew] monoidal structures

$$(\mathscr{C}, I, \bullet), \quad (\mathscr{C}, J, *)$$

equipped with tensorial strengths

$$J \bullet Z \to J, \quad (X * Y) \bullet Z \to (X \bullet Z) * (Y \bullet Z)$$

▶ **Formally**: $((\mathscr{C}, \bullet), J, *)$ is a pseudomonoid in the 2-category of $(\mathscr{C}, I, \bullet)$-actegories, strong functors, and strong natural transformations

Example: Every cartesian category with

$$I = J = 1 \quad \text{and} \quad \bullet = * = \times$$

Def: A near-semiring object in a near-semiring category is an object S with monoid structures

$$I \to S \leftarrow S \bullet S \quad , \quad J \to S \leftarrow S * S$$

compatible in that

$$\begin{tikzcd} J \bullet S \arrow[r] \arrow[d] & J \arrow[d] \\ S \bullet S \arrow[r] & S \end{tikzcd} \qquad \begin{tikzcd} (S * S) \bullet S \arrow[r] \arrow[d] & (S \bullet S) * (S \bullet S) \arrow[r] & S * S \arrow[d] \\ S \bullet S \arrow[rr] & & S \end{tikzcd}$$

- connects to algebraic combinatorics [discussed with J. Kock]

Example: For every monoid object $(M, 1, \times)$ in a cartesian closed category, the endo-exponential $[M \Rightarrow M]$ is a near-semiring object.

structure

$$\mathrm{id} = \lambda x.x \quad , \quad f \circ g = \lambda x. f(g(x))$$

$$j = \lambda x.1 \quad , \quad f * g = \lambda x. f(x) \times g(x)$$

laws

$$j \circ h = j \quad , \quad (f * g) \circ h = (f \circ h) * (g \circ h)$$

- connects to the algebraic theory of the λ-calculus

- near-semiring categories
  the universe of discourse for $\bullet$-monoids with $\bullet$-strong $*$-monoidal algebraic theories
- near-semiring objects
  $\bullet$-monoids with $\bullet$-strong $*$-monoid structure
- Monadic Theory [F. & Saville, FSCD 2017]
  monoids with compatible $\mathbb{T}$-algebraic structure for a strong monad $\mathbb{T}$

Cor. (of the monadic theory [F. & Saville, FSCD 2017])

For

a nsr-category with finite coproducts and colimits of ω-chains both of which are preserved by $- \bullet X$ and $- * X$,

the $*$-list object on the $\bullet$-unit

$$L_*(I) = \mu X.\ J + I * X$$

is an initial nsr-object

Algebraic Combinatorial Framework

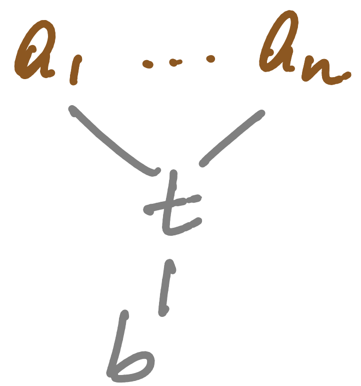
- $(\mathbb{A}, \mathbb{B})$-species [F. & Gambino & Hyland & Winskel] between small categories

$$T : !\mathbb{A} \times \mathbb{B}^{\circ} \longrightarrow \mathbf{Set}$$

$!$ = free symmetric monoidal completion

<u>idea:</u>

$$T(a_1 \dots a_n ; b) = \{ \quad \}$$



- connects to synthetic HoTT [F. & Horlick]

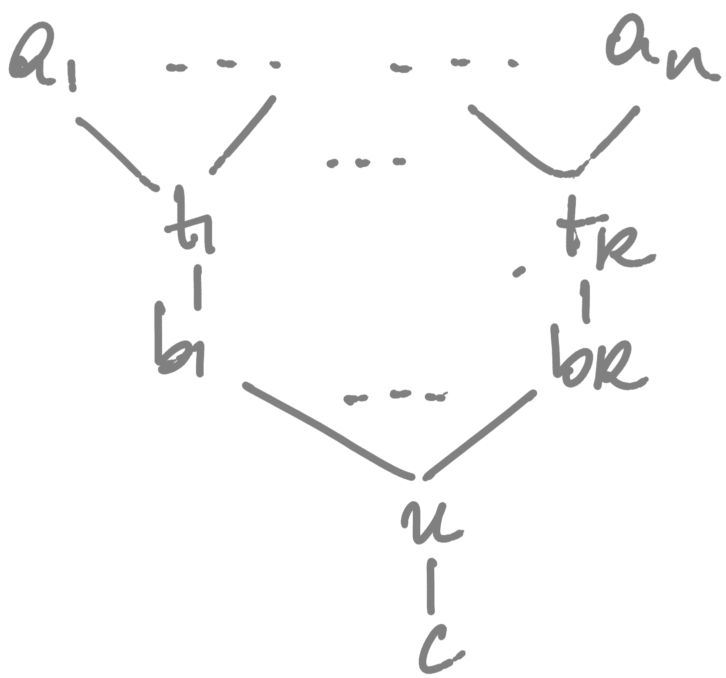
▶ Composition

$$\frac{T : {!\mathbb{A}} \times \mathbb{B}^{o} \longrightarrow \mathrm{Set} \qquad U : {!\mathbb{B}} \times \mathbb{C}^{o} \longrightarrow \mathrm{Set}}{U \circ T : {!\mathbb{A}} \times \mathbb{C}^{o} \longrightarrow \mathrm{Set}}$$

<u>Idea</u>:

formal composites

$$(U \circ T)(a_1 \text{---} a_n ; c) = \left\{ \quad \right\}$$



Thm [F. & Gambino & Hyland & Winskel]

We have a cartesian closed bicategory of generalised species of structure Esp.

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Structure levels

GLOBAL

products $\mathbb{A} \sqcap \mathbb{B} = \mathbb{A} \uplus \mathbb{B}$

exponentials $[\mathbb{A} \Rightarrow \mathbb{B}] = !\mathbb{A}^{\circ} \times \mathbb{B}$

LOCAL

$$\mathrm{Esp}(\mathbb{A}, \mathbb{B}) = \mathrm{Set}^{[\mathbb{A} \Rightarrow \mathbb{B}]}$$

$(\mathrm{Esp}(\mathbb{A}, \mathbb{A}), \mathrm{Id}, \circ)$ monoidal

Example:

GLOBAL

monad in Esp

$$
\begin{tikzcd}[column sep=large, row sep=large]
A \arrow[rr, bend left=70, "\mathrm{Id}"] \arrow[rr, "T"] \arrow[rr, phantom, bend left=35, "\Downarrow"] \arrow[dr, bend right, "T"'] & & A \\
& A \arrow[ur, bend right, "T"'] \arrow[u, phantom, "\Uparrow"] &
\end{tikzcd}
$$

LOCAL

monoid in Esp(A, A)

$$\mathrm{Id} \Rightarrow T \Leftarrow T \circ T$$

Example:

GLOBAL

monad in Esp

$$
\begin{tikzcd}
A \arrow[rr, bend left=70, "\mathrm{Id}"] \arrow[rr, bend left=35, phantom, "\Downarrow"] \arrow[rr, "T"] \arrow[dr, "T"'] & & A \\
& A \arrow[ur, "T"'] \arrow[u, phantom, "\Uparrow"] &
\end{tikzcd}
$$

LOCAL

monoid in Esp(A, A)

$$\mathrm{Id} \Rightarrow T \Leftarrow T \circ T$$

- generalised symmetric operads

idea.

$$T \circ T \longrightarrow T$$

formal composites $\longmapsto$ actual compositions

Iterating monads in Esp

└ an algebraic generalization of
the slice construction of [Baez & Dolan]

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(1) $T \in \mathrm{Esp}(\mathbb{A}, \mathbb{A})$ a monoid

$T^{+} \in \mathrm{Esp}(\int T, \int T)$ a monoid

Iterating monads in Esp

└ an algebraic generalization of the slice construction of [Baez & Dolan, Szawiel & Zawadowski]

(1) $T \in \mathrm{Esp}(A,A)$ a monoid.

(2) $\mathrm{Esp}(A,A)/_T$ is a monoidal category

$$\begin{array}{c} Id \\ \downarrow \\ T \end{array} \quad , \qquad \begin{array}{c} P \\ \downarrow \\ T \end{array} \circ/_T \begin{array}{c} Q \\ \downarrow \\ T \end{array} = \begin{array}{c} P \circ Q \\ \downarrow \\ T \circ T \\ \downarrow \\ T \end{array}$$

Iterating monads in Esp

└ an algebraic generalization of the slice construction of [Baez & Dolan, Szawiel & Zawadowski]

(1) $T \in \mathrm{Esp}(\mathbb{A},\mathbb{A})$ a monoid.

(2) $\mathrm{Esp}(\mathbb{A},\mathbb{A})/_T$ is a monoidal category

$$\mathrm{Esp}(\mathbb{A},\mathbb{A})/_T \cong \mathrm{Psh}(\textstyle\int T)$$

$$\mathrm{Psh}(\mathbb{C})/_P \simeq \mathrm{Psh}(\textstyle\int P)$$

$\int T$ has elements $t \in T(a_1, \ldots, a_n; a)$ as objects

Iterating monads in Esp

└ an algebraic generalization of the slice construction of [Baez & Dolan, Szawiel & Zawadowski]

(1) $T \in \mathrm{Esp}(\mathbb{A}, \mathbb{A})$ a monoid

(2) $\mathrm{Esp}(\mathbb{A}, \mathbb{A})/T$ is a monoidal category

(3) $\mathbb{1} \longrightarrow \mathrm{PSh}(\int T) \longleftarrow \mathrm{PSh}(\int T) \times \mathrm{PSh}(\int T)$

monoidal structure

$$
\begin{tikzcd}
(3) & \mathbb{1} \arrow[r] & \mathrm{PSh}(\textstyle\int T) \arrow[d, squiggly, "\text{internalization}"'] & \mathrm{PSh}(\textstyle\int T) \times \mathrm{PSh}(\textstyle\int T) \arrow[l] \\
(4) & 1 \arrow[r, "s"] & \textstyle\int T & \textstyle\int T \sqcap \textstyle\int T \arrow[l, "s"'] \arrow[u, squiggly, "\text{analytic externalization}"']
\end{tikzcd}
$$

monoidal structure

<u>Thm</u> [F.]: a pseudo-monoid in Esp

(3) $\mathbb{1} \longrightarrow \mathrm{PSh}(\int T) \longleftarrow \mathrm{PSh}(\int T) \times \mathrm{PSh}(\int T)$

monoidal structure

internalization

analytic externalization

(4) $1 \xrightarrow{s} \int T \xleftarrow{s} \int T \sqcap \int T$

Thm [F.]: a pseudo-monoid in Esp

(5) **Thm** [F. & Saville]: The endoexponential $[\int T \Rightarrow \int T]$ is a pseudo near-semiring object in Esp

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$\Big\}$ externalization

(6) $Esp(\int T, \int T)$ is a near-semiring category

(5) <u>Thm</u> [F. & Saville]: The endoexponential $[\int T \Rightarrow \int T]$ is a pseudo near-semiring object in Esp

externalization

(6) $\mathrm{Esp}(\int T, \int T)$ is a near-semiring category

(7) The initial near-semiring object $T^+ \in \mathrm{Esp}(\int T, \int T)$ is a monoid

(5) <u>Thm</u> [F. & Saville]: The endoexponential $[\int T \Rightarrow \int T]$ is a pseudo near-semiring object in Esp

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(8) GOTO (1) with $A := \int T$ and $T := T^+$

(5) <u>Thm</u> [F. & Saville]: The endoexponential $[\int T \Rightarrow \int T]$ is a pseudo near-semiring object in Esp

externalization

(6) $Esp(\int T, \int T)$ is a near-semiring category

(7) The initial nsr-object $T^+ \in Esp(\int T, \int T)$ is a monoid

(8) GOTO (1) with $A := \int T$ and $T := T^+$

<u>Example</u>: Opetopes arise from the identity monad on $\mathbb{1}$

Categorical opetopic structures

(generalizing opetopic sets)

$$\underline{\mathrm{Nsr}}(X) = \mu Z.\, L_*(I + X \bullet Z)$$

$$\begin{tikzcd}
 & & & T_{i+1} \in \mathrm{Mon}(\mathbb{A}_{i+1}) \arrow[dd, dotted, no head] \\
 & \text{free } \mathrm{Nsr}(X_i) \arrow[urr, mapsto] & & \\
 & & & \mathbb{A}_{i+1} \arrow[dl] \\
 & X_i \arrow[uu, mapsto] & & \\
\int T_i & & \int T_i \arrow[ll, "s"'] &
\end{tikzcd}$$

$T_i \in \mathrm{Mon}(\mathbb{A}_i)$

RESEARCH PANORAMA

Second-order Abstract Syntax | Higher-dimensional structure

Type Theory ⟷ Higher-dimensional category theory

Abstract syntax
- L variables
- L parameterised metavariables
- L binding operators

Trees
- L leaves
- L nodes
- ?

Context
- L cartesian

Indexing
- L linear

Levels

└ 2

Dimensionality

└ ∞

Algebraic Theories

└ equational $\rightsquigarrow$ Algebraic structure

└ monoids

Rewriting Theory

└ CRSs [Klop] $\rightsquigarrow$?

Algebraic Translations $\rightsquigarrow$?

? $\leftsquigarrow$ Geometry

Cartesian Distributors $\leftsquigarrow$ Framework

└ Generalised species

[F. & Joyal]

CT2015