Towards a Mathematical Theory of Substitution

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Substitution

Examples:

- Logic/algebra/rewriting.

$$t\,[{}^{u}\!/\!_{x}] \qquad t\,[{}^{u_1}\!/\!_{x_1}, \ldots, {}^{u_n}\!/\!_{x_n}]$$

- Type theory.

$$T\,[{}^{t}\!/\!_{x}]$$

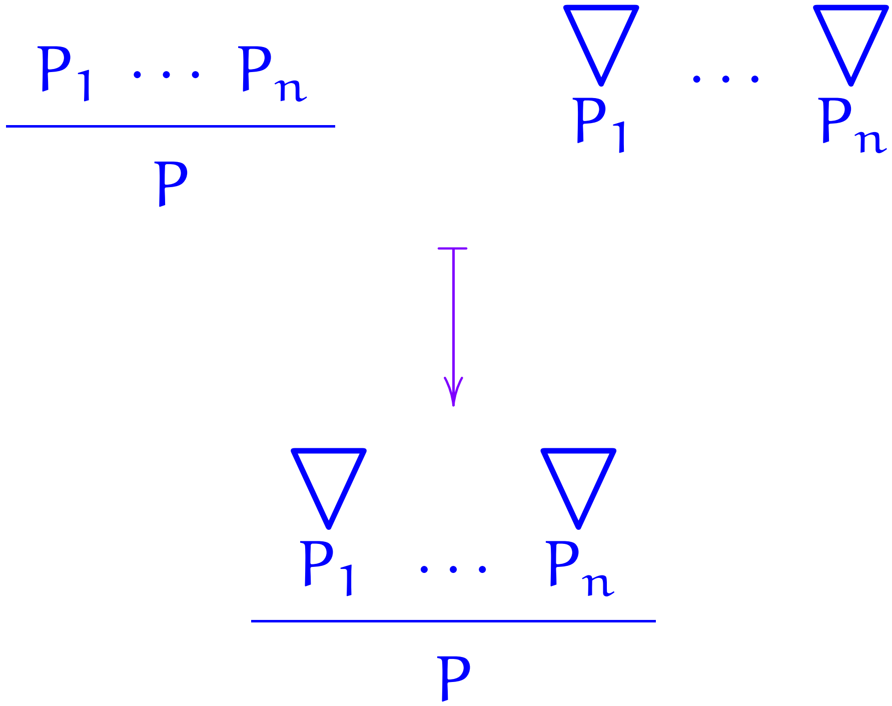
- Formal languages.

$$w_0 X_1 w_1 \ldots X_n w_n \qquad X_i \mapsto W_i$$

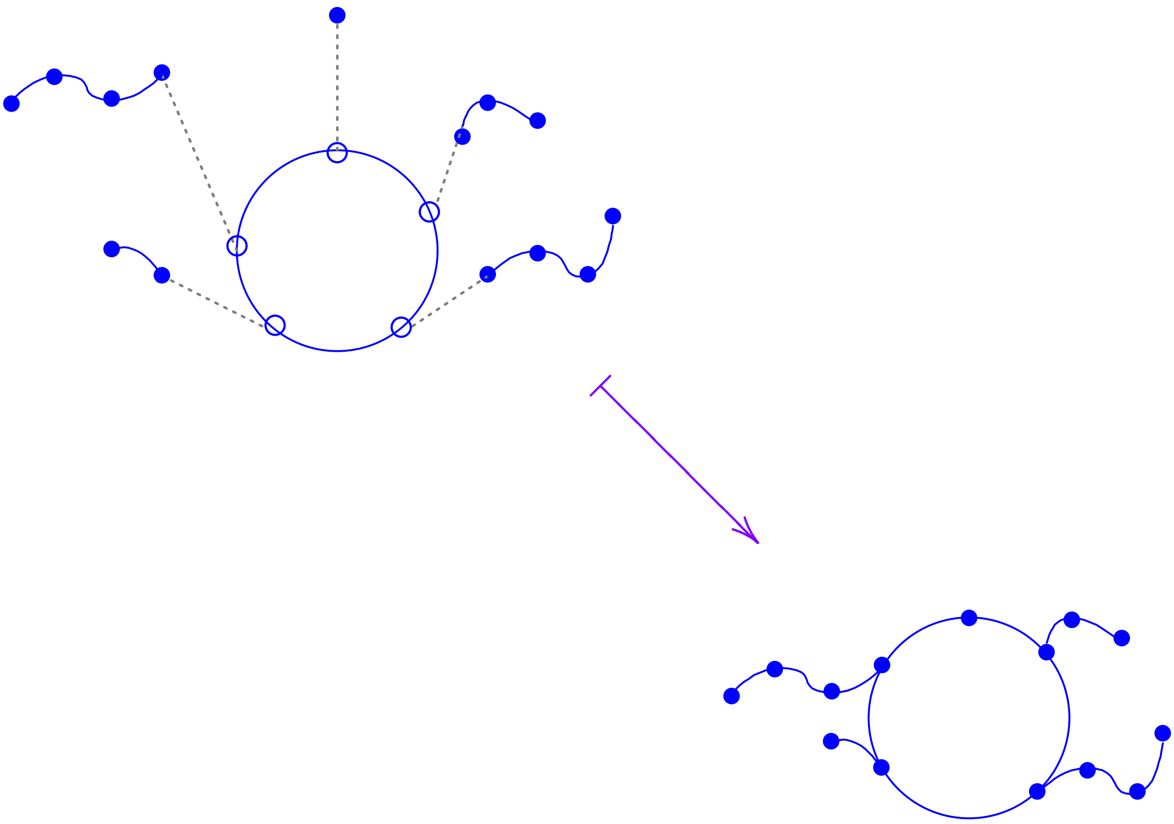
$$\downarrow$$

$$w_0 W_1 w_1 \ldots W_n w_n$$

- Proof theory.



- Structural combinatorics.



Substitution

Aspects

- syntactic *vs.* semantic models
- homogeneous *vs.* heterogeneous
- typed *vs.* untyped
- variables *vs.* occurrences
- single *vs.* simultaneous
- binding
- higher order
- algorithms

Substitution

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- variables *vs.* occurrences
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Plan

ANALYSE substitution from a foundational standpoint in a variety of scenarios and SYNTHESISE a mathematical theory.

Algebraic theories

Clone of operations $\{ C_n \times (C_m)^n \to C_m \mid \cdots \}$

$\equiv$

Lawvere theories

$\equiv$

Finitary monads

$\equiv$

Monoids for the substitution tensor product

Algebraic theories

Clone of operations $\{ C_n \times (C_m)^n \to C_m \mid \cdots \}$

$\equiv$

Lawvere theories

$\equiv$

Finitary monads

$\equiv$

Monoids for the substitution tensor product

Substitution tensor product on $\mathbf{Set}^{\mathsf{F}}$

finite sets and functions

$\uparrow$

$$\mathsf{Endo}_{\mathsf{fin}}(\mathbf{Set}) \simeq \mathbf{Set}^{\mathsf{F}}$$

$$\mathsf{Id}, \circ \leftrightarrow \mathsf{V}, \bullet$$

$$\begin{cases} \mathsf{V}(n) = n \\ (\mathsf{X} \bullet \mathsf{Y})(n) = \int^{k \in \mathsf{F}} \mathsf{X}(k) \times (\mathsf{Y}n)^k \end{cases}$$

Cartesian mono-sorted substitution

monoid structure for the substitution tensor product on $\mathrm{Set}^{\mathbb{F}}$

Examples:

- Finitary algebraic syntax.

  Σ = signature of operators with arities in $\mathbb{N}$

  $\Sigma^\star$ = free monad on $\Sigma(X) = \coprod_{o \in \Sigma} X^{|o|}$

  SUBSTITUTION STRUCTURE:

  - $n \to \Sigma^\star(n)$
  - $\Sigma^\star(n) \times (\Sigma^\star m)^n \to \Sigma^\star(m)$

  NB: Arises from the universal property of $\Sigma^\star$ by structural recursion ($\rightsquigarrow$ correct substitution algorithm).

see *e.g.* [31]

- Lambda-calculus syntax.

$\Lambda(n) = \{\,\lambda\text{-terms with free variables in } n\,\}$

with functorial action given by (capture-avoiding) variable renaming

$$\left\{\quad \frac{x \in n}{x \in \Lambda(n)} \qquad \frac{t_1, t_2 \in \Lambda(n)}{t_1(t_2) \in \Lambda(n)} \qquad \frac{t \in \Lambda(n \uplus \{x\})}{\lambda x.\, t \in \Lambda(n)}\,(\dagger) \right.$$

$(\dagger)$ SUBTLETY: α-equivalence

- Lambda-calculus syntax.

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$(\dagger)$ SUBTLETY: α-equivalence

SUBSTITUTION STRUCTURE:

- $n \to \Lambda(n)$
- $\Lambda(n) \times (\Lambda m)^n \quad \to \quad \Lambda(m)$

  $t, (i \mapsto t_i)_{i \in n} \quad \mapsto \quad t\,[\,{}^{t_i}/_{i}\,]_{i \in n}$

  ↳ (capture-avoiding) simultaneous substitution

- Clone of maps.

  The clone of maps $\langle C, C \rangle$ on an object C in a cartesian category is given by

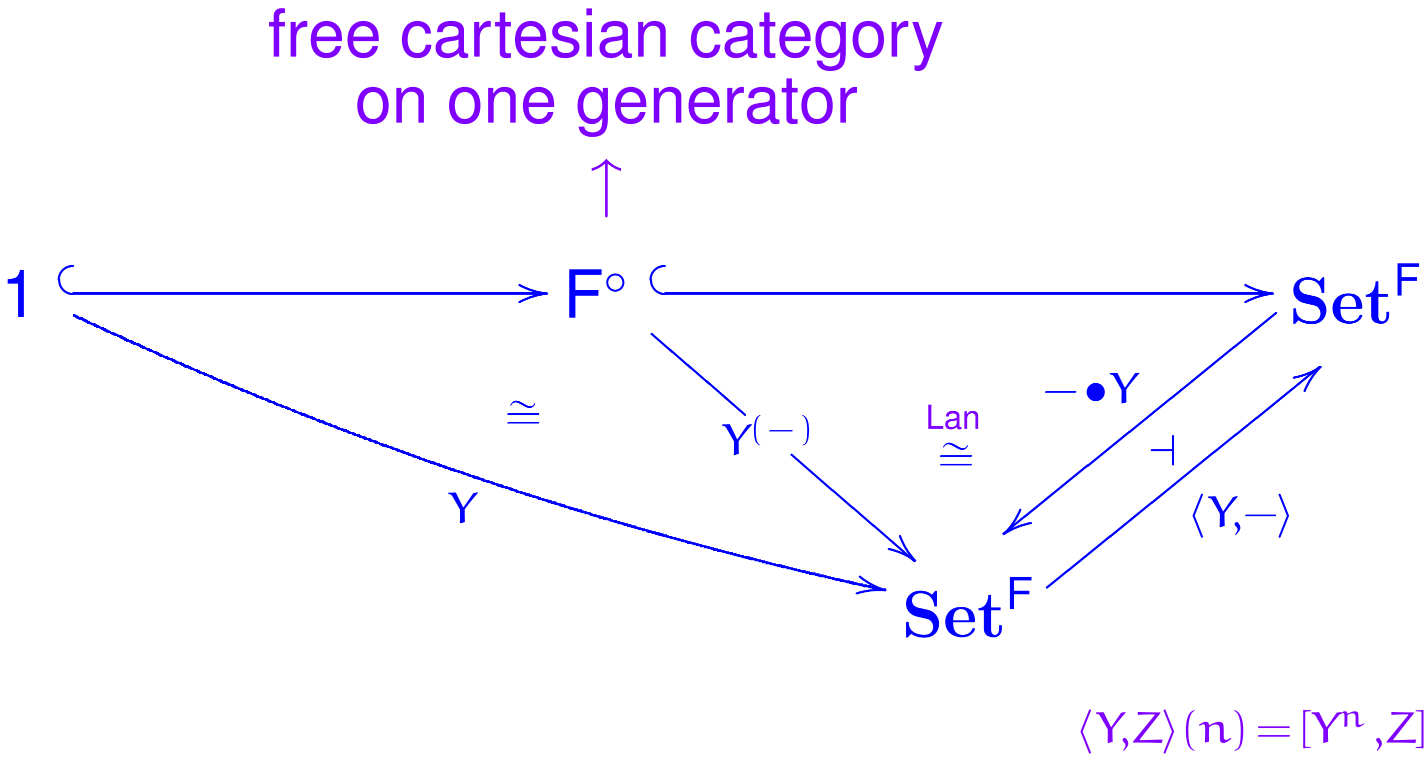
$$\langle C, C \rangle(n) = [C^n, C]$$

<u>SUBSTITUTION STRUCTURE</u>:

- $n \longrightarrow [C^n, C] : i \mapsto \pi_i$
- $[C^n, C] \times [C^m, C]^n \longrightarrow [C^m, C]$

$$[C^n, C] \times [C^m, C]^n \xrightarrow{\cong} [C^n, C] \times [C^m, C^n] \xrightarrow{\circ} [C^m, C]$$

The substitution tensor product ...



... is closed

Algebraic theories in $\mathrm{Set}^{\mathsf{F}}$

syntax with variable binding

Example: $\Sigma_\lambda = \{\,\mathsf{app} : 2, \mathsf{abs} : \mathsf{V}\,\}$

NB: $\mathsf{V} = \mathsf{y}(1)$

Algebraic theories in $\mathbf{Set}^{\mathsf{F}}$

syntax with variable binding

Example: $\Sigma_\lambda = \{\,\mathsf{app} : 2, \mathsf{abs} : \mathsf{V}\,\}$ **NB:** $\mathsf{V} = \mathsf{y}(1)$

Then,

$$\mathbf{Set}^{\mathsf{F}} \circlearrowleft \Sigma_\lambda(\mathsf{X}) = \mathsf{X}^2 + \mathsf{X}^{\mathsf{V}}$$

and

$$(\Sigma_\lambda)^\star \mathsf{V} \;=\; \mu \mathsf{X}.\, \mathsf{V} + \mathsf{X}^2 + \mathsf{X}^{\mathsf{V}} \;\cong\; \Lambda$$

see [16, 31]

Algebraic theories in $\mathbf{Set}^{\mathsf{F}}$

syntax with variable binding

Example: $\Sigma_\lambda = \{\, \mathsf{app} : 2, \mathsf{abs} : \mathsf{V} \,\}$ **NB:** $\mathsf{V} = \mathsf{y}(1)$

Then,

$$\mathbf{Set}^{\mathsf{F}} \circlearrowleft \; \Sigma_\lambda(X) = X^2 + X^{\mathsf{V}}$$

and

$$(\Sigma_\lambda)^\star \mathsf{V} \;=\; \mu X.\, \mathsf{V} + X^2 + X^{\mathsf{V}} \;\cong\; \Lambda$$

see [16, 31]

NB:

$$X^2 \to X \;\equiv\; \{\, (Xn)^2 \to Xn \mid \cdots \,\}$$

$$X^{\mathsf{V}} \to X \;\equiv\; \{\, X(n+1) \to Xn \mid \cdots \,\}$$

α-equivalence

$$\begin{array}{ccc} \mathsf{F}^\circ & \hookrightarrow & \mathbf{Set}^{\mathsf{F}} \\ {\scriptstyle (-)+1}\downarrow & & {\scriptstyle (-)\times\mathsf{V}}\downarrow \dashv \uparrow{\scriptstyle (-)^{\mathsf{V}} = ((-)+1)^*} \\ \mathsf{F}^\circ & \hookrightarrow & \mathbf{Set}^{\mathsf{F}} \end{array}$$

Λ is (universally characterised as) the free Σ_λ-algebra on V , and its substitution structure is derived by parameterised structural recursion as follows:

$$
\begin{array}{ccccc}
\Sigma_\lambda(\Lambda) \bullet \Lambda & \longrightarrow & \Sigma_\lambda(\Lambda \bullet \Lambda) & \xrightarrow{\Sigma_\lambda(s)} & \Sigma_\lambda(\Lambda) \\
\downarrow & & & & \downarrow \\
\Lambda \bullet \Lambda & & \xrightarrow{\quad s \quad} & & \Lambda \\
\uparrow & & \cong & \nearrow & \\
V \bullet \Lambda & & & &
\end{array}
$$

$\rightsquigarrow$ correct (capture-avoiding) simultaneous substitution algorithm

see [16, 31]

Λ is (universally characterised as) the free Σ_λ-algebra on $V^{(\dagger)}$, and its substitution structure is derived by parameterised structural recursion as follows:

$$
\begin{array}{ccccc}
\Sigma_\lambda(\Lambda) \bullet \Lambda & \xrightarrow{(\ddagger)} & \Sigma_\lambda(\Lambda \bullet \Lambda) & \xrightarrow{\Sigma_\lambda(s)} & \Sigma_\lambda(\Lambda) \\
\downarrow & & & & \downarrow \\
\Lambda \bullet \Lambda & & \xrightarrow{\quad s \quad} & & \Lambda \\
\uparrow & & & \nearrow_{\cong} & \\
V \bullet \Lambda & & & &
\end{array}
$$

$\leadsto$ correct (capture-avoiding) simultaneous substitution algorithm

see [16, 31]

$^{(\dagger)}$ yields an induction principle — see [19, 31]

$^{(\ddagger)}$ SUBTLETY: *pointed* strength

→ capture avoidance

General theory

SETTING:

$$\mathrm{I}/\mathscr{C} \xrightarrow{\ \mathrm{U}\ } \Sigma \circlearrowright \mathscr{C}$$

a monoidal closed category

an endofunctor with a U-strength:

$$\Sigma(X) \otimes Y \xrightarrow{\sigma_{X,(I \to Y)}} \Sigma(X \otimes Y)$$

General theory

SETTING:

$$\begin{array}{c} I/\mathscr{C} \\ \downarrow U \\ \Sigma \circlearrowright \mathscr{C} \end{array}$$

a monoidal closed category

an endofunctor with a U-strength:

$$\Sigma(X) \otimes Y \xrightarrow{\sigma_{X,(I \to Y)}} \Sigma(X \otimes Y)$$

MODELS: (Σ, σ)-monoids.

$$\begin{array}{ccccc} & & \Sigma X & & \\ & & \downarrow \xi & & \\ X \otimes X & \xrightarrow{m} & X & \xleftarrow{e} & I \end{array}$$

an algebra

a monoid

such that

$$\begin{array}{ccccc} \Sigma(X) \otimes X & \xrightarrow{\sigma_{X,e}} & \Sigma(X \otimes X) & \xrightarrow{\Sigma m} & \Sigma X \\ \downarrow \xi \otimes X & & & & \downarrow \xi \\ X \otimes X & & \xrightarrow{m} & & X \end{array}$$

The

free Σ-algebra and free monoid

constructions

$$\Sigma\text{-alg}(\mathscr{C}) \underset{\longleftarrow}{\overset{\longrightarrow}{\top}} \mathscr{C} \underset{\longrightarrow}{\overset{\longleftarrow}{\top}} \mathrm{Mon}(\mathscr{C})$$

$$\mu X.\, C + \Sigma X \longleftarrow\!\!\shortmid\; C \;\longmapsto \mu X.\, I + C \otimes X$$

unify to

$$(\Sigma, \sigma)\text{-Mon}(\mathscr{C}) \underset{\longrightarrow}{\overset{\longleftarrow}{\dashv}} \mathscr{C} \circlearrowleft \mathcal{S}$$

where

$$\mathcal{S}(C) = \mu X.\, I + C \otimes X + \Sigma X$$

NB: The initial (Σ, σ)-monoid has underlying object $\mathcal{S}0 = \mu X.\, I + \Sigma X = \Sigma^{\star} I$.

see [26, 30]

Initial-algebra semantics with substitution

The unique (Σ, σ)-monoid homomorphism from the initial (Σ, σ)-monoid provides an initial-algebra semantics that is both compositional and respects substitution.

see [16, 24]

Example: Lambda calculus.

For $\mathrm{D} \lhd \mathrm{D}^{\mathrm{D}}$ in a cartesian closed category, the clone of maps $\langle \mathrm{D}, \mathrm{D} \rangle$ has a canonical Σ_λ-algebra structure

$$\begin{array}{ccc} \langle \mathrm{D}, \mathrm{D} \rangle \times \langle \mathrm{D}, \mathrm{D} \rangle & \longrightarrow & \langle \mathrm{D}, \mathrm{D} \rangle \\ \downarrow & & \uparrow \\ \langle \mathrm{D}, \mathrm{D} \rangle \times \langle \mathrm{D}, \mathrm{D}^{\mathrm{D}} \rangle & \xrightarrow{\cong} & \langle \mathrm{D}, \mathrm{D} \times \mathrm{D}^{\mathrm{D}} \rangle \end{array}$$

$$\begin{array}{ccc} \langle \mathrm{D}, \mathrm{D} \rangle^{\mathsf{V}} & \longrightarrow & \langle \mathrm{D}, \mathrm{D} \rangle \\ & \searrow^{\cong} \quad \nearrow & \\ & \langle \mathrm{D}, \mathrm{D}^{\mathrm{D}} \rangle & \end{array}$$

making it into a Σ_λ-monoid for the canonical pointed strength.

The induced initial-algebra semantics amounts to the standard interpretation of the λ-calculus.

see [16, 19]

Single-variable and simultaneous substitution

The theory of monoids in $\mathbf{Set}^{\mathsf{F}}$ for the substitution tensor product is enriched algebraic for the cartesian closed structure.

$\mathsf{Mon}_{\mathsf{V},\bullet}(\mathbf{Set}^{\mathsf{F}})$ is (equivalent to) the category of algebras X with operations

$$\mathsf{X}^{\mathsf{V}+1} \to \mathsf{X} \quad \text{and} \quad 1 \to \mathsf{X}^{\mathsf{V}}$$

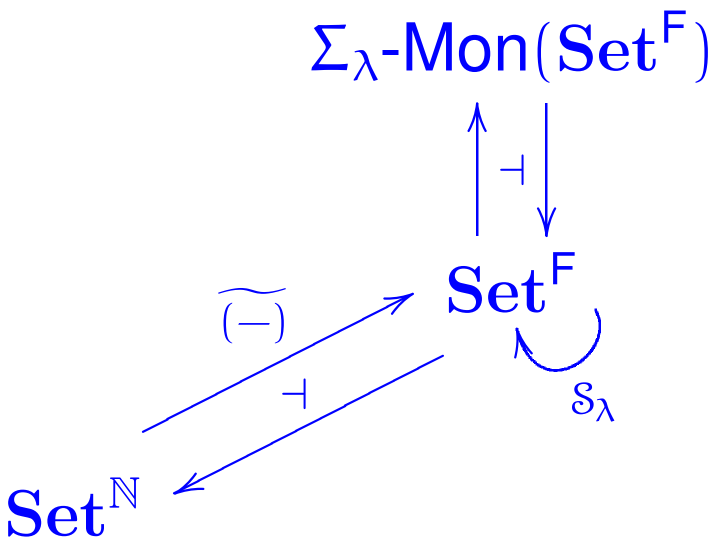
subject to

$$\ldots \quad \text{4 axioms} \quad \ldots$$

see [16]

Second-order syntax with variable binding and substitution

Example:



For $\mathrm{M} \in \mathbf{Set}^{\mathbb{N}}$,

$$\mathcal{S}_\lambda(\widetilde{\mathrm{M}}) \;\cong\; \mathsf{V} + \widetilde{\mathrm{M}} \bullet \mathcal{S}_\lambda(\widetilde{\mathrm{M}}) + \Sigma_\lambda(\mathcal{S}_\lambda \widetilde{\mathrm{M}})$$

see [22]

$\mathcal{S}_\lambda(\widetilde{M})$ can be syntactically presented as follows:

$$\frac{x \in n}{\mathsf{var}(x) \in \mathcal{S}_\lambda(\widetilde{M})(n)}$$

$$\frac{t_1, t_2 \in \mathcal{S}_\lambda(\widetilde{M})(n)}{\mathsf{app}(t_1, t_2) \in \mathcal{S}_\lambda(\widetilde{M})(n)}$$

$$\frac{t \in \mathcal{S}_\lambda(\widetilde{M})\big(n \uplus \{x\}\big)}{\mathsf{abs}\big((x)t\big) \in \mathcal{S}_\lambda(\widetilde{M})(n)} \quad \text{(up to } \alpha\text{-equivalence)}$$

$$\frac{t_1, \ldots, t_k \in \mathcal{S}_\lambda(\widetilde{M})(n)}{T[t_1, \ldots, t_k] \in \mathcal{S}_\lambda(\widetilde{M})(n)} \quad \big(T \in M(k)\big)$$

and the monoid multiplication structure

$$\mathcal{S}_\lambda(\widetilde{M}) \bullet \mathcal{S}_\lambda(\widetilde{M}) \to \mathcal{S}_\lambda(\widetilde{M})$$

amounts to (capture-avoiding) simultaneous substitution.

see [31]

MOREOVER, the action of $\mathcal{S}_\lambda$ enriches over $\mathbf{Set}^{\mathsf{F}}$ with respect to its cartesian closed structure, and we obtain a further substitution structure

$$\mathcal{S}_\lambda(\widetilde{M}) \times (\mathcal{S}_\lambda \widetilde{N})^{\widetilde{M}} \longrightarrow \mathcal{S}_\lambda(\widetilde{N})$$

that amounts to SECOND-ORDER SUBSTITUTION for metavariables.

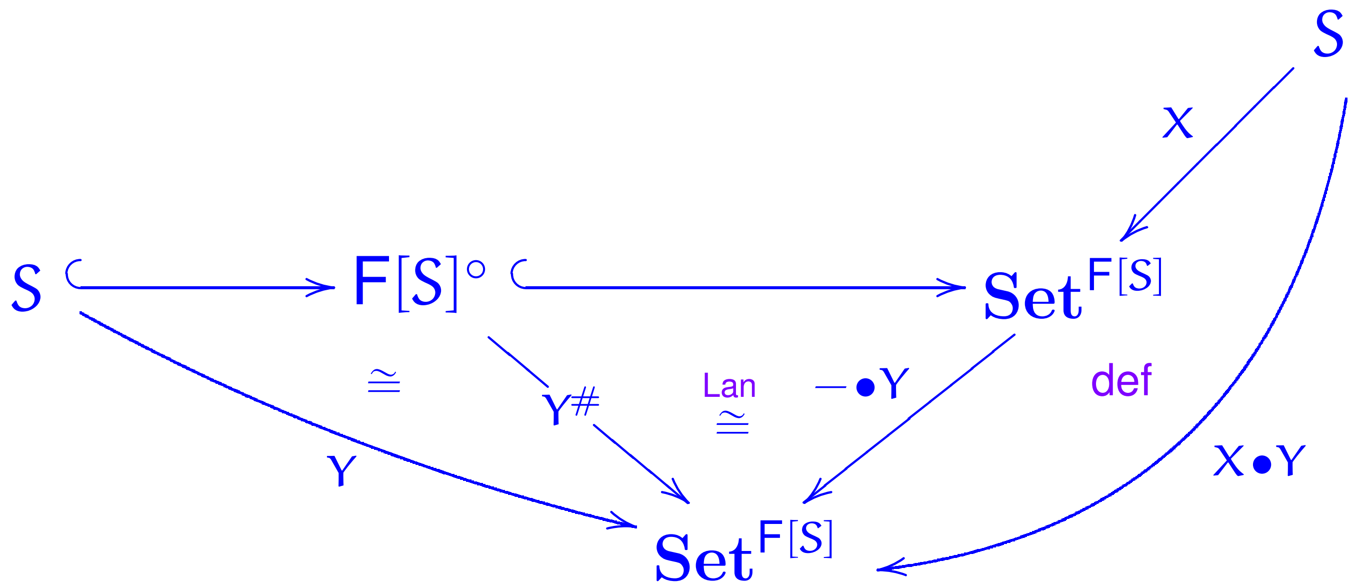
As usual, this arises by universal properties and is given by parameterised structural recursion; yielding a correct substitution algorithm.

see [31]

Many-sorted contexts

For S a set of sorts, let $\mathsf{F}[S]$ be the free cocartesian category on S.

The substitution tensor product on $\left(\mathbf{Set}^{\mathsf{F}[S]}\right)^S$ is given as follows:



That is,

$$(X \bullet Y)_\sigma(\Gamma) \;=\; \int^{\Delta \in \mathsf{F}[S]} X_\sigma(\Delta) \times \prod_{\tau \in \Delta} Y_\tau(\Gamma)$$

see [19, 23, 24, 26]

Simply typed lambda calculus, algebraically

Let T be a set of base types, and let $\overline{\mathsf{T}}$ be its closure under 1, $*$, $\Rightarrow$.

Consider

$$\Sigma \circlearrowright \left(\mathbf{Set}^{\mathsf{F}[\overline{\mathsf{T}}]}\right)^{\overline{\mathsf{T}}}$$

induced by

$$
\begin{array}{ll}
(\dagger) & \mathsf{app}^{(\sigma,\tau)} : (\sigma{\Rightarrow}\tau, \sigma) \longrightarrow \tau \\
(\ddagger) & \mathsf{abs}^{(\sigma,\tau)} : ((\sigma)\tau) \longrightarrow \sigma{\Rightarrow}\tau \\
 & \mathsf{proj1}^{(\sigma,\tau)} : (\sigma,\tau) \longrightarrow \sigma \,, \quad \mathsf{proj2}^{(\sigma,\tau)} : (\sigma,\tau) \longrightarrow \tau \\
 & \mathsf{pair}^{(\sigma,\tau)} : (\sigma,\tau) \longrightarrow \sigma{*}\tau \\
 & \mathsf{ter} : () \longrightarrow 1
\end{array}
$$

$(\dagger)\ X_{\sigma\Rightarrow\tau} \times X_{\sigma} \longrightarrow X_{\tau}$

$(\ddagger)\ {X_{\tau}}^{V_{\sigma}} \longrightarrow X_{\sigma\Rightarrow\tau}$

see [19, 23]

Furthermore, let CC be the following equational theory for Σ-monoids:

$$(\beta) \quad F : [\sigma]\tau, T : []\sigma$$

$$\vdash \quad \mathsf{app}\big(\mathsf{abs}\big((x:\sigma)F[\mathsf{var}(x)]\big), T[]\big)$$

$$=$$

$$F\,[T[]] \;:\; \tau$$

$$(\eta) \quad F : [](\sigma{\Rightarrow}\tau)$$

$$\vdash \quad \mathsf{abs}\big((x:\sigma)\mathsf{app}\big(F[], \mathsf{var}(x)\big)\big)$$

$$=$$

$$F[] \;:\; \sigma{\Rightarrow}\tau$$

Furthermore, let CC be the following equational theory for Σ-monoids:

$$(\beta) \quad F : [\sigma]\tau, T : [\,]\sigma \vdash \mathsf{app}\big(\mathsf{abs}\big((x:\sigma)F[\mathsf{var}(x)]\big), T[\,]\big) = F\,[T[\,]] \ : \tau$$

<u>informally</u>:

$$(\lambda x.\, F)T = F\,[{}^{T}/_{x}]$$

$$(\eta) \quad F : [\,](\sigma{\Rightarrow}\tau) \vdash \mathsf{abs}\big((x:\sigma)\mathsf{app}\big(F[\,], \mathsf{var}(x)\big)\big) = F[\,] \ : \sigma{\Rightarrow}\tau$$

$$\lambda x.\, Fx = F$$

$$\big(x \notin \mathsf{FV}(F)\big)$$

$$(\mathsf{proj}) \quad M : [\,]\sigma, N : [\,]\tau$$

$$\vdash \ \mathsf{proj1}(M[\,], N[\,]) = M[\,] \ : \sigma$$

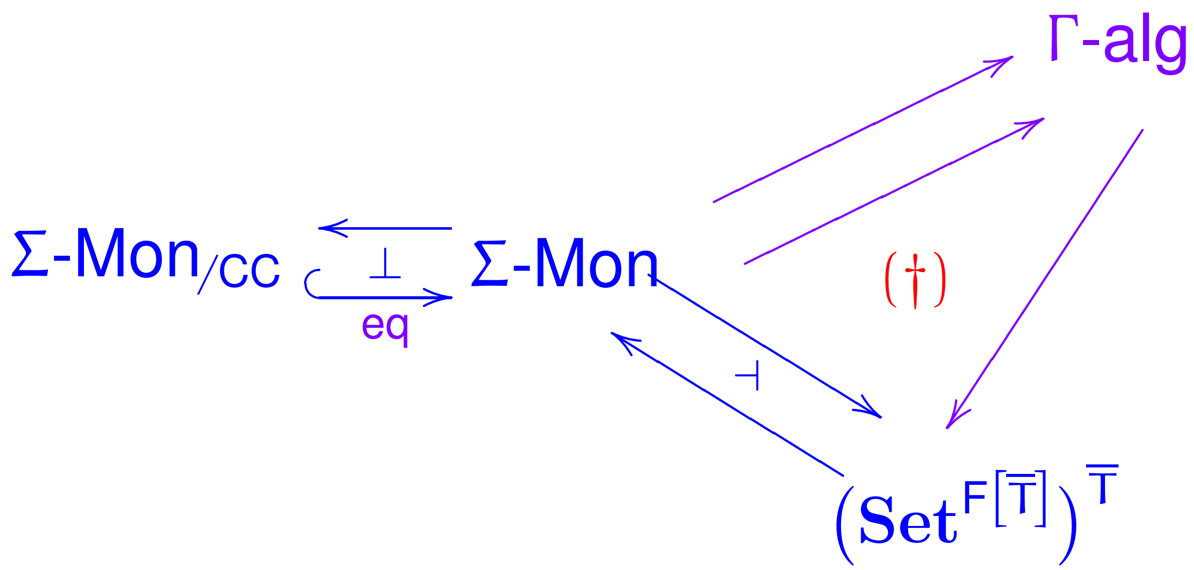
$$\vdash \ \mathsf{proj2}(M[\,], N[\,]) = N[\,] \ : \tau$$

$$(\mathsf{pair}) \quad T : [\,](\sigma * \tau)$$

$$\vdash \quad \mathsf{pair}\big(\mathsf{proj1}(T[\,]), \mathsf{proj2}(T[\,])\big) = T[\,] \ : \sigma * \tau$$

$$(\mathsf{ter}) \quad T : [\,]1 \ \vdash \ T[\,] = \mathsf{ter} : \ 1$$

Then



and the Lawvere theory associated to the initial $\Sigma\text{-Mon}_{/\mathsf{CC}}$ is the free cartesian closed category on T.

(†) induced by CC

[**NB:** This generalises to free cartesian closed categories on small categories.]

$^{(\dagger)}$**Example:** The parallel pair induced by (β).

For $\mathbb{N}[X] = \coprod_{n\in\mathbb{N}} X^n$, let $M \in \left(\mathbf{Set}^{\mathbb{N}[\overline{T}]}\right)^{\overline{T}}$ be defined from the context of (β) as

$$M_\tau(\sigma) = \{\, F \,\} \quad , \qquad M_\sigma() = \{\, T \,\}$$

and empty otherwise.

The terms of (β) correspond to global elements

$$1 \longrightarrow \mathcal{S}(\widetilde{M})$$

that induce functors

$$\Sigma\text{-Mon} \simeq \mathcal{S}\text{-alg} \longrightarrow (-)^{\widetilde{M}}\text{-alg}$$

over $\left(\mathbf{Set}^{F[\overline{T}]}\right)^{\overline{T}}$ as follows

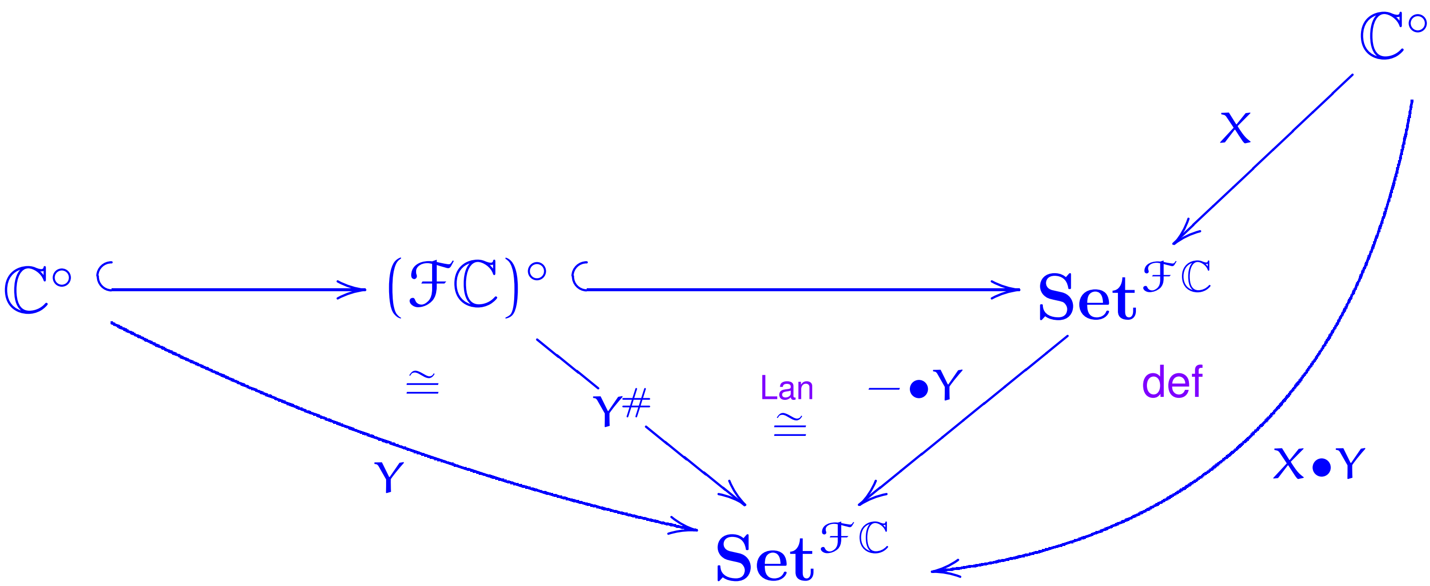
$$\begin{array}{ccc} & & X^{\widetilde{M}} \\ & & \downarrow \\ \mathcal{S}(X) & & \mathcal{S}(X)^{\mathcal{S}(\widetilde{M})} \\ \downarrow & \mapsto & \downarrow \\ X & & \mathcal{S}(X)^1 \\ & & \downarrow \\ & & X \end{array}$$

see [30]

Dependent sorts

For a small category $\mathbb{C}$, let $\mathfrak{F}\mathbb{C} \simeq (\mathbf{Set}^{\mathbb{C}^\circ})_{\mathsf{fp}}$ be the free finite colimit completion of $\mathbb{C}$.

The substitution tensor product on $(\mathbf{Set}^{\mathfrak{F}\mathbb{C}})^{\mathbb{C}^\circ}$ is given as follows:



That is,

$$(\mathsf{X} \bullet \mathsf{Y})_{\mathsf{C}}(\Gamma) = \int^{\Delta \in \mathfrak{F}\mathbb{C}} \mathsf{X}_{\mathsf{C}}(\Delta) \times \lim_{\mathsf{D} \in \mathsf{El}(\Delta)} \mathsf{Y}_{\mathsf{p}_\Delta \mathsf{D}}(\Gamma)$$

PROGRAMME

The various developments of the previous slides carry over to this more general setting.

Following Makkai [13], after Lawvere [9] and Otto [11], the syntactic theory is considered for *simple* categories (= skeletal and one-way, with finite fan-out).

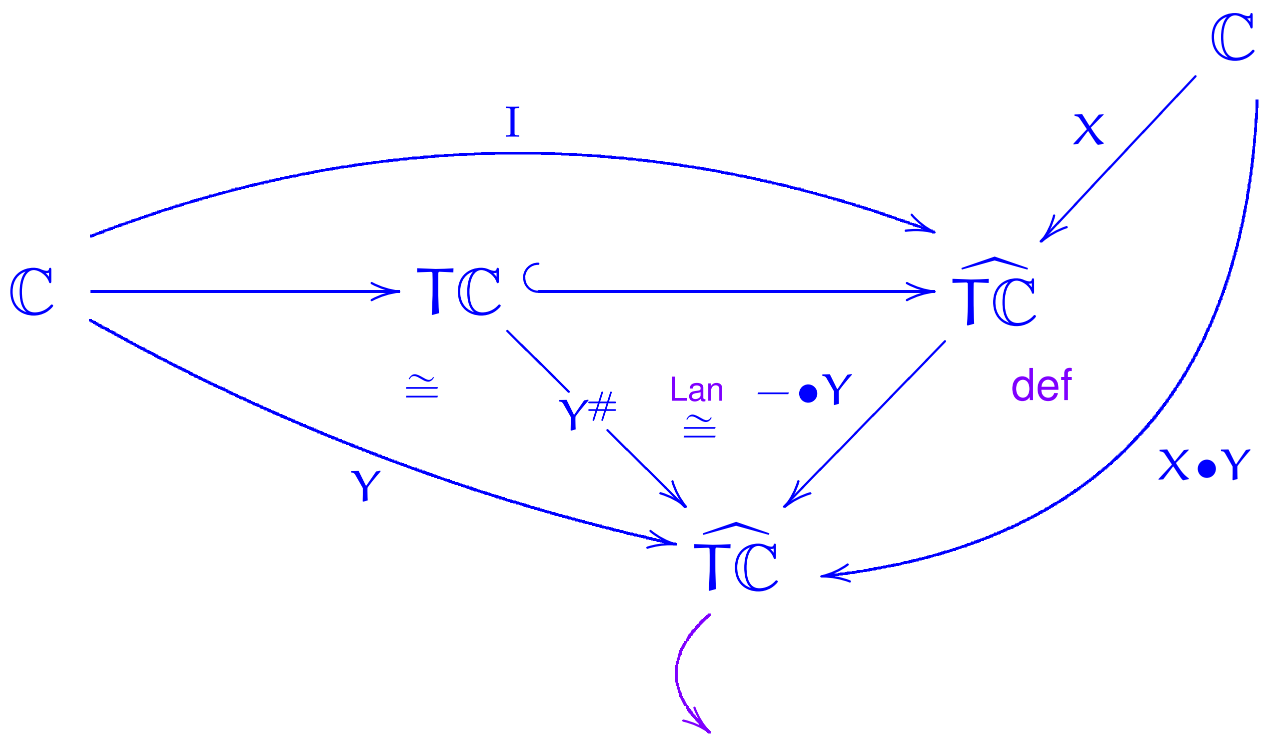
This amounts to extending the theory to incorporate DEPENDENT SORTS.

NB: The limit in the substitution tensor product accounts for the heavy dependency required in the substitution operation.

see [26]

General theory : Idea

For T a 2-monad on **CAT**, consider



equipped with a T-algebra structure

Examples:

- T = identity
- T = free cartesian completion
- T = free finite limit completion

- T = free monoidal completion
- T = free symmetric monoidal completion

- T = free monoidal completion
- T = free symmetric monoidal completion

the T-algebra structure on $\widehat{\mathsf{T}\mathbb{C}}$ is given by Day's tensor product [2, 7]

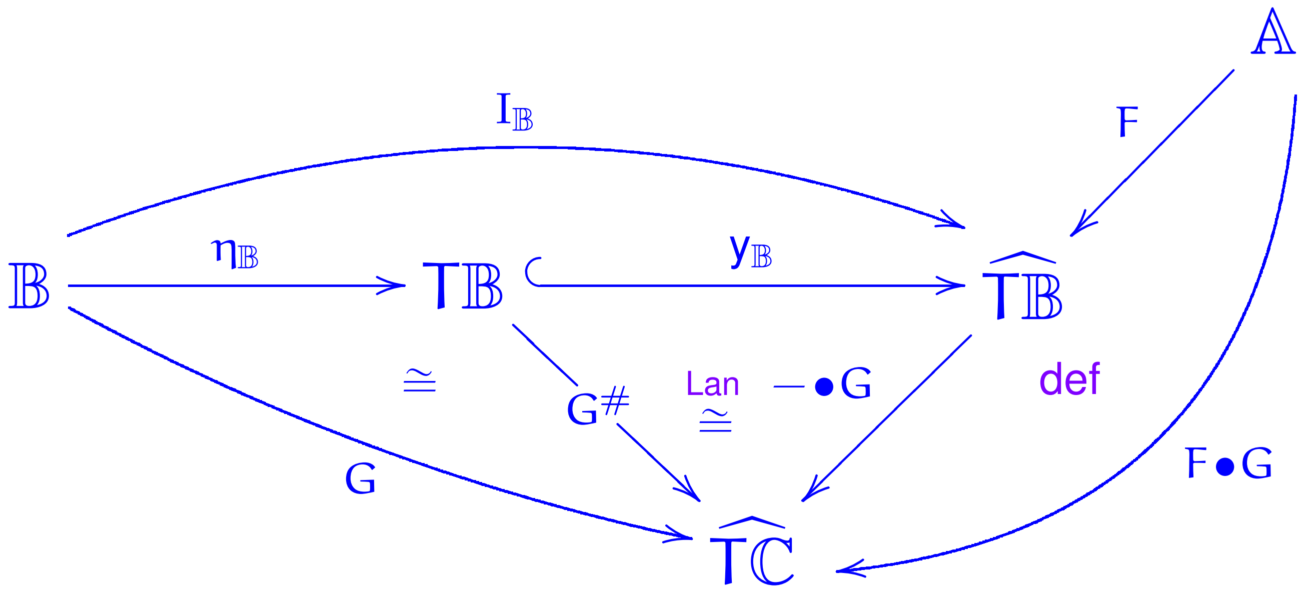
$(\mathrm{I}, \bullet)$-monoids = planar / symmetric operads

see *e.g.* [3, 15, 20]

QUESTION: Are there applications of the previous theory to the theory of operads!?

More generally : From substitution to composition

For (T, η, μ) a 2-monad on $\mathbf{CAT}$, consider



with respect to T-algebra structures

$$\tau_{\mathbb{C}} : T\left(\widehat{T\mathbb{C}}\right) \to \widehat{T\mathbb{C}}$$

such that

$$y_{\mathbb{C}} : (T\mathbb{C}, \mu_{\mathbb{C}}) \to (\widehat{T\mathbb{C}}, \tau_{\mathbb{C}})$$

and

$$\mathrm{Lan}_{y_{\mathbb{X}}}(h) : (\widehat{T\mathbb{C}}, \tau_{\mathbb{C}}) \to (\widehat{T\mathbb{D}}, \tau_{\mathbb{D}})$$

$$\text{for all } h : (T\mathbb{C}, \mu_{\mathbb{C}}) \to (\widehat{T\mathbb{D}}, \tau_{\mathbb{D}})$$

NB: The above can be axiomatised further.

[Hyland, Gambino, Fiore]

(see also [24])

Kleisli bicategory

$$\frac{\mathrm{T}\mathbb{C} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{B} \qquad \mathrm{T}\mathbb{B} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{A}}{\mathrm{T}\mathbb{C} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{A}}$$

- T = identity
  $\leadsto$ profunctors
- T = free symmetric monoidal completion
  $\leadsto$ JOYAL SPECIES OF STRUCTURES[6, 8] arise as the endomorphisms of 1
  $\leadsto$ GENERALISED SPECIES OF STRUCTURES
  see [23, 25]

Coherence (idea):

$$\begin{aligned} &\mathrm{Lan}_y\big((\mathrm{Lan}_y\, H^{\#})\, G\big)^{\#} \\ &\quad\cong \mathrm{Lan}_y\big((\mathrm{Lan}_y\, H^{\#})\, G^{\#}\big) \\ &\quad\cong (\mathrm{Lan}_y\, H^{\#})\,(\mathrm{Lan}_y\, G^{\#}) \end{aligned}$$

Kleisli bicategory

$$\frac{\mathsf{T}\mathbb{C} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{B} \qquad \mathsf{T}\mathbb{B} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{A}}{\mathsf{T}\mathbb{C} \longrightarrow\!\!\!\!\!\!\!+\;\; \mathbb{A}}$$

- T = identity
  $\rightsquigarrow$ profunctors

- T = free symmetric monoidal completion
  $\rightsquigarrow$ JOYAL SPECIES OF STRUCTURES[6, 8] arise as the endomorphisms of 1
  $\rightsquigarrow$ GENERALISED SPECIES OF STRUCTURES(†)
  see [23, 25]

Coherence (idea):

$$\begin{aligned}
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&\quad \cong \ \mathrm{Lan}_{y}\left(\left(\mathrm{Lan}_{y}\,H^{\#}\right)G^{\#}\right) \\
&\quad \cong \ \left(\mathrm{Lan}_{y}\,H^{\#}\right)\left(\mathrm{Lan}_{y}\,G^{\#}\right)
\end{aligned}$$

(†) The coherence isomorphisms can also be given *formally* (compare [3]) in a theory of Lawvere's generalized logic [4], providing a *logical* view of coherence. The coherence laws can be then established elementwise.

PROGRAMME: Extend generalized logic to a type theory within which the coherence laws may be also established formally.

Linear models

Substitution operations on linear species[6]

finite linear orders and monotone bijections
↑

For $X, Y \in \mathbf{Set}^{\mathsf{L}}$ with $Y(\emptyset) = \emptyset$:

1. $$(X \bullet Y)(L) = \sum_{P \in \textbf{LinPart}(L)} X(P) \times \prod_{\ell \in P} Y(\ell)$$

   ⤳ composition of ordinary generating series [Joyal]

   see [6]

2. $$(X \bullet Y)(L) = \sum_{P \in \textbf{Part}(L)} X(P) \times \prod_{\ell \in P} Y(\ell)$$

   ⤳ composition of exponential generating series [Foata]

   see [1, 14]

Substitution tensor products on linear species

For $X, Y \in \mathbf{Set}^{\mathsf{L}}$:

1. $(X \bullet Y)(\ell)$

$$= \int^{P \in \mathsf{L}} X(P) \times \int^{\ell_p \, (p \in P)} \prod_{p \in P} Y(\ell_p) \times \mathsf{L}(\oplus_{p \in P} \ell_p, \ell)$$

   arises from the general theory for T the free monoidal completion, noting that L is (equivalent to) the free monoidal category on one object.

Substitution tensor products on linear species

For $X, Y \in \mathbf{Set}^{\mathsf{L}}$:

1. $(X \bullet Y)(\ell)$

$$= \int^{P \in \mathsf{L}} X(P) \times \int^{\ell_p \, (p \in P)} \prod_{p \in P} Y(\ell_p) \times \mathsf{L}(\oplus_{p \in P} \ell_p, \ell)$$

arises from the general theory for T the free monoidal completion, noting that L is (equivalent to) the free monoidal category on one object.

2. $(X \bullet Y)(\ell)$

$$= \int^{P \in \mathsf{L}} X(P) \times \int^{\ell_p \, (p \in P)} \prod_{p \in P} Y(\ell_p) \times \mathsf{Mon}_{\mathsf{bij}}\Big(\bigvee_{p \in P} \ell_p, \ell \Big)$$

where $\bigvee_{p \in P} \ell_p$ has underlying set $\biguplus_{p \in P} \ell_p$ ordered by $(p, x) \leq (p', x')$ iff either $p = p'$ and $x \leq x'$, or $p < p'$ and $x = \min(\ell_p)$ and $x' = \min(\ell_{p'})$.

$\rightsquigarrow$ GENERALISED LINEAR SPECIES OF STRUCTURES

Mixed models

Example: DILL = Dual Intuitionistic Linear Logic

see [12]

$$\Gamma;\Delta \vdash t : \tau$$

intuitionistic (cartesian) context ← Γ ; Δ → linear (symmetric monoidal) context

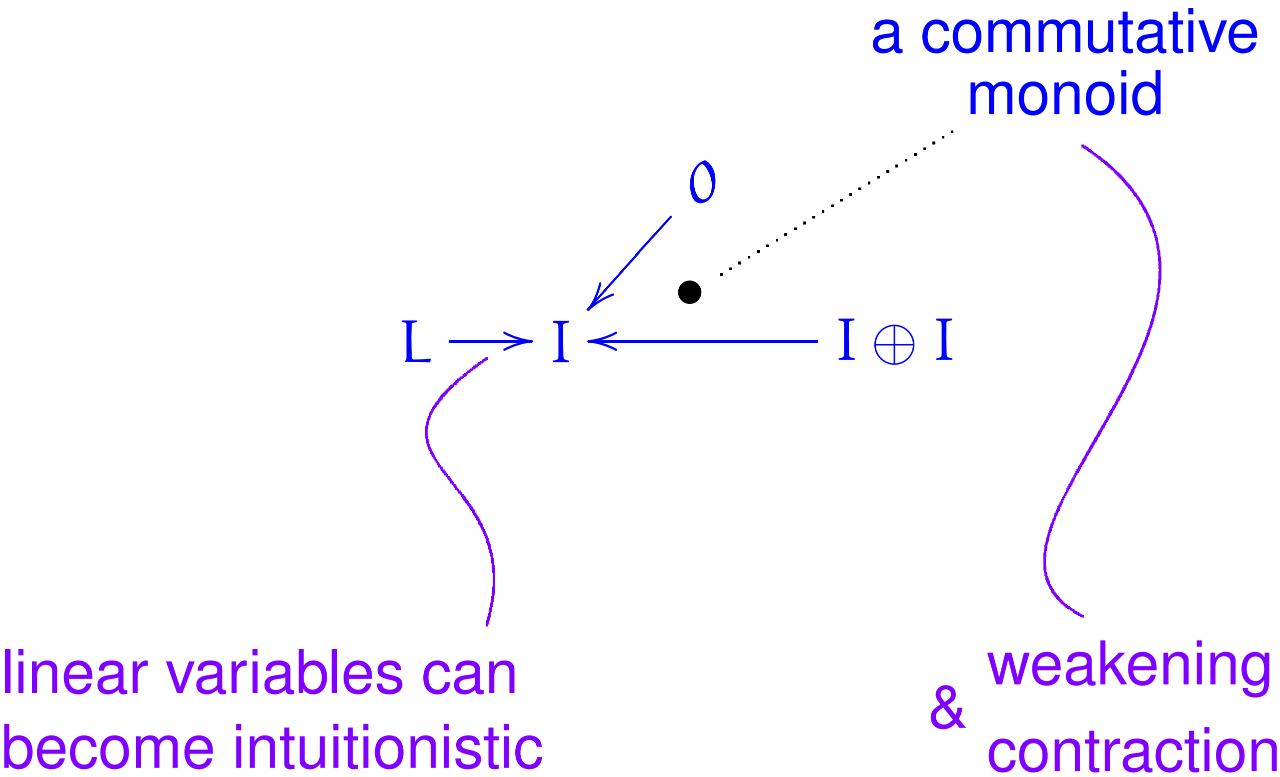
CUT RULE:

$$\frac{x_1 : \sigma_1, \ldots, x_m : \sigma_m ; y_1 : \tau_1, \ldots, y_n : \tau_n \vdash t : \alpha \qquad \Gamma; - \vdash u_i : \sigma_i \quad (1 \leq i \leq m) \qquad \Gamma; \Delta_j \vdash v_j : \tau_j \quad (1 \leq j \leq n)}{\Gamma; \Delta_1, \ldots, \Delta_n \vdash t\,[{}^{u_i}/_{x_i}, {}^{v_j}/_{y_j}]_{1\leq i\leq m, 1\leq j\leq n} : \alpha}$$

▶ NEW FEATURE absent in mathematical examples

MATHEMATICAL MODEL

The category of (mono-sorted) *mixed contexts* M is the free symmetric monoidal category over the following symmetric monoidal theory:



type theoretically:

$$\frac{\Gamma\,;\, x, \Delta \vdash t}{\Gamma, x\,;\, \Delta \vdash t}$$

CONTEXT INDEXING

$$\begin{array}{ccccc} & & 1 & & \\ & & \nearrow & & \\ \mathsf{M} & \hookleftarrow & \mathsf{F} & \longrightarrow & \mathsf{F} \times \mathsf{F} \end{array} \qquad \text{in } \mathbf{Cat}$$

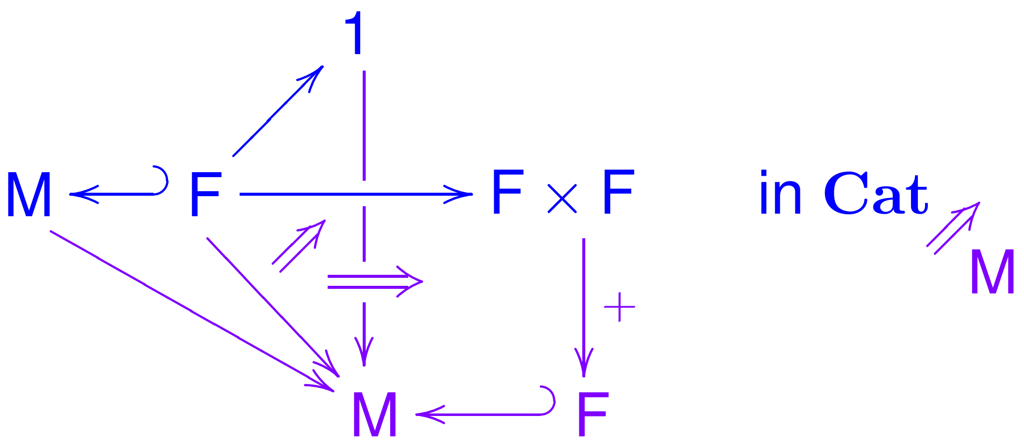
induces

$$\mathsf{M}^{\circ} \xrightarrow{\mathcal{M}} \mathbf{Cat}$$

$$\cdots \oplus \mathrm{L} \oplus \cdots \oplus \mathrm{I} \oplus \cdots \longmapsto \ldots \times \mathsf{M} \times \ldots \times \mathsf{F} \times \ldots$$

CONTEXT INDEXING

In fact



induces

$$\mathsf{M}^{\circ} \xrightarrow{\;\mathcal{M}\;} \mathbf{Cat}\;{\Nearrow}\;\mathsf{M}$$

$$\cdots \oplus \mathsf{L} \oplus \cdots \oplus \mathrm{I} \oplus \cdots \longmapsto \ldots \times \mathsf{M} \times \ldots \times \mathsf{F} \times \ldots$$

$$\downarrow$$

$$\mathsf{M}$$

Mixed substitution tensor product

For $X, Y \in \mathbf{Set}^{M}$:

$(X \bullet Y)(D)$

$$= \int^{C \in M} X(C) \times \int^{\Delta \in \mathcal{M}(C)} \prod_{i \in |C|} Y(\Delta_i) \times M\big(\oplus_C(\Delta), D\big)$$

NEW FEATURE

$$\mathcal{M}(C) \xrightarrow{\ \oplus_C\ } M$$

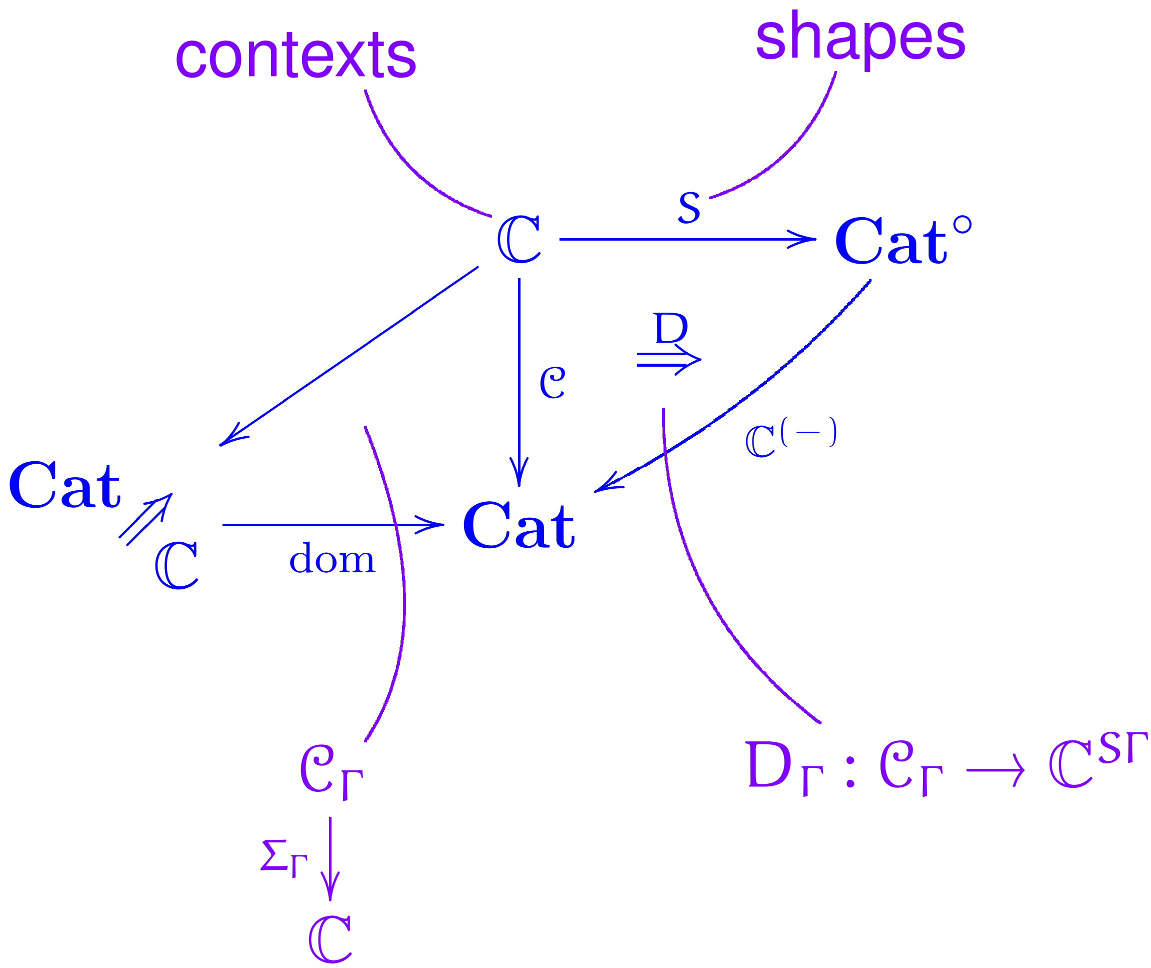
- Monoids = Mixed operads

  generalise and combine Lawvere theories and (symmetric) operads

- A combinatorial model of DILL

  . . . and more

A unifying framework



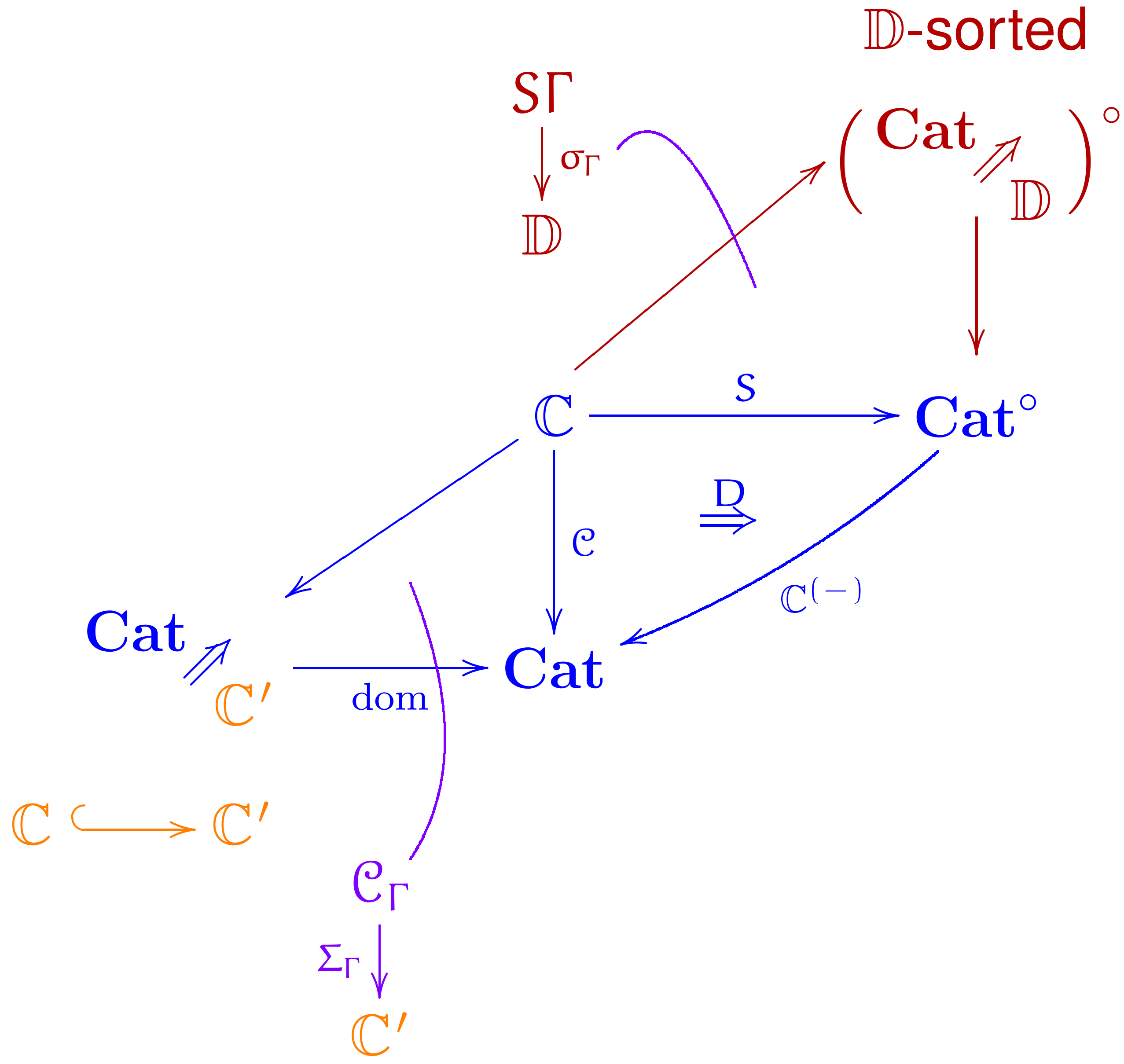
- Substitution tensor product **?**

For $X, Y \in \mathbf{Set}^{\mathbb{C}^{\circ}}$:

$$(X \bullet Y)(C)$$

$$= \int^{\Gamma \in \mathbb{C}} X(\Gamma) \times \int^{\Delta \in \mathcal{C}_\Gamma} \left(\lim_{i \in S\Delta} Y(D_\Gamma \Delta)_i \right) \times [\Sigma_\Gamma \Delta, C]$$

Further generalisations



- Substitution tensor product **?**

For $X, Y \in \left(\mathbf{Set}^{\mathbb{C}^\circ}\right)^{\mathbb{D}}$:

$$(X \bullet Y)_\tau(C)$$

$$= \int^{\Gamma \in \mathbb{C}} X_\tau(\Gamma) \times \int^{\Delta \in \mathcal{C}_\Gamma} \left(\lim_{i \in S\Delta} Y_{\sigma_\Delta(i)}(D_\Gamma \Delta)_i \right) \times [\Sigma_\Gamma \Delta, C]$$

PROGRAMME

Obtain a substitution tensor product from (cartesian (compare [5, 10])) monad structure on

$$\widetilde{\mathrm{S}}(\mathbb{X}) = S \nearrow\!\!\!\nearrow_{\mathbb{X}}$$

NB: $\widetilde{\mathrm{S}}(1) \cong \mathbb{C}$

induced by structure on $\mathbb{C}$.

Developments

- Mathematical theory of substitution
  - typed *vs.* untyped
  - homogeneous *vs.* heterogeneous see [18]
  - single variable vs. simultaneous substitution
  - cartesian, linear, mixed, *etc.* substitution
  - specification and algorithms
  - syntax and semantics
- Reduction of type theory to algebra
  - admissibility of cut
  - second-order theories
  - dependent sorts

- Equational and inequational theories
  - free constructions
  - modularity
  - rewriting
- Structural combinatorics
  - Generalised species
    - cartesian closed
    - differential[see [23]]

    (cartesian closed, differential: structure)
  - Groupoids and generalised analytic functors[see [27]]
- Profunctors
  - Groupoids and strong (= †) compact closure[see [28]]
  - Annihilation/creation operators

see [28, 32] (and also [17, 29])

Programme

- Categories of contexts as free monoidal theories
- Comparison with/extension to Kelly's clubs[5, 10]
- Generalized logic type-theoretically and coherence
- Extraction of syntactic theory from model theory
- Applications
  - Theory of operads
  - Combinatorics
  - Domain Theory[see 27]

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