Second-Order Algebraic Theories

(Extended Abstract)

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Abstract. Fiore and Hur [10] recently introduced a conservative extension of universal algebra and equational logic from first to second order. *Second-order universal algebra* and *second-order equational logic* respectively provide a model theory and a formal deductive system for languages with variable binding and parameterised metavariables. This work completes the foundations of the subject from the viewpoint of categorical algebra. Specifically, the paper introduces the notion of *second-order algebraic theory* and develops its basic theory. Two categorical equivalences are established: at the syntactic level, that of second-order equational presentations and second-order algebraic theories; at the semantic level, that of second-order algebras and second-order functorial models. Our development includes a mathematical definition of syntactic translation between second-order equational presentations. This gives the first formalisation of notions such as encodings and transforms in the context of languages with variable binding.

1 Introduction

Algebra started with the study of a few sample algebraic structures: groups, rings, lattices, *etc.* Based on these, Birkhoff [3] laid out the foundations of a general unifying theory, now known as universal algebra.

Birkhoff's formalisation of the notion of algebra starts with the introduction of equational presentations. These constitute the syntactic foundations of the subject. Algebras are then the semantics or model theory, and play a crucial role in establishing the logical foundations. Indeed, Birkhoff introduced equational logic as a sound and complete formal deductive system for reasoning about algebraic structure.

The investigation of algebraic structure was further enriched by the advent of category theory, with the fundamental work of Lawvere on algebraic theories [18] and of Linton on finitary monads [17]. These approaches give a presentation-independent treatment of the subject. Algebraic theories correspond to the syntactic line of development; monads to the semantic one (see *e.g.* [15]).

We contend that it is only by looking at algebraic structure from all of the above perspectives, and the ways in which they interact, that the subject is properly understood. In the context of computer science, for instance, consider that: (i) initial-algebra semantics provides canonical compositional interpretations [14]; (ii) free constructions amount to abstract syntax [19], that is amenable to proofs by structural induction and definitions by structural recursion [4];

(*iii*) equational presentations can be regarded as (bidirectional) rewriting theories, and studied from a computational point of view [16]; (*iv*) algebraic theories come with an associated notion of algebraic translation [18], whose syntactic counterpart provides the right notion of syntactic translation between equational presentations [12, 13]; (*v*) strong monads have an associated metalogic from which equational logics can be synthesised [9, 10].

The realm of universal algebra is restricted to first-order languages. In particular, this leaves out languages with variable binding. Variable-binding constructs are at the core of fundamental calculi and theories in computer science and logic [5, 6], and incorporating them into algebra has been a main foundational research problem. The present work develops such a programme from the viewpoint of algebraic theories.

Our presentation is in two parts. The first part (Sections 2 and 3) sets up the necessary background; the second part (Sections 4 to 6) constitutes the contribution of the paper.

The background material gives an introduction to the work of Fiore and Hur [10] on a conservative extension of universal algebra and its equational logic from first to second order, *i.e.* to languages with variable binding and parameterised metavariables. Our summary recalls: (*i*) the notion of second-order equational presentation, that allows the specification of equational theories by means of schematic identities over signatures of variable-binding operators; (*ii*) the model theory of second-order equational presentations by means of second-order algebras; and (*iii*) the deductive system underlying formal reasoning about second-order algebraic structure.

The crux of our work is the notion of *second-order algebraic theory* (Definition 4.1). At the syntactic level, the correctness of our definition is established by showing a categorical equivalence between second-order equational presentations and second-order algebraic theories (Theorem 5.2). This involves distilling a notion of syntactic translation between second-order equational presentations that corresponds to the canonical notion of morphism between second-order algebraic theories. These syntactic translations provide a mathematical formalisation of notions such as encodings and transforms. On top of the syntactic correspondence, we furthermore establish a semantic one, by which *second-order functorial semantics* is shown to correspond to the model theory of second-order universal algebra (Theorem 6.1 and Corollary 6.1).

2 Second-Order Equational Logic

We briefly present *Second-Order Equational Logic* as introduced by Fiore and Hur [10] together with the syntactic machinery that surrounds it. For succinctness, our exposition is restricted to the unityped setting. The general multi-typed framework can be found in [10].

Signatures. A (unityped second-order) *signature* $\Sigma = (O, |-|)$ is specified by a set of operators O and an arity function $|-| : O \longrightarrow \mathbb{N}^*$, see [1, 2]. For $\mathsf{o} \in O$, we write $\mathsf{o} : (n_1, \ldots, n_k)$ whenever $|\mathsf{o}| = (n_1, \ldots, n_k)$. The intended meaning is that the operator o takes k arguments with the i^{th} argument binding n_i variables.

Example 2.1. The signature of the λ-*calculus* has operators $\mathsf{abs} : (1)$ and $\mathsf{app} : (0, 0)$.

Terms. We consider terms in contexts with two zones, respectively declaring metavariables and variables. Metavariables come with an associated natural number arity. A metavariable M of arity m, denoted $\mathrm{M} : [m]$, is to be parameterised by m terms. We represent contexts as $\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_n$ where the metavariables M_i and the variables x_j are assumed distinct.

Signatures give rise to terms in context. Terms are built up by means of operators from both variables and metavariables, and hence referred to as second-order. The judgement for *terms* in context ($\Theta \rhd \Gamma \vdash -$) is defined by the following rules.

(Variables) For $x \in \Gamma$,

$$\frac{}{\Theta \rhd \Gamma \vdash x}$$

(Metavariables) For $(\mathrm{M} : [m]) \in \Theta$,

$$\frac{\Theta \rhd \Gamma \vdash t_i \ \ (1 \leq i \leq m)}{\Theta \rhd \Gamma \vdash \mathrm{M}[t_1, \ldots, t_m]}$$

(Operators) For $\mathsf{o} : (n_1, \ldots, n_k)$,

$$\frac{\Theta \rhd \Gamma, \vec{x_i} \vdash t_i \ (1 \leq i \leq k)}{\Theta \rhd \Gamma \vdash \mathsf{o}\big((\vec{x_1})\, t_1, \ldots, (\vec{x_k})\, t_k\big)}$$

where $\vec{x_i}$ stands for $x_{i,1}, \ldots, x_{i,n_i}$.

Second-order terms are considered up the α-equivalence relation induced by stipulating that, for every operator o, in the term $\mathsf{o}\big(\ldots, (\vec{x_i})t_i, \ldots\big)$ the $\vec{x_i}$ are bound in t_i.

Example 2.2. Two terms for the λ-calculus signature (Example 2.1) follow:

$$\mathrm{M} : [1], \mathrm{N} : [0] \rhd \cdot \vdash \mathsf{app}\big(\mathsf{abs}\big((x)\mathrm{M}[x]\big), \mathrm{N}[\,]\big) \quad , \quad \mathrm{M} : [1], \mathrm{N} : [0] \rhd \cdot \vdash \mathrm{M}[\mathrm{N}[\,]] \ .$$

Substitution calculus. The second-order nature of the syntax requires a two-level substitution calculus [1, 8]. Each level respectively accounts for the substitution of variables and metavariables, with the latter operation depending on the former.

The operation of capture-avoiding simultaneous *substitution* of terms for variables maps

$$\Theta \rhd x_1, \ldots, x_n \vdash t \quad \text{and} \quad \Theta \rhd \Gamma \vdash t_i \ \ (1 \leq i \leq n)$$

to

$$\Theta \rhd \Gamma \vdash t[{}^{t_i}/_{x_i}]_{1 \leq i \leq n}$$

according to the following inductive definition:

– $x_j[{}^{t_i}/_{x_i}]_{1 \leq i \leq n} = t_j$

– $\big(\mathrm{M}[\ldots,s,\ldots]\big)[{}^{t_i}\!/\!{}_{x_i}]_{1\le i\le n} = \mathrm{M}\big[\ldots,s[{}^{t_i}\!/\!{}_{x_i}]_{1\le i\le n},\ldots\big]$

– $\big(\mathsf{o}(\ldots,(y_1,\ldots,y_k)s,\ldots)\big)[{}^{t_i}\!/\!{}_{x_i}]_{1\le i\le n}$
$= \mathsf{o}\big(\ldots,(z_1,\ldots,z_k)s[{}^{t_i}\!/\!{}_{x_i},{}^{z_j}\!/\!{}_{y_j}]_{1\le i\le n,1\le j\le k},\ldots\big)$
with $z_j \notin \mathrm{dom}(\Gamma)$ for all $1 \le j \le k$

The operation of *metasubstitution* of abstracted terms for metavariables maps

$$\mathrm{M}_1 : [m_1],\ldots,\mathrm{M}_k : [m_k] \rhd \Gamma \vdash t \quad\text{and}\quad \Theta \rhd \Gamma,\vec{x_i} \vdash t_i \quad (1 \le i \le k)$$

to

$$\Theta \rhd \Gamma \vdash t\{\mathrm{M}_i := (\vec{x_i})t_i\}_{1\le i\le k}$$

according to the following inductive definition:

– $x\{\mathrm{M}_i := (\vec{x_i})t_i\}_{1\le i\le k} = x$

– $\big(\mathrm{M}_\ell[s_1,\ldots,s_m]\big)\{\mathrm{M}_i := (\vec{x_i})t_i\}_{1\le i\le k} \;= t_\ell[{}^{s'_j}\!/\!{}_{x_{i,j}}]_{1\le j\le m}$
where, for $1 \le j \le m$, $s'_j = s_j\{\mathrm{M}_i := (\vec{x_i})t_i\}_{1\le i\le k}$

– $\big(\mathsf{o}(\ldots,(\vec{x})s,\ldots)\big)\{\mathrm{M}_i := (\vec{x_i})t_i\}_{1\le i\le k} \;= \mathsf{o}\big(\ldots,(\vec{x})s\{\mathrm{M}_i := (\vec{x_i})t_i\}_{1\le i\le k},\ldots\big)$

Presentations. An *equational presentation* is specified by a signature together with a set of axioms over it, each of which is a pair of terms in context.

Example 2.3. The equational presentation of the λ-calculus extends the signature of Example 2.1 with the following equations.

$$(\beta)\quad \mathrm{M}:[1],\mathrm{N}:[0] \rhd \cdot \vdash \mathsf{app}\big(\mathsf{abs}(\,(x)\mathrm{M}[x]\,),\mathrm{N}[\,]\big) \equiv \mathrm{M}\big[\mathrm{N}[\,]\big]$$
$$(\eta)\quad \mathrm{F}:[0] \rhd \cdot \vdash \mathsf{abs}\big(\,(x)\mathsf{app}(\mathrm{F}[\,],x)\,\big) \equiv \mathrm{F}[\,]$$

Logic. The rules of *Second-Order Equational Logic* are given in Figure 1. Besides the rules for axioms and equivalence, it consists of just one additional rule stating that the operation of metasubstitution in extended variable contexts is a congruence.

We note the following basic result from [10]: *Second-Order Equational Logic* is a conservative extension of *(First-Order) Equational Logic*.

3 Second-Order Universal Algebra

The model theory of Fiore and Hur [10] for second-order equational presentations is recalled. This is presented here in concrete elementary terms, but could have also been given in abstract monadic terms. The reader is referred to [10] for the latter perspective.

Semantic universe. We write $\mathbf{F}$ for the free cocartesian category on an object. Explicitly, it has set of objects $\mathbb{N}$ and morphisms $m \longrightarrow n$ given by functions $\|m\| \longrightarrow \|n\|$, where, for $\ell \in \mathbb{N}$, $\|\ell\| = \{1,\ldots,\ell\}$.

We will work within and over the semantic universe $\boldsymbol{\mathcal{S}et}^{\mathbf{F}}$ of sets in variable contexts [11]. We write $\boldsymbol{y}$ for the Yoneda embedding $\mathbf{F}^{\mathrm{op}} \hookrightarrow \boldsymbol{\mathcal{S}et}^{\mathbf{F}}$.

(Axiom)

$$\frac{(\Theta \rhd \Gamma \vdash s \equiv t) \in E}{\Theta \rhd \Gamma \vdash s \equiv t}$$

(Equivalence)

$$\frac{\Theta \rhd \Gamma \vdash t}{\Theta \rhd \Gamma \vdash t \equiv t} \qquad \frac{\Theta \rhd \Gamma \vdash s \equiv t}{\Theta \rhd \Gamma \vdash t \equiv s} \qquad \frac{\Theta \rhd \Gamma \vdash s \equiv t \qquad \Theta \rhd \Gamma \vdash t \equiv u}{\Theta \rhd \Gamma \vdash s \equiv u}$$

(Extended metasubstitution)

$$\frac{\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd \Gamma \vdash s \equiv t \qquad \Theta \rhd \Delta, \vec{x_i} \vdash s_i \equiv t_i \quad (1 \leq i \leq k)}{\Theta \rhd \Gamma, \Delta \vdash s\{\mathrm{M}_i := (\vec{x_i})s_i\}_{1 \leq i \leq k} \equiv t\{\mathrm{M}_i := (\vec{x_i})t_i\}_{1 \leq i \leq k}}$$

Fig. 1. *Second-Order Equational Logic.*

Substitution. We recall the *substitution monoidal structure* in semantic universes [11]. It has tensor unit and tensor product respectively given by $\boldsymbol{y}1$ and $X \bullet Y = \int^{k \in \mathbf{F}} X(k) \times Y^k$.

A monoid $\boldsymbol{y}1 \xrightarrow{\nu} A \xleftarrow{\varsigma} A \bullet A$ for the substitution monoidal structure equips A with substitution structure. In particular, the map $\nu_k = (\boldsymbol{y}k \cong (\boldsymbol{y}1)^k \xrightarrow{\nu^k} A^k)$ induces the embedding

$$\big(A^{\boldsymbol{y}n} \times A^n\big)(k) \longrightarrow A(k+n) \times A^k(k) \times A^n(k) \longrightarrow \big(A \bullet A\big)(k)$$

which together with the multiplication yield a *substitution operation*

$$\varsigma_n : A^{\boldsymbol{y}n} \times A^n \longrightarrow A \enspace .$$

These substitution operations provide the interpretation of metavariables.

Algebras. Every signature Σ induces a *signature endofunctor* on $\boldsymbol{\mathcal{S}et}^{\mathbf{F}}$ given by $\mathcal{F}_\Sigma X = \coprod_{\mathsf{o}:(n_1,\ldots,n_k) \text{ in } \Sigma} \prod_{1 \leq i \leq k} X^{\boldsymbol{y}n_i}$. $\mathcal{F}_\Sigma$-algebras $\mathcal{F}_\Sigma X \longrightarrow X$ provide an interpretation $[\![\mathsf{o}]\!]_X : \prod_{1 \leq i \leq k} X^{\boldsymbol{y}n_i} \longrightarrow X$ for every operator $\mathsf{o} : (n_1, \ldots, n_k)$ in Σ.

We note that there are canonical natural isomorphisms

$$\textstyle\coprod_{i \in I}(X_i \bullet Y) \cong \big(\coprod_{i \in I} X_i\big) \bullet Y$$

$$\textstyle\big(\prod_{1 \leq i \leq n} X_i\big) \bullet Y \cong \prod_{1 \leq i \leq n}(X_i \bullet Y)$$

and, for all points $\eta : \boldsymbol{y}1 \longrightarrow Y$, natural extension maps

$$\eta^{\#_n} : X^{\boldsymbol{y}n} \bullet Y \longrightarrow (X \bullet Y)^{\boldsymbol{y}n} \enspace .$$

These constructions equip every signature endofunctor with a *pointed strength* $\varpi_{X, \boldsymbol{y}1 \to Y} : \mathcal{F}_\Sigma(X) \bullet Y \longrightarrow \mathcal{F}_\Sigma(X \bullet Y)$. See [8] for details.

Models. The models that we are interested in (referred to as *Σ-monoids* in [11, 8]) are algebras equipped with a compatible substitution structure. For a signature Σ, we let Σ-Mod be the category of *Σ-models* with objects $A \in \boldsymbol{\mathcal{S}et}^{\mathbf{F}}$

equipped with an $\mathcal{F}_\Sigma$-algebra structure $\alpha : \mathcal{F}_\Sigma A \longrightarrow A$ and a monoid structure $\boldsymbol{y}1 \xrightarrow{\nu} A \xleftarrow{\varsigma} A \bullet A$ that are compatible in the sense that the diagram

$$\begin{tikzcd}
\mathcal{F}_\Sigma(A) \bullet A \arrow[r, "\varpi_{A,\nu}"] \arrow[d, "\alpha \bullet A"'] & \mathcal{F}_\Sigma(A \bullet A) \arrow[r, "\mathcal{F}_\Sigma \varsigma"] & \mathcal{F}_\Sigma(A) \arrow[d, "\alpha"] \\
A \bullet A \arrow[rr, "\varsigma"'] & & A
\end{tikzcd}$$

commutes. Morphisms are maps that are both $\mathcal{F}_\Sigma$-algebra and monoid homomorphims.

Semantics. For $\Theta = (\mathrm{M}_1 : [m_1] \ldots, \mathrm{M}_k : [m_k])$ and $\Gamma = (x_1, \ldots, x_n)$, the interpretation of a term $\Theta \rhd \Gamma \vdash t$ in a model A is a morphism

$$[\![\Theta \rhd \Gamma \vdash t]\!]_A : [\![\Theta \rhd \Gamma]\!]_A \longrightarrow A \ ,$$

where $[\![\Theta \rhd \Gamma]\!]_A = \prod_{1 \le i \le k} A^{\boldsymbol{y} m_i} \times \boldsymbol{y} n$, given by structural induction as follows:

- $[\![\Theta \rhd \Gamma \vdash x_j]\!]_A$ is the composite $[\![\Theta \rhd \Gamma]\!]_A \xrightarrow{\pi_2} \boldsymbol{y} n \xrightarrow{\nu_n} A^n \xrightarrow{\pi_j} A$.
- $[\![\Theta \rhd \Gamma \vdash \mathrm{M}_i[t_1, \ldots, t_{m_i}]]\!]_A$ is the composite

  $$[\![\Theta \rhd \Gamma]\!]_A \xrightarrow{\langle \pi_i\, \pi_1, f \rangle} A^{\boldsymbol{y} m_i} \times A^{m_i} \xrightarrow{\varsigma_{m_i}} A$$

  where $f = \big\langle [\![\Theta \rhd \Gamma \vdash t_j]\!] \big\rangle_{1 \le j \le m_i}$.
- For $\mathsf{o} : (n_1, \ldots, n_\ell)$,

  $$[\![\Theta \rhd \Gamma \vdash \mathsf{o}\big((\vec{y_1})t_1, \ldots, (\vec{y_\ell})t_\ell\big)]\!]$$

  is the composite $[\![\Theta \rhd \Gamma]\!]_A \xrightarrow{\langle f_j \rangle_{1 \le j \le \ell}} \prod_{1 \le j \le \ell} A^{\boldsymbol{y} n_j} \xrightarrow{[\![\mathsf{o}]\!]_A} A$ where f_j is the exponential transpose of

$$\textstyle\prod_{1 \le i \le k} A^{\boldsymbol{y} m_i} \times \boldsymbol{y} n \times \boldsymbol{y} n_j \cong \prod_{1 \le i \le k} A^{\boldsymbol{y} m_i} \times \boldsymbol{y}(n + n_j) \xrightarrow{[\![\Theta \rhd \Gamma, \vec{y_j} \vdash t_j]\!]_A} A \ .$$

Equational models. We say that a model A *satisfies* $\Theta \rhd \Gamma \vdash s \equiv t$, for which we use the notation $A \models (\Theta \rhd \Gamma \vdash s \equiv t)$, iff $[\![\Theta \rhd \Gamma \vdash s]\!]_A = [\![\Theta \rhd \Gamma \vdash t]\!]_A$.

For an equational presentation (Σ, E), we write (Σ, E)-Mod for the full subcategory of Σ-Mod consisting of the Σ-models that satisfy the axioms E.

Soundness and completeness [10].

For an equational presentation (Σ, E), the judgement $\Theta \rhd \Gamma \vdash s \equiv t$ is derivable from E iff $A \models (\Theta \rhd \Gamma \vdash s \equiv t)$ for all (Σ, E)-models A.

4 Second-Order Algebraic Theories

We introduce the notion of unityped second-order algebraic theory and establish it as the categorical counterpart to that of second-order equational presentation. The generalisation to the multi-typed case should be evident.

Remark. Having omitted the monadic view of second-order universal algebra, the important role played by the monadic perspective in our development will not be considered here.

Theory of equality. The theory of equality plays a pivotal role in the definition of algebraic theory. Thus, we proceed first to identify the second-order algebraic theory of equality. This we do both in syntactic and semantic terms. The (first-order) algebraic theory of equality is then considered from this new perspective.

The syntactic viewpoint leads us to define the category $\mathbf{M}$ with set of objects $\mathbb{N}^*$ and morphisms $(m_1, \ldots, m_k) \longrightarrow (n_1, \ldots, n_\ell)$ given by tuples

$$\langle\, \mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_{n_i} \vdash t_i \,\rangle_{i \in \|\ell\|}$$

of terms under the empty signature. The identity on $(m_1, \ldots, m_k)$ is given by

$$\langle\, \mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_{m_i} \vdash \mathrm{M}_i[x_1, \ldots, x_{m_i}] \,\rangle_{i \in \|k\|} \;\; ;$$

whilst the composition of

$$\langle\, \mathrm{M}_1 : [\ell_1], \ldots, \mathrm{M}_i : [\ell_i] \rhd x_1, \ldots, x_{m_p} \vdash s_p \,\rangle_{p \in \|j\|} : (\ell_1, \ldots, \ell_i) \longrightarrow (m_1, \ldots, m_j)$$

and

$$\langle\, \mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_j : [m_j] \rhd x_1, \ldots, x_{n_q} \vdash t_q \,\rangle_{q \in \|k\|} : (m_1, \ldots, m_j) \longrightarrow (n_1, \ldots, n_k)$$

is given by metasubstitution as follows:

$$\langle\, \mathrm{M}_1 : [\ell_1], \ldots, \mathrm{M}_i : [\ell_i] \rhd x_1, \ldots, x_{n_q} \vdash t_q\{\mathrm{M}_p := (x_1, \ldots, x_{m_p})s_p\}_{p \in \|j\|} \,\rangle_{q \in \|k\|} \;\; .$$

The category $\mathbf{M}$ is strict cartesian, with terminal object given by the empty sequence and binary products given by concatenation. Furthermore, the object $(0) \in \mathbf{M}$ is exponentiable. Indeed, the exponential object $(0) \Rightarrow (m_1, \ldots, m_k)$ is $(m_1 + 1, \ldots, m_k + 1)$ with evaluation map

$$(m_1 + 1, \ldots, m_k + 1, 0) \longrightarrow (m_1, \ldots, m_k)$$

given by

$$\left\langle \begin{array}{l} \mathrm{M}_1 : [m_1 + 1], \ldots, \mathrm{M}_k : [m_k + 1], \mathrm{M}_{k+1} : [0] \rhd x_1, \ldots, x_{m_i} \\ \qquad \vdash \;\; \mathrm{M}_i\big[x_1, \ldots, x_{m_i}, \mathrm{M}_{k+1}[\,]\big] \end{array} \right\rangle_{i \in \|k\|}$$

In fact, this structure provides a semantic characterisation of $\mathbf{M}$.

Lemma 4.1 (Universal property of M). *The category* $\mathbf{M}$*, together with the object* $(0) \in \mathbf{M}$*, is initial amongst cartesian categories equipped with an exponentiable object (with respect to cartesian functors that preserve the exponentiable object).*

Loosely speaking, then, $\mathbf{M}$ is the free (strict) cartesian category on an exponentiable object.

Algebraic theories. We extend Lawvere's fundamental notion of (first-order) algebraic theory [18] to second order.

Definition 4.1 (Second-order algebraic theories). *A* second-order algebraic theory *consists of a cartesian category* $\mathbb{T}$ *and a strict cartesian identity-on-objects functor* $\mathbf{M} \longrightarrow \mathbb{T}$ *that preserves the exponentiable object* (0)*.*

The most basic example is the *second-order algebraic theory of equality* given by $\mathbf{M}$ (together with the identity functor).

Every second-order algebraic theory has an underlying (first-order) algebraic theory. To formalise this, recall that the (first-order) algebraic theory of equality $\mathbf{L} = \mathbf{F}^{\mathrm{op}}$ is the free (strict) cartesian category on an object and consider the unique cartesian functor $\mathbf{L} \longrightarrow \mathbf{M}$ mapping the generating object to the exponentiable object. Then, the (first-order) algebraic theory underlying $\mathbf{M} \longrightarrow \mathbb{T}$ is $\mathbf{L} \longrightarrow \mathbb{T}_0$ for $\mathbf{L} \longrightarrow \mathbb{T}_0 \hookrightarrow \mathbb{T}$ the identity-on-objects/full-and-faithful factorisation of $\mathbf{L} \longrightarrow \mathbf{M} \longrightarrow \mathbb{T}$. In particular, $\mathbf{L}$ underlies $\mathbf{M}$.

The theory of a presentation. For a second-order equational presentation $\mathcal{E}$, the *classifying category* $\mathbf{M}(\mathcal{E})$ has set of objects $\mathbb{N}^*$ and morphisms $\vec{m} \longrightarrow \vec{n}$, say with $\vec{m} = (m_1, \ldots, m_k)$ and $\vec{n} = (n_1, \ldots, n_\ell)$, given by tuples

$$\Big\langle \, [\, \mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_{n_i} \vdash t_i \,]_{\mathcal{E}} \, \Big\rangle_{i \in \|\ell\|}$$

of equivalence classes of terms under the equivalence relation that identifies two terms iff they are provably equal from $\mathcal{E}$ in *Second-Order Equational Logic*. (Identities and composition are defined on representatives as in $\mathbf{M}$.)

Lemma 4.2. *For a second-order equational presentation $\mathcal{E}$, the category $\mathbf{M}(\mathcal{E})$ together with the canonical functor $\mathbf{M} \longrightarrow \mathbf{M}(\mathcal{E})$ is a second-order algebraic theory.*

We refer to $\mathbf{M} \longrightarrow \mathbf{M}(\mathcal{E})$ as the second-order algebraic theory of $\mathcal{E}$.

The presentation of a theory. The *internal language* $\mathcal{E}(T)$ of a second-order algebraic theory $T : \mathbf{M} \longrightarrow \mathbb{T}$ is the second-order equational presentation defined as follows:

(Operators) For every $f : (m_1, \ldots, m_k) \longrightarrow (n)$ in $\mathbb{T}$, we have an operator o_f of arity $(m_1, \ldots, m_k, \underbrace{0, \ldots, 0}_{n \text{ times}})$.

(Equations) Setting

$$\mathsf{t}_f = \mathsf{o}_f\big((x_1, \ldots, x_{m_1})\mathrm{M}_1[x_1, \ldots, x_{m_1}], \ldots, (x_1, \ldots, x_{m_k})\mathrm{M}_k[x_1, \ldots, x_{m_k}], x_1, \ldots, x_n\big)$$

for every $f : (m_1, \ldots, m_k) \longrightarrow (n)$ in $\mathbb{T}$, we have

- $\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_n \vdash s \equiv \mathsf{t}_{T\langle s \rangle}$
  for every $\langle s \rangle : (m_1, \ldots, m_k) \longrightarrow (n)$ in $\mathbf{M}$,
- $\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_n \vdash \mathsf{t}_h \equiv \mathsf{t}_g\{\mathrm{M}_i := (x_1, \ldots, x_{n_i})\mathsf{t}_{f_i}\}_{1 \le i \le \ell}$
  for every $h : (m_1, \ldots, m_k) \longrightarrow (n)$, $g : (n_1, \ldots, n_\ell) \longrightarrow (n)$, and $f_i : (m_1, \ldots, m_k) \longrightarrow (n_i)$, $1 \le i \le \ell$, such that $h = g \circ \langle f_1, \ldots, f_\ell \rangle$ in $\mathbb{T}$.

Algebraic translations. For second-order algebraic theories $T : \mathbf{M} \longrightarrow \mathbb{T}$ and $T' : \mathbf{M} \longrightarrow \mathbb{T}'$, a second-order *algebraic translation* $T \longrightarrow T'$ is a functor $F : \mathbb{T} \longrightarrow \mathbb{T}'$ such that $T' = FT$. We write $\boldsymbol{\mathcal{SOAT}}$ for the category of second-order algebraic theories and algebraic translations.

Theorem 4.1 (Theory/presentation correspondence). *Every second-order algebraic theory $T : \mathbf{M} \longrightarrow \mathbb{T}$ is isomorphic to the second-order algebraic theory of its associated equational presentation $\mathbf{M} \longrightarrow \mathbf{M}(\mathcal{E}(T))$.*

5 Second-Order Syntactic Translations

We introduce the notion of *syntactic translation* between second-order equational presentations. This we justify by establishing its equivalence with that of algebraic translation between the associated second-order algebraic theories.

Signature translations. A *syntactic translation* $\tau : \Sigma \longrightarrow \Sigma'$ between second-order signatures is given by a mapping from the operators of Σ to the terms of Σ' as follows:

$$\mathsf{o} : (m_1, \ldots, m_k) \quad \longmapsto \quad \mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd \cdot \vdash \tau_{\mathsf{o}} \quad .$$

Note that the term associated to an operator has an empty variable context and that the metavariable context is determined by the arity of the operator.

A translation $\tau : \Sigma \longrightarrow \Sigma'$ extends to a mapping from the terms of Σ to the terms of Σ'

$$\Theta \rhd \Gamma \vdash t \quad \longmapsto \quad \Theta \rhd \Gamma \vdash \tau(t)$$

according to the following inductive definition:

- $\tau(x) = x$
- $\tau\big(\mathrm{M}[t_1, \ldots, t_m]\big) = \mathrm{M}\big[\tau(t_1), \ldots, \tau(t_m)\big]$
- $\tau\big(\mathsf{o}\big((\vec{x_1})t_1, \ldots, (\vec{x_k})t_k\big)\big) = \tau_{\mathsf{o}}\{\mathrm{M}_i := (\vec{x_i})\tau(t_i)\}_{1 \leq i \leq k}$

Lemma 5.1 (Compositionality). *The extension of a syntactic translation between second-order signatures commutes with substitution and metasubstitution.*

Example 5.1 (Continutation Passing Style). A formalisation of the CPS transform for the λ-calculus as a syntactic translation due to Plotkin [20] follows. We provide it in informal notation for ease of readability.

$$\mathsf{app} : (0, 0) \quad \longmapsto \quad \mathrm{M} : [0], \mathrm{N} : [0] \rhd \cdot \vdash \lambda k.\, \mathrm{M}[\,]\, \big(\lambda m.\, m\, (\lambda \ell.\, \mathrm{N}[\,]\, \ell)\, k\big)$$

$$\mathsf{abs} : (1) \quad \longmapsto \quad \mathrm{F} : [1] \rhd \cdot \vdash \lambda k.\, k\, \big(\lambda x.\, (\lambda \ell.\, \mathrm{F}[x]\, \ell)\big)$$

Equational translations. A *syntactic translation* between second-order equational presentations $\tau : (\Sigma, E) \longrightarrow (\Sigma,' E')$ is a translation $\tau : \Sigma \longrightarrow \Sigma'$ such that, for every axiom $\Theta \rhd \Gamma \vdash s \equiv t$ in E, the judgement $\Theta \rhd \Gamma \vdash \tau(s) \equiv \tau(t)$ is derivable from E'.

Lemma 5.2. *The extension of a syntactic translation between second-order equational presentations preserves second-order equational derivability.*

We write $\boldsymbol{\mathcal{SOEP}}$ for the category of second-order equational presentations and syntactic translations. (The identity syntactic translation maps an operator $\mathsf{o} : (m_1, \ldots, m_k)$ to the term $\mathsf{o}\big(\ldots, (x_1, \ldots, x_{m_i})\mathrm{M}_i[x_1, \ldots, x_{m_i}], \ldots\big)$; whilst the composition of τ followed by τ' maps o to $\tau'(\tau_{\mathsf{o}})$.)

Theorem 5.1 (Presentation/theory correspondence). *Every second-order equational presentation $\mathcal{E}$ is isomorphic to the second-order equational presentation of its associated algebraic theory $\mathcal{E}(\mathbf{M}(\mathcal{E}))$.*

Syntactic and algebraic translations. A syntactic translation $\tau : \mathcal{E} \longrightarrow \mathcal{E}'$ induces the algebraic translation $\mathbf{M}(\tau) : \mathbf{M}(\mathcal{E}) \longrightarrow \mathbf{M}(\mathcal{E}')$, mapping $\langle\, [t_1]_{\mathcal{E}}, \ldots, [t_\ell]_{\mathcal{E}} \,\rangle$ to $\langle\, [\tau(t_1)]_{\mathcal{E}'}, \ldots, [\tau(t_\ell)]_{\mathcal{E}'} \,\rangle$. This gives a functor $\mathcal{SOEP} \longrightarrow \mathcal{SOAT}$. Conversely, an algebraic translation $F : T \longrightarrow T'$ induces the syntactic translation $\mathcal{E}(F) : \mathcal{E}(T) \longrightarrow \mathcal{E}(T')$, mapping an operator o_f, for $f : (m_1, \ldots, m_k) \longrightarrow (n)$ in $\mathbb{T}$, to the term $\mathsf{t}_{Ff}\big[{}^{\mathrm{M}_{k+1}[\,]}/x_1, \ldots, {}^{\mathrm{M}_{k+n}[\,]}/x_n\big]$. This gives a functor $\mathcal{SOAT} \longrightarrow \mathcal{SOEP}$.

Theorem 5.2. *The categories $\mathcal{SOAT}$ and $\mathcal{SOEP}$ are equivalent.*

6 Second-Order Functorial Semantics

We extend Lawvere's functorial semantics for algebraic theories [18] from first to second order.

Functorial models. The category $\mathcal{M}\boldsymbol{od}(T)$ of (set-theoretic) *functorial models* of a second-order algebraic theory $T : \mathbf{M} \longrightarrow \mathbb{T}$ is the category of cartesian functors $\mathbb{T} \longrightarrow \mathcal{S}\boldsymbol{et}$ and natural transformations between them.

Every $\mathcal{E}$-model A, for a second-order equational presentation $\mathcal{E}$, provides a functorial model $\mathbf{M}(\mathcal{E}) \longrightarrow \mathcal{S}\boldsymbol{et}$ as follows:

- on objects, $(m_1, \ldots, m_k)$ is mapped to $\prod_{1 \le i \le k} A(m_i)$;
- on morphisms, $\langle\, [\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_{n_i} \vdash t_j]_{\mathcal{E}} \,\rangle_{j \in \|\ell\|}$ is mapped to $\langle\, (f_j)_0 \rangle_{1 \le j \le \ell}$ where $f_j : \prod_{1 \le i \le k} A^{\boldsymbol{y}m_i} \longrightarrow A^{\boldsymbol{y}n_j}$ is the exponential transpose of $[\![\mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_{n_j} \vdash t_j]\!]_A$.

As we proceed to show, every functorial model essentially arises in this manner (see Corollary 6.1).

Clones. We need recall and develop some aspects of the theory of *clones* from universal algebra (see *e.g.* [7]).

Let C be an exponentiable object in a cartesian category $\mathscr{C}$. Recall that the family $\langle C \rangle = \{C^n \Rightarrow C\}_{n \in \mathbb{N}}$ has a canonical clone structure

$$\iota_i^{(n)} : 1 \longrightarrow \langle C \rangle_n \ (1 \le i \le n \in \mathbb{N}) \ \ , \quad \varsigma_{m,n} : \langle C \rangle_m \times {\langle C \rangle_n}^m \longrightarrow \langle C \rangle_n \ (m, n \in \mathbb{N})$$

known as the *clone of operations* on C. Thus, as it is the case with every clone, the family $\langle C \rangle$ canonically extends to a functor $\mathbf{F} \longrightarrow \mathscr{C} : n \longmapsto \langle C \rangle_n$.

For every $m_1, \ldots, m_k \in \mathbb{N}$ (for $k \in \mathbb{N}$), $n \in \mathbb{N}$, and $f : \prod_{1 \le i \le k} \langle C \rangle_{m_i} \longrightarrow \langle C \rangle_n$ in $\mathscr{C}$ let $\widetilde{f} = \{\widetilde{f}_\ell\}_{\ell \in \mathbb{N}}$ be given by setting

$$\widetilde{f}_\ell = \Big(\textstyle\prod_{1 \le i \le k} \langle C \rangle_{\ell + m_i} \cong C^\ell \Rightarrow \prod_{1 \le i \le k} \langle C \rangle_{m_i} \xrightarrow{C^\ell \Rightarrow f} C^\ell \Rightarrow \langle C \rangle_n \cong \langle C \rangle_{\ell + n} \Big).$$

The family $\widetilde{f}$ is a natural transformation $\prod_{1 \le i \le k} \langle C \rangle_{(-) + m_i} \longrightarrow \langle C \rangle_{(-) + n}$ and commutes with the clone structure. The latter in the sense that, for

$$w_\ell = \Big({\langle C \rangle_q}^p \cong {\langle C \rangle_q}^p \times 1 \xrightarrow{{\langle C \rangle_\jmath}^p \times \langle \iota_{q+i}^{(q+\ell)} \rangle_{1 \le i \le \ell}} {\langle C \rangle_{q+\ell}}^p \times {\langle C \rangle_{q+\ell}}^\ell \cong {\langle C \rangle_{q+\ell}}^{p+\ell} \Big)$$

where $\jmath$ denotes the inclusion $\|q\| \hookrightarrow \|q + \ell\|$, the diagram

$$
\begin{tikzcd}
 & \prod_{1\leq i\leq k}\langle C\rangle_{p+m_i} \times {\langle C\rangle_{q+m_i}}^{p+m_i} \arrow[dr, "\prod_{1\leq i\leq k}\varsigma_{p+m_i,q+m_i}"] & \\
\prod_{1\leq i\leq k}\langle C\rangle_{p+m_i} \times {\langle C\rangle_{q}}^{p} \arrow[ur, "\langle \mathrm{id}\times w_{m_i}\rangle_{1\leq i\leq k}"] \arrow[d, "\widetilde{f}_p\times w_n"'] & & \prod_{1\leq i\leq k}\langle C\rangle_{q+m_i} \arrow[d, "\widetilde{f}_q"] \\
\langle C\rangle_{p+n} \times {\langle C\rangle_{q+n}}^{p+n} \arrow[rr, "\varsigma_{p+n,q+n}"'] & & \langle C\rangle_{q+n}
\end{tikzcd}
$$

commutes for all $p, q \in \mathbb{N}$.

Let Σ be a second-order signature, and consider a functorial model $S : \mathbf{M}(\Sigma) \longrightarrow \boldsymbol{\mathcal{S}et}$. Then, the image under the cartesian functor S of the clone of operations induced by the exponentiable object $(0) \in \mathbf{M}(\Sigma)$ together with the family $\{\widetilde{f_{\mathsf{o}}}\}_{\mathsf{o}:(m_1,\ldots,m_k)\text{ in }\Sigma}$, where $f_{\mathsf{o}} = \langle\, \mathsf{o}(\ldots,(x_1,\ldots,x_{m_i})\mathrm{M}_i[x_1,\ldots,x_{m_i}],\ldots)\,\rangle$, yields a Σ-model $\underline{S} \in \boldsymbol{\mathcal{S}et}^{\mathbf{F}}$.

Furthermore, for all $f = \langle\, \mathrm{M}_1 : [m_1], \ldots, \mathrm{M}_k : [m_k] \rhd x_1, \ldots, x_n \vdash t\,\rangle$ in $\mathbf{M}(\Sigma)$ we have that the image of $\widetilde{f}$ under $S : \mathbf{M}(\Sigma) \longrightarrow \boldsymbol{\mathcal{S}et}$ amounts to the interpretation of t in $\underline{S}$. Thus, for all second-order equational presentations $\mathcal{E} = (\Sigma, E)$, the Σ-model induced by the restriction of a functorial model $\mathbf{M}(\mathcal{E}) \longrightarrow \boldsymbol{\mathcal{S}et}$ to $\mathbf{M}(\Sigma)$ is an $\mathcal{E}$-model.

The above constructions between functorial and algebraic models provide an equivalence.

Theorem 6.1. *For every second-order equational presentation $\mathcal{E}$, the category of algebraic models $\mathcal{E}$-*Mod *and the category of functorial models $\boldsymbol{\mathcal{M}od}(\mathbf{M}(\mathcal{E}))$ are equivalent.*

Corollary 6.1. *For every second-order algebraic theory T, the category of functorial models $\boldsymbol{\mathcal{M}od}(T)$ and the category of algebraic models $\mathcal{E}(T)$-*Mod *are equivalent.*

Algebraic functors. As in the first-order case, every algebraic translation $F : T \longrightarrow T'$ between second-order algebraic theories contravariantly induces an *algebraic functor* $\boldsymbol{\mathcal{M}od}(T') \longrightarrow \boldsymbol{\mathcal{M}od}(T) : S \longmapsto S\,F$ between the corresponding categories of models. We also have the following fundamental result.

Theorem 6.2. *The algebraic functor $\boldsymbol{\mathcal{M}od}(T') \longrightarrow \boldsymbol{\mathcal{M}od}(T)$ induced by a second-order algebraic translation $T \longrightarrow T'$ has a left adjoint.*

7 Concluding Remarks

We have introduced *second-order algebraic theories* (Section 4): (i) showing them to be the presentation-independent categorical syntax of second-order equational presentations (Theorems 4.1, 5.1, and 5.2), and (ii) establishing that their functorial semantics amounts to second-order universal algebra (Theorem 6.1 and Corollary 6.1). In the context of (i), our development included a notion of second-order syntactic translation (Section 5), which, in the context of (ii), contravariantly gives rise to algebraic functors between categories of models (Theorem 6.2).

With this theory in place, one is now in a position to: (a) consider constructions on second-order equational presentations in a categorical setting, and indeed the developments for (first-order) algebraic theories on limits, colimits, and tensor product carry over to the second-order setting; (b) investigate conservative-extension results for second-order equational presentations in a mathematical framework; and (c) study Morita equivalence for second-order algebraic theories.

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