Polymorphic Algebraic Theories

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Joint work with Makoto Hamana

Method and Technique

- Interplay between:

  mathematical structure and formal language.

- Categorical modelling tools:

  categorical algebra, presheaf categories,
  Grothendieck construction, Kan extensions,
  discrete generalised polynomial functors.

This Talk

One more step on ***Algebraic Foundations for Type Theories***.

- Second-order algebraic theories. [CSL'10, MFCS'10]

  (e.g. untyped and simply-typed λ-calculus)

- Polymorphic algebraic theories. [LICS'13]

  (e.g. System F)

- Dependent algebraic theories.

  (e.g. MLTT)

Polymorphic Algebraic Theories

- Theory:

  Mathematical foundations.

- Application:

  Mechanised formalisation.

Algebraic Desiderata

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- Logical framework = sound and complete equational logic.
- Translations = algebraic homomorphisms.
- Equational theories = invariant presentations.

Polymorphic Type Theories

Example: Polymorphic FPC

Types:

$$\tau \quad ::= \quad \alpha \mid \tau_1 + \tau_2 \mid \tau_1 \times \tau_2 \mid \tau_1 \Rightarrow \tau_2 \mid \mu(\alpha.\tau) \mid \forall(\alpha.\tau)$$

Terms:

$$t \quad ::= \quad \cdots \mid \lambda(x : \tau.t) \mid t_1(t_2) \mid \Lambda(\alpha.t) \mid t(\tau)$$

Polymorphic Type Theories

Example: Polymorphic FPC

Types: $\boxed{\textsc{Level 1}}$

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Terms: $\boxed{\textsc{Level 2}}$

$$t \quad ::= \quad \cdots \mid \lambda(x : \tau.t) \mid t_1(t_2) \mid \Lambda(\alpha.t) \mid t(\tau)$$

Polymorphic Type Theories

Example: Polymorphic FPC with equi-recursive types

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$$\tau \quad ::= \quad \alpha \mid \tau_1 + \tau_2 \mid \tau_1 \times \tau_2 \mid \tau_1 \Rightarrow \tau_2 \mid \mu(\alpha.\tau) \mid \forall(\alpha.\tau)$$

$$\mu(\alpha.\tau) = \tau[\mu(\alpha.\tau)/\alpha]$$

Terms: $\boxed{\textsc{Level 2}}$

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Second-Order Algebraic Theories

- Binding signatures.

  Example:

operator		arity	specification
λ	:	$(*)* \longrightarrow *$	one binding argument
$@$	:	$*, * \longrightarrow *$	two non-binding arguments

Second-Order Algebraic Theories

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$@$	:	$*, * \longrightarrow *$	two non-binding arguments

- Pre-models = Σ-monoids in $\mathcal{S}et^{\mathbb{F}}$.

  Example:

$$
\begin{tikzcd}
 & A^V \arrow[dr, "\lambda"] & & A \times A \arrow[dl, "@"'] & \\
V \arrow[rr, "v"'] & & A & & A \bullet A \arrow[ll, "s"]
\end{tikzcd}
$$

algebra

$\updownarrow$ compatible structures

Lawvere theory

- Free pre-models
  = abstract syntax with variable binding and metavariables

$$\begin{tikzcd} \Sigma\text{-Mon} \arrow[d, shift left=4ex] \arrow[d, phantom, "\dashv"] \\ \boldsymbol{Set}^{\mathbb{F}} \arrow[u, shift left=4ex] \arrow[loop left, "\mathcal{M}"] \end{tikzcd}$$

- Free pre-models
  = abstract syntax with variable binding and metavariables

$$
\begin{tikzcd}
\Sigma\text{-Mon} \arrow[d, shift left=3ex] \\
\mathcal{S}et^{\mathbb{F}} \arrow[u] \arrow[loop left, "\mathcal{M}"]
\end{tikzcd}
\qquad \dashv
$$

Example: $t \in \mathcal{M}(\underline{X})(\Gamma)$ iff $X \rhd \Gamma \vdash t$

$$
\frac{X \rhd \Gamma, x \vdash t}{X \rhd \Gamma \vdash \lambda(x.t)}
\qquad
\frac{X \rhd \Gamma \vdash t_1 \quad \Gamma \vdash t_2}{X \rhd \Gamma \vdash t_1 @ t_2}
$$

$$
\frac{}{X \rhd \Gamma \vdash x}\ (x \in \Gamma)
\qquad
\frac{X \rhd \Gamma \vdash t_i (1 \leq i \leq n)}{X \rhd \Gamma \vdash \mathsf{M}[t_1, \cdots, t_n]}\ (\mathsf{M} \in X(n))
$$

- Second-order equational presentations

  Example:

$$(\beta) \quad \mathsf{M} : [*]* \;,\; \mathsf{N} : * \;\vdash\; \lambda(x.\mathsf{M}[x]) \mathbin{@} \mathsf{N} \;=\; \mathsf{M}[\mathsf{N}]$$

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- Second-order models

  Example: premodels A satisfying

$$[\![\,\lambda(x.\mathsf{M}[x])\ @\ \mathsf{N}[\,]\,]\!]\ =\ [\![\,\mathsf{M}[\mathsf{N}[\,]]\,]\!]\ :\ A^V\times A\to A$$

Coincides with Martin Hyland's notion of *semiclosed algebraic theory* as a model of the λ-calculus

- Second-Order Equational Logic

  Judgements

$$X \triangleright \Gamma \vdash t_1 = t_2$$

  subject to congruence rules of meta substitution and extension.

Polymorphic Algebraic Theories

Types: $\boxed{\text{LEVEL 1}}$

Second-order equational presentations

Example:

$$+, \times, \Rightarrow \;:\; *, * \longrightarrow * \qquad\qquad \forall, \exists, \mu \;:\; (*)* \longrightarrow *$$

$$\mathsf{T} : [*]* \;\vdash\; \mu(x.\mathsf{T}[x]) \;=\; \mathsf{T}\big[\mu(x.\mathsf{T}[x])\big]$$

Terms: $\boxed{\text{LEVEL 2}}$

Example: System F

Vernacular rules.

$$\frac{\alpha_i \mid x_j : \tau_j, x : \sigma \vdash t : \tau}{\alpha_i \mid x_j : \tau_j \vdash \lambda(x : \sigma.t) : \sigma \Rightarrow \tau}$$

$$\frac{\alpha_i, \alpha \mid x_i : \tau_i \vdash t : \tau}{\alpha_i \mid x_i : \tau_i \vdash \Lambda(\alpha.t) : \forall(\alpha.\tau)}$$

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Polymorphic signature operators.

$$\lambda \quad :: \quad \mathsf{S}, \mathsf{T} : * \quad \rhd \quad (\mathsf{S})\mathsf{T} \;\longrightarrow\; \mathsf{S} \Rightarrow \mathsf{T}$$

$$\Lambda \quad :: \quad \mathsf{T} : [*]* \quad \rhd \quad \langle\alpha\rangle\mathsf{T}[\alpha] \;\longrightarrow\; \forall(\alpha.\mathsf{T}[\alpha])$$

Example: Existential λ-calculus

Vernacular rules.

$$\frac{\alpha_i \mid x_j : \tau_j \vdash s : \sigma[\tau/\alpha]}{\alpha_i \mid x_j : \tau_j \vdash \mathsf{pack}(\tau, s) : \exists(\alpha.\sigma)}$$

$$\frac{\alpha_i \mid x_j : \tau_j \vdash s : \exists(\alpha.\sigma) \qquad \alpha_i, \alpha \mid x_j : \tau_j, x : \sigma \vdash t : \tau}{\alpha_i \mid x_j : \tau_j \vdash \mathsf{unpack}\ s\ \mathsf{as}\ (\alpha, x)\ \mathsf{in}\ t : \tau} \ (\alpha \# \tau_j, \tau)$$

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Polymorphic signature operators.

$$\mathsf{pack} \;::\; \mathsf{S} : [*]*, \mathsf{T} : * \;\rhd\; \mathsf{S}[\mathsf{T}] \longrightarrow \exists(\alpha.\mathsf{S}[\alpha])$$

$$\mathsf{unpack} \;::\; \mathsf{S} : [*]*, \mathsf{T} : * \;\rhd\; \exists(\alpha.\mathsf{S}[\alpha]), \langle\alpha\rangle(\mathsf{S}[\alpha])\mathsf{T} \longrightarrow \mathsf{T}$$

Contexts for Polymorphism

Contexts

$$\underset{\text{type context}}{\alpha_1, \ldots, \alpha_n} \quad \mid \quad \underset{\text{term context}}{\Gamma} \quad \vdash \quad \underset{\text{term}}{t} \quad : \quad \underset{\text{type}}{\tau}$$

with types in a universe $\mathcal{U}$ are modelled by the Grothendieck construction

[Hamana FOSSACS'11]

$$G\mathcal{U} \;=\; \int^{n \in \mathbb{F}} \mathbb{F}{\downarrow}(\mathcal{U}n) \times \mathcal{U}n$$

Polymorphic Signatures

- A *polymorphic signature* consists of
  1. a second-order signature Σ_1 and equational presentation E_1 for type structure, and
  2. a polymorphic signature Σ_2 for term structure.

Polymorphic Signatures and Structures

- A *polymorphic signature* consists of
  1. a second-order signature Σ_1 and equational presentation E_1 for type structure, and
  2. a polymorphic signature Σ_2 for term structure.
- A *polymorphic structure* consist of
  1. a type universe U modelling (Σ_1, E_1), and
  2. compatible algebraic structure in $\mathcal{S}\boldsymbol{et}^{\mathrm{G}U}$ as follows . . .

$$
\begin{tikzcd}
 & \Sigma_2 A \arrow[d] & \uparrow A \arrow[dl] & \\
V \arrow[r, "v"'] & A & & A \bullet A \arrow[ll, "s"]
\end{tikzcd}
$$

where Σ_2 and $\uparrow$ are endofunctors on $\mathcal{S}et^{\mathrm{GU}}$ whose algebras respectively model the polymorphic signature operators and the operation of type-in-term substitution.

$$\begin{tikzcd} & \Sigma_2 A \arrow[d] & \uparrow A \arrow[dl] \\ V \arrow[r, "v"'] & A & A \bullet A \arrow[l, "s"'] \end{tikzcd}$$

where Σ_2 and $\uparrow$ are endofunctors on $\mathcal{S}et^{\mathrm{GU}}$ whose algebras respectively model the polymorphic signature operators and the operation of type-in-term substitution.

NB: These constructions are performed within the theory of discrete generalised polynomial functors on presheaf categories[ICALP'12] and hence support free constructions, which provide the abstract syntax of polymorphic algebraic theories.

Polymorphic Equational Presentations

Example: System F

Vernacular.

$$(\beta) \quad \Gamma \vdash \lambda(x:\sigma.M)@N = M[N/x] : \tau$$

$$(\beta') \quad \Gamma \vdash \Lambda(\alpha.M)\,(\sigma) = M[\sigma/\alpha] : \tau[\sigma/\alpha]$$

Formal.

$$(\beta) \quad \mathsf{S},\mathsf{T}:* \;\triangleright\; \mathsf{M}:[\mathsf{S}]\mathsf{T}\,,\;\mathsf{N}:\mathsf{S}$$

$$\vdash\; @_{\mathsf{S},\mathsf{T}}(\,\lambda_{\mathsf{S},\mathsf{T}}(x.\mathsf{M}[x])\,,\,\mathsf{N}\,) = \mathsf{M}[\mathsf{N}] : \mathsf{T}$$

$$(\beta') \quad \mathsf{S}:*\,,\;\mathsf{T}:[*]* \;\triangleright\; \mathsf{M}:\{\alpha\}\mathsf{T}[\alpha]$$

$$\vdash\; @'_{\mathsf{S},\mathsf{T}}(\,\Lambda_{\mathsf{T}}(\alpha.\mathsf{M}\{\alpha\})\,) = \mathsf{M}\{\mathsf{S}\} : \mathsf{T}[\mathsf{S}]$$

Example: Existential λ-calculus.

Vernacular.

$$(\exists\beta) \quad \Gamma \ \vdash \quad \mathsf{unpack}\ \big(\mathsf{pack}(\iota, \mathrm{N})\big)\ \mathsf{as}\ (\alpha, x)\ \mathsf{in}\ M \quad = \quad M[\iota/\alpha, \mathrm{N}/x] \quad : \quad \tau$$

$$(\exists\eta) \quad \Gamma \ \vdash \quad \mathsf{unpack}\ \mathrm{N}\ \mathsf{as}\ (\alpha, x)\ \mathsf{in}\ M[\mathsf{pack}(\alpha, x)/z] \quad = \quad M[\mathrm{N}/z] \quad : \quad \tau$$

Formal.

$$(\exists\beta) \quad \mathsf{S} : [*]* \,,\ \mathsf{T}, \mathsf{U} : * \quad \rhd \quad \mathsf{M} : \{\alpha\}\big[\mathsf{S}[\alpha]\big]\mathsf{T} \,,\ \mathsf{N} : \mathsf{S}[\mathsf{U}]$$

$$\vdash \quad \mathsf{unpack}_{\mathsf{S},\mathsf{T}}(\ \mathsf{pack}_{\mathsf{S},\mathsf{U}}(\mathsf{N})\,,\ \alpha.x.\mathsf{M}\{\alpha\}[x]\ ) \quad = \quad \mathsf{M}\{\mathsf{U}\}[\mathsf{N}] \quad : \quad \mathsf{T}$$

$$(\exists\eta) \quad \mathsf{S} : [*]* \,,\ \mathsf{T} : * \quad \rhd \quad \mathsf{M} : \big[\exists(\alpha.\mathsf{S}[\alpha])\big]\mathsf{T} \,,\ \mathsf{N} : \exists(\alpha.\mathsf{S}[\alpha])$$

$$\vdash \quad \mathsf{unpack}_{\mathsf{S},\mathsf{T}}(\ \mathsf{N}\,,\ \alpha.x.\mathsf{M}[\mathsf{pack}_{\mathsf{S},\alpha}(x)]\ ) \quad = \quad \mathsf{M}[\mathsf{N}] \quad : \quad \mathsf{T}$$

Algebraic Models

An *algebraic model* of a polymorphic equational presentation is a polymorphic structure satisfying the axioms.

Example: Seely's PL-category semantics of System F yields an algebraic model of the equational presentation of System F.

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Polymorphic Translations

Polymorphic translations $(\mathfrak{U}, A) \to (\mathfrak{U}', A')$ are given by homomorphisms $\varphi : \mathfrak{U} \to \mathfrak{U}'$ and $\vartheta : A \to \varphi^* A'$.

Example: Fujita's CPS translation from System F to the Existential λ-calculus is a polymorphic translation.

Polymorphic Equational Logic

Judgements:

$$Z \;\triangleright\; \vec{\alpha} \;\mid\; \Gamma \;\vdash_{\mathfrak{U}}\; s = t \;:\; \tau$$

where $\mathfrak{U}$ is a type universe, Z is a set of metavariable declarations, $\vec{\alpha} \mid \Gamma \vdash \tau$ is a context, and $Z \triangleright \vec{\alpha} \mid \Gamma \vdash s, t : \tau$ are meta-terms.

Polymorphic Equational Logic

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Rules:

Congruence of all the algebraic structure plus *universe shift*:

$$\frac{Z \ \triangleright \ \vec{\alpha} \ | \ \Gamma \ \vdash_{\mathfrak{U}} \ s = t \ : \ \tau}{(\!|\, Z \,|\!) \ \triangleright \ \vec{\alpha} \ | \ (\!|\, \Gamma \,|\!) \ \vdash_{\mathfrak{V}} \ (\!|\, s \,|\!) = (\!|\, t \,|\!) \ : \ (\!|\, \tau \,|\!)} \ (\!|\, - \,|\!) : \mathfrak{U} \to \mathfrak{V}$$