Lie Structure and Composition

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Topics

- Algebraic structure

  On generalised species of structures (*viz.* symmetric sequences and generalisations)

- Notions of composition

  - Multi-categorical (or operadic) [Lambek; May]

    $$(A_1, \ldots, A_n) \to B$$

  - Poly-categorical (or dioperadic) [Szabo; Gan]

    $$(A_1, \ldots, A_n) \to (B_1, \ldots, B_m)$$

Topics

Characterisations of symmetric multi and poly composition by means of combinatorial interpretations of Lie structure

- Algebraic structure

  On generalised species of structures (*viz.* symmetric sequences and generalisations)

- Notions of composition

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    $$(A_1, \ldots, A_n) \to B$$

  - Poly-categorical (or dioperadic) [Szabo; Gan]

    $$(A_1, \ldots, A_n) \to (B_1, \ldots, B_m)$$

Multi-composition

- Simultaneous

- Partial

- **NB** Equivalent in the presence of identities, but not necessarily otherwise. [Markl]

Multi-composition

- Simultaneous

$$\frac{\begin{array}{c} f_i : (A_1^{(i)}, \ldots, A_{n_i}^{(i)}) \to B_i \\ g : (B_1, \ldots, B_m) \to C \quad (1 \leq i \leq m) \end{array}}{g \circ (f_1, \ldots, f_m) : (A_1^{(1)}, \ldots, A_{n_1}^{(1)}, \cdots, A_{n_1}^{(m)}, \ldots, A_{n_m}^{(m)}) \to C}$$

- Partial

$$\frac{f : (A_1, \ldots, A_n) \to B_i \qquad g : (B_1, \ldots, B_m) \to C}{g \circ_i f : (B_1, \ldots, B_{i-1}, A_1, \cdots, A_n, B_{i+1}, \ldots, B_m) \to C} \; (1 \leq i \leq n)$$

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Simultaneous multi-composition is semigroup structure

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- **Example**
  - $\mathbb{B}$ = (the category of finite cardinals and bijections)
  - $\boldsymbol{\mathcal{S}et}^{\mathbb{B}}$ = (the category of symmetric sequences)
  - Substitution tensor product $\bullet$ on symmetric sequences.[Joyal]

$$\big(P \bullet Q\big)(n)$$
$$= \int^{k \in \mathbb{B}} P(k) \times \int^{n_1,\ldots,n_k \in \mathbb{B}} \prod_{1 \le i \le k} Q(n_i) \times \mathbb{B}\big(n_1 + \cdots + n_k, n\big)$$

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  - Substitution tensor product $\bullet$ on symmetric sequences. [Joyal]
  - $\bullet$-semigroups (or associative $\bullet$-algebras, or α-algebras) $\mathrm{P} \bullet \mathrm{P} \to \mathrm{P}$ are non-unital symmetric operads. [Kelly]

$$
\begin{tikzcd}
(\mathrm{P} \bullet \mathrm{P}) \bullet \mathrm{P} \arrow[rr, "\alpha_{\mathrm{P},\mathrm{P},\mathrm{P}}", "\cong"'] \arrow[d] & & \mathrm{P} \bullet (\mathrm{P} \bullet \mathrm{P}) \arrow[d] \\
\mathrm{P} \bullet \mathrm{P} \arrow[r] & \mathrm{P} & \mathrm{P} \bullet \mathrm{P} \arrow[l]
\end{tikzcd}
$$

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- **NB**

  For non-unital *coloured* symmetric operads, one generalises to $\mathbb{B}_{\mathrm{C}}$ = (the free symmetric monoidal category on a set of colours C). [Fiore-Gambino-Hyland-Winskel]

What about partial multi-composition ?

- **Remark**

  Non-unital symmetric operads with partial composition are symmetric sequences

  $$P \in \boldsymbol{\mathcal{S}et}^{\mathbb{B}}$$

  equipped with *single composition* operations

  $$\circ_{n,m} : P(n+1) \times P(m) \to P(n+m) \qquad (n, m \in \mathbb{B})$$

  subject to

  equivariance , associativity , exchange

  laws.

The calculus of species

[Joyal]

- For symmetric sequences P, Q:
  - Multiplication

    $$(P \cdot Q)(n) = \int^{n_1, n_2 \in \mathbb{B}} P(n_1) \times Q(n_2) \times \mathbb{B}(n_1 + n_2, n)$$

    [Day]
  - Derivation

    $$\partial P(n) = P(n+1) \qquad \partial P = \big(y(1) \multimap P\big)$$

The calculus of species

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  - Derivation
    $$\partial P(n) = P(n+1)$$ $$\partial P = \big(y(1) \multimap P\big)$$

- **Remark**

  Non-unital symmetric operads with partial composition are symmetric sequences
  $$P \in \boldsymbol{\mathcal{S}et}^{\mathbb{B}}$$
  equipped with a *single composition* operation
  $$\circ : \partial(P) \cdot P \to P$$
  subject to

  associativity , exchange ‣ the laws

  laws.

The pre-Lie product

- **Definition**
  The *pre-Lie product* of symmetric sequences P, Q is defined as

$$P \star Q = \partial(P) \cdot Q$$

  that is,

$$\big(P \star Q\big)(k) = \int^{n,m \in \mathbb{B}} P(n+1) \times Q(m) \times \mathbb{B}(n+m, k)$$

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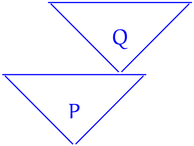
  structures of which may be graphically represented as



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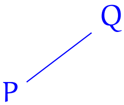

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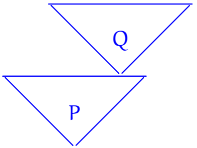

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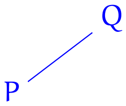

$\boxed{?}$ How do $(P \star Q) \star R$ and $P \star (Q \star R)$ compare?

- **Combinatorial interpretation of the additive pre-Lie identity**

  There is a canonical natural *pre-Lie isomorphism*

  $$\pi_{P,Q,R} : (P \star Q) \star R + P \star (R \star Q) \cong P \star (Q \star R) + (P \star R) \star Q$$

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- **Definition**

  A *π-algebra* is a symmetric sequence P equipped with a structure map $P \star P \to P$ compatible with the pre-Lie isomorphism.

$$\begin{tikzcd}
(P \star P) \star P + P \star (P \star P) \arrow[rr, "\pi_{P,P,P}", "\cong"'] \arrow[d] & & P \star (P \star P) + (P \star P) \star P \arrow[d] \\
P \star P \arrow[r] & P & P \star P \arrow[l]
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\end{tikzcd}$$

- **Theorem**

  The notions of non-unital symmetric operad with single composition and of π-algebra are equivalent.

Algebraic perspective

- **Definition**
  - A *pre-Lie algebra* is a vector space V equipped with an operation $\star : V \otimes V \to V$ subject to

$$(x \star y) \star z - x \star (y \star z) = (x \star z) \star y - x \star (z \star y)$$

[Gerstenhaber]

- **Remark**
  The commutator associated to a pre-Lie product is a Lie bracket.

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    [Gerstenhaber]

- A *Novikov algebra* is a pre-Lie algebra that further satisfies
  $$x \star (y \star z) = y \star (x \star z)$$
  [Dzhumadil'daev-Lofwall]

- A CAD_1 *algebra* is a commutative associative algebra $(V, \cdot)$ equipped with a derivation operation $\partial : V \to V$.

- **Remark**
  The commutator associated to a pre-Lie product is a Lie bracket.

- **Proposition**

  Every CAD_1 algebra $(V, \cdot, \partial)$ induces two Novikov algebra structures on V by setting

  $$x \star y \;=\; \partial(x) \cdot y$$

  and

  $$x \circledast y \;=\; x \cdot y + x \star y \;=\; (x + \partial x) \cdot y$$

- **Theorem**

  For symmetric sequences, a $\circledast$-algebra is a non-unital symmetric operad equipped with a binary operation that is pre-commutative and associative.

Directions

- Axiomatisation of pre-Lie products.
  - **Corollary**

    Every distributive symmetric-monoidal category with a strong endofunctor D such that the canonical map

    $$D(X) \otimes Y + X \otimes D(Y) \longrightarrow D(X \otimes Y)$$

    is an isomorphism has pre-Lie products

    $$X \star Y = D(X) \otimes Y$$

    and

    $$X \circledast Y = X \otimes Y + X \star Y$$

- Further examples. (SDG?)
- Generalisations. (braided? monoidal actions?)

Poly-composition

- Partial

[Gentzen; Szabo; Gan]

$$\frac{f : \vec{A} \to \vec{B}, \overset{j}{O}, \vec{C} \qquad g : \vec{X}, \overset{i}{O}, \vec{Y} \to \vec{Z}}{g \;{}_j\circ_i\; f \;:\; \vec{X}, \vec{A}, \vec{Y} \to \vec{B}, \vec{Z}, \vec{C}}$$

- Simultaneous

Requires a combinatorial definition of *matching* codomains and domains of poly-maps. Characterised as associative algebra structure for a composition tensor product. [Koslowski; Garner]

- **Remark**

  A *non-unital symmetric dioperad* is a symmetric matrix

$$\mathrm{P} \in \boldsymbol{\mathcal{S}et}^{\mathbb{B}^\circ \times \mathbb{B}}$$

equipped with a single composition operation

$$\circ : \partial_1(\mathrm{P}) \cdot \partial_2(\mathrm{P}) \to \mathrm{P}$$

subject to

associativity , exchange

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- **Remark**
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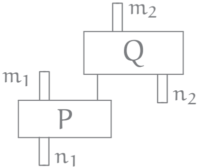
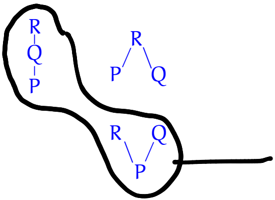


- **NB** $P * Q = \partial_1(P) \cdot \partial_2(Q)$

  $$\big(P * Q\big)(m, n) = \int^{m_1, m_2 \in \mathbb{B}^\circ, n_1, n_2 \in \mathbb{B}} P(m_1{+}1, n_1) \times Q(m_2, n_2{+}1) \times \mathbb{B}(m, m_1{+}m_2) \times \mathbb{B}(n_1{+}n_2, n)$$

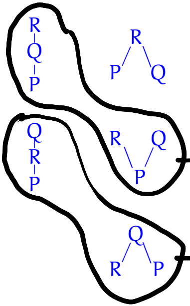
P (Q R)



(P Q) R

P (Q R)

(P R) Q



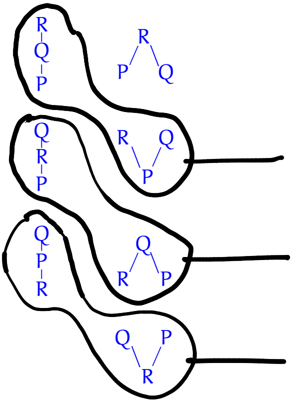
(P Q) R

P (R Q)

P (Q R)

(P R) Q

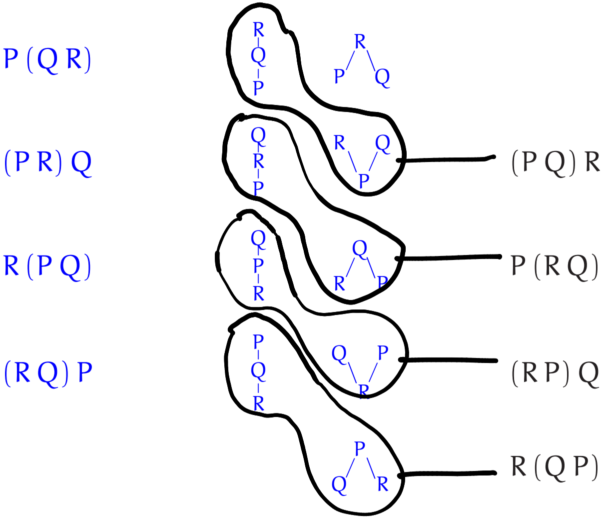
R (P Q)



(P Q) R

P (R Q)

(R P) Q


P (Q R)
(P R) Q
R (P Q)
(R Q) P
(P Q) R
P (R Q)
(R P) Q
R (Q P)

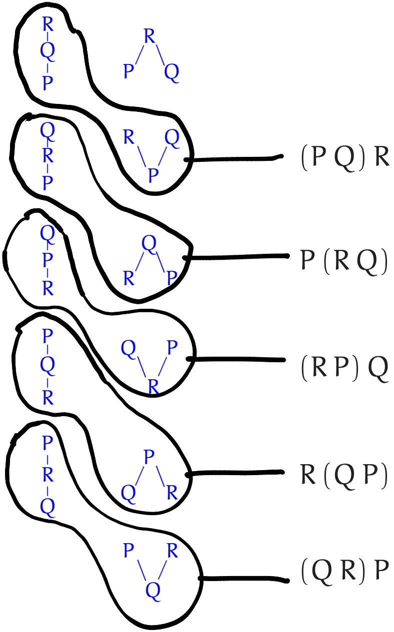
P (Q R)

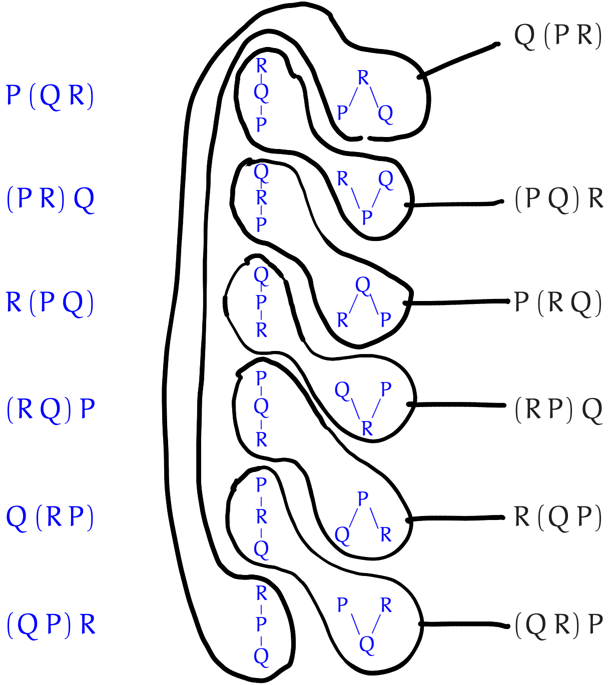
(P R) Q

R (P Q)

(R Q) P

Q (R P)




P (Q R)
(P R) Q
R (P Q)
(R Q) P
Q (R P)
(Q P) R
Q (P R)
(P Q) R
P (R Q)
(R P) Q
R (Q P)
(Q R) P

- **Combinatorial interpretation of the additive Lie-admissible identity**

  There is a canonical natural *Lie-admissible isomorphism*

  $$\rho_{P,Q,R} : F(P, Q, R) \cong G(P, Q, R)$$

  for

  $$\begin{aligned} F(P, Q, R) = \; & P * (Q * R) + (P * R) * Q + R * (P * Q) \\ & + (R * Q) * P + Q * (R * P) + (Q * P) * R \end{aligned}$$

  and

  $$\begin{aligned} G(P, Q, R) = \; & (P * Q) * R + Q * (P * R) + (Q * R) * P \\ & + R * (Q * P) + (R * P) * Q + P * (R * Q) \end{aligned}$$

  heuristically generated by commuting and re-bracketing

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  $$\begin{aligned} F(P,Q,R) = \; & P*(Q*R) + (P*R)*Q + R*(P*Q) \\ & + (R*Q)*P + Q*(R*P) + (Q*P)*R \end{aligned}$$

  and

  $$\begin{aligned} G(P,Q,R) = \; & (P*Q)*R + Q*(P*R) + (Q*R)*P \\ & + R*(Q*P) + (R*P)*Q + P*(R*Q) \end{aligned}$$

  heuristically generated by commuting and re-bracketing

- **Theorem**

  The notions of non-unital symmetric dioperad with single composition and of ρ-algebra are equivalent.

Algebraic perspective

- **Definition**
  - A *Lie-admissible algebra* is a vector space V equipped with an operation $* : V \otimes V \to V$ whose commutator is a Lie bracket. [Albert]
  - A CAD_2 *algebra* is a commutative associative algebra $(V, \cdot)$ equipped with two commuting derivation operations $\partial_1, \partial_2 : V \to V$.
- **Proposition**

  Every CAD_2 algebra $(V, \cdot, \partial_1, \partial_2)$ induces a Lie-admissible algebra structure on V by setting

$$x * y = \partial_1(x) \cdot \partial_2(y)$$

Appendix

Associativity and exchange laws

$$
\begin{tikzcd}
\partial(P)\cdot\partial(P)\cdot P \arrow[rr] \arrow[d] & & \partial\big(\partial(P)\cdot P\big)\cdot P \arrow[d] \\
\partial(P)\cdot P \arrow[r] & P & \partial(P)\cdot P \arrow[l]
\end{tikzcd}
$$

$$
\begin{tikzcd}
\partial\partial(P)\cdot P\cdot P \arrow[rr, "\cong" description] \arrow[d] & & \partial\partial(P)\cdot P\cdot P \arrow[d] \\
\partial\big(\partial(P)\cdot P\big)\cdot P \arrow[d] & & \partial\big(\partial(P)\cdot P\big)\cdot P \arrow[d] \\
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$$

▸ single composition