Domains in $\mathcal{H}$

Marcelo P. Fiore
COGS, University of Sussex

Giuseppe Rosolini
DISI, Università di Genova

Abstract

We give various internal descriptions of the category ω-**Cpo** of ω-complete posets and ω-continuous functions in the model $\mathcal{H}$ of Synthetic Domain Theory introduced in [8]. It follows that the ω-cpos lie between the two extreme synthetic notions of domain given by repleteness and well-completeness.

Introduction

Synthetic Domain Theory aims at giving a few simple axioms to be added to an intuitionistic set theory in order to obtain domain-like sets. The idea at the core of this study was proposed by Dana Scott in the late 70's: domains should be certain "sets" in a mathematical universe where domain theory would be available. In particular, domains would come with intrinsic notions of approximation and passage to the limit with respect to which all functions will be continuous. Various suggestions for the notion of domain (typically within a set-theoretic universe given by an elementary topos with natural numbers object [17]) appeared in the literature, *e.g.* in [11, 26, 10, 23, 20, 16]. All these notions support the constructions needed in denotational semantics, *e.g.* sums, products, exponentials, lifting, fixed-point operators and, in the studied models, also the solution of recursive domain equations.

In [8], the authors introduced two Grothendieck toposes, $\mathcal{H}$ and $\mathcal{H}_2$, and studied them as models of SDT. Each model respectively embeds the following standard categories of domains: the category ω-**Cpo** of ω-complete posets and ω-continuous functions, and the category of ω-complete posets with ω-continuous pullbacks (= ω-continuous bounded binary meets) and ω-stable (= ω-continuous and pullback preserving) functions.

In each topos it was possible to check that the replete objects [11, 26] were included among the embedded domains; thus showing that the synthetic theory encompasses some of the traditional models used in denotational semantics. In this paper we characterize the embedding of ω-**Cpo** in the model $\mathcal{H}$, and use it to analyze the notions of repleteness and well-completeness which are two extreme synthetic notions of domain respectively suggested in [11, 26] and [15, 16].

The paper is organized as follows. First, we recall some category-theoretic background material and some internal notions in a topos. Then, we recall some facts of Synthetic Domain Theory, and the presentations of the topos $\mathcal{H}$. Finally, we give various internal characterizations of the embedding ω-**Cpo** $\hookrightarrow \mathcal{H}$ and compare it with the full subcategories of the replete objects and of the well-complete ones.

1 Basic concepts

Lifting monad. A monad (L, η, μ) internalizing possibly non-terminating computations is usually called *lifting* and it is axiomatized by requiring that the unit η classifies (certain) partial maps

that are closed under composition; see *e.g.* [21, 4]. Precisely, in a category with terminal object 1, the conditions are that the naturality diagrams for the unit η are pullbacks (*i.e.* the natural transformation η is *cartesian*) and that, in the situation

$$\begin{array}{ccc} 1 & \longleftarrow & D \longrightarrow B \\ {\scriptstyle \eta_1}\downarrow & \text{pb} & \downarrow \\ L1 & \longleftarrow & A \end{array} \quad ,$$

there exists a unique *classifying* map $A \longrightarrow LB$ such that the diagram

$$\begin{array}{ccc} D & \longrightarrow & B \\ \downarrow & & \downarrow{\scriptstyle \eta_B} \\ A & \longrightarrow & LB \end{array} \qquad (1)$$

is a pullback. Monos obtained as pullbacks of η_1 (as, for example, $D \rightarrowtail A$ above) are called *admissible* and are denoted with $\hookrightarrow$. Note that they form a subcategory, see [11].

Examples of lifting monads on a topos are: the monad classifying total maps (*i.e.* the identity monad), the monad classifying partial maps defined on a decidable subobject (*i.e.* the monad on $(-)+1$), and the monad classifying *all* partial maps (see *e.g.* [14]).

An example that will illustrate at an elementary level what will follow is given by an extension of the lifting monad $(-)_\perp$ on posets —that classifies partial maps defined on upper-closed subsets— to simplicial sets. Let **Pos** be the category of posets and monotone functions, and let $\mathbf{\Delta}$ be the category of monotone functions between the finite non-empty ordinals $\underline{n}$ ($\stackrel{\text{def}}{=} 0 \leq \cdots \leq n-1$). As $\mathbf{\Delta}$ is a dense generator in **Pos**, we have a full and faithful "Yoneda embedding" (or *nerve functor*) Y from **Pos** into the topos $\widehat{\mathbf{\Delta}}$ of *simplicial sets* (*i.e.* presheaves on $\mathbf{\Delta}$, see [17]) mapping a poset P to the simplicial set $\mathrm{Y}(P) \stackrel{\text{def}}{=} \mathbf{Pos}(J\,\underline{\ }\,, P)$, where $J\colon \mathbf{\Delta} \hookrightarrow \mathbf{Pos}$ is the inclusion functor. The lifting monad $(-)_\perp$ on **Pos** extends to $\widehat{\mathbf{\Delta}}$ by left Kan extension as in the diagram below.

$$\begin{array}{ccc} \mathbf{Pos} & \stackrel{\mathrm{Y}}{\hookrightarrow} & \widehat{\mathbf{\Delta}} \\ {\scriptstyle (-)_\perp}\downarrow & \stackrel{\text{Lan}}{\cong} & \downarrow{\scriptstyle L} \\ \mathbf{Pos} & \stackrel{\mathrm{Y}}{\hookrightarrow} & \widehat{\mathbf{\Delta}} \end{array}$$

Explicitly, one can describe L as follows: for a simplicial set A,

- $(LA)(\underline{n}) = \biguplus_{0 \leq i \leq n} A(\underline{i})$ where, by convention, we set $A(\underline{0}) = 1$; and
- for $f\colon \underline{m} \longrightarrow \underline{n}$ in $\mathbf{\Delta}$, the action $(LA)(f)\colon L(A)(\underline{n}) \longrightarrow L(A)(\underline{m})$ in **Set** is given by the map

  $$\left[A(\phi_i)\colon A(\underline{i}) \longrightarrow A(\underline{m_i})\right]_{0 \leq i \leq n}$$

  where the diagram

  $$\begin{array}{ccc} \underline{m_i} & \stackrel{\phi_i}{\longrightarrow} & \underline{i} \\ \downarrow & & \downarrow \\ \underline{m} & \stackrel{f}{\longrightarrow} & \underline{n} \end{array}$$

  is a pullback in **Pos**, and where, thanks to the convention, $A(\underline{0} \longrightarrow \underline{i})$ is the unique map $A(\underline{i}) \longrightarrow 1$.

The natural transformation $\eta_A : A \longrightarrow LA$ which, at the ordinal $\underline{n}$, injects $A(\underline{n})$ into $(LA)(\underline{n})$ classifies partial maps defined on admissible subobjects and yields *the* lifting monad on $\widehat{\mathbf{\Delta}}$.

Orthogonality. The crucial property of a model of Synthetic Domain Theory is stated in terms of *orthogonality*. This is a very useful notion that appeared in relation with factorization systems and limit-preserving functors. Recall from [9] that an object A in a category is said to be *orthogonal* to a map $D \longrightarrow C$ whenever, for every map $D \longrightarrow A$, there exists a unique map $C \longrightarrow A$ such that $(D \longrightarrow C \longrightarrow A) = (D \longrightarrow A)$. Further, in a cartesian closed category, an object A is *internally orthogonal* to a map $f : D \longrightarrow C$ if the arrow $A^f : A^C \longrightarrow A^D$ is an iso. Internal orthogonality corresponds to a *parameterized* version of orthogonality.

Proposition 1.1 *In a cartesian closed category $\mathcal{C}$, the following are equivalent:*

(i) *A is internally orthogonal to $f : D \longrightarrow C$.*

(ii) *A is orthogonal to $f \times \mathrm{id}_X : D \times X \longrightarrow C \times X$, for every X in $\mathcal{C}$.*

If $\mathcal{C}$ is a (pre)topos, another equivalent condition is

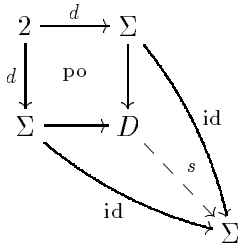
(iii) $\mathcal{C} \Vdash \forall\, \delta \in A^D .\ \exists!\, \gamma \in A^C .\ \delta = \gamma \circ f$.

The hypothesis that $\mathcal{C}$ be a pretopos is only used to interpret the formula. In fact, one could just assume that $\mathcal{C}$ is regular, but should then use languages with unique-existence quantifiers as in [3].

Σ-separation and path-transitivity. Consider a topos with a lifting monad on it such that its underlying functor L has a global element (*i.e.* a natural transformation from the functor constantly valued 1 into L) $\bot : 1 \stackrel{\cdot}{\longrightarrow} L$ for which $1 \xrightarrow{\bot_1} L1 \xhookleftarrow{\eta_1} 1$ are disjoint. We shall write $\bot$ for the map $\bot_1 : 1 \longrightarrow L1$ and $\top$ for the map $\eta_1 : 1 \hookrightarrow L1$.

For $n \geq 0$, set $\Sigma_n \stackrel{\text{def}}{=} L^n 1$ (omitting the index in the case $n = 1$) and let $d \stackrel{\text{def}}{=} [\bot, \top] : 2 \longrightarrow \Sigma$, where $2 \stackrel{\text{def}}{=} 1 + 1$. For an object A, we define the *path relation* $P_A \rightarrowtail A \times A$ as the image of the map $A^\Sigma \xrightarrow{A^d} A^2 \xrightarrow{\cong} A \times A$. We say that A is Σ-*separated* whenever $A^d : A^\Sigma \longrightarrow A^2$ is a mono; that is, roughly speaking, when A^Σ is the path relation P_A on A. For example, in the topos of simplicial sets, Σ_n is (isomorphic to the image under the Yoneda embedding of) the ordinal $\underline{n+1}$; and, for a poset P, the path relation on the simplicial set $\mathrm{Y}(P)$ is (isomorphic to) $\mathrm{Y}(P^\Sigma) \cong \mathrm{Y}(P)^\Sigma$, showing that $\mathrm{Y}(P)$ is Σ-separated.

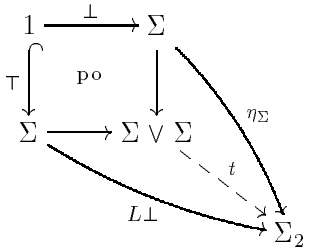
Note that Σ-separation is an orthogonality condition: an object is Σ-separated if and only if it is internally orthogonal to the unique mediating map s in



,

cf. [8].

Consider the following notion, used implicitly in [5, 6]: an object is *path-transitive* when it is internally orthogonal to the unique mediating map t in



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Examples of path-transitive objects are the simplicial sets $\mathrm{Y}(P)$, for every poset P. The name "path-transitive" is justified by the fact that, if an object is path-transitive, then the path relation on it is transitive. We remark that the converse need not hold. Indeed, in the topos of simplicial sets, the subobject $A \subseteq \Sigma_2$ defined by

$$A(\underline{n}) \stackrel{\text{def}}{=} \{\, f \in \mathbf{\Delta}(\underline{n}, \underline{3}) \mid \#\,\mathrm{im}(f) \leq 2 \,\}$$

has a transitive path relation $P_A \subseteq A \times A$, given by

$$P_A(\underline{n}) = \{\, (f, g) \in A(\underline{n}) \times A(\underline{n}) \mid f \leq g \colon \underline{n} \longrightarrow \underline{3} \,\} \quad ,$$

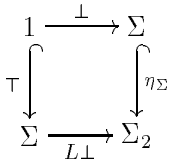
but it is not path-transitive.

As follows from the proposition below, in all examples of interest $\Sigma \vee \Sigma$ is *not* path-transitive.

Proposition 1.2 *The following are equivalent.*

(i) $\Sigma \vee \Sigma$ *is path-transitive.*

(ii) *The diagram*



is a pushout.

(iii) *Every object is path-transitive.*

(iv) $1 \xrightarrow{\perp} \Sigma \xleftarrow{\top} 1$ *is a coproduct diagram.*

Double-negation topology. The *double-negation topology* is the (Lawvere-Tierney) topology $\neg\neg \colon \Omega \longrightarrow \Omega$ defined in terms of the internal logical negation, see [17].

Given a topology j, the internal definition of the j-closure $\overline{m} \colon \overline{R} \rightarrowtail A$ of $m \colon R \rightarrowtail A$ is given in terms of the classifying maps into Ω: if $\chi \colon A \longrightarrow \Omega$ classifies m, then $j \circ \chi \colon A \longrightarrow \Omega$ classifies $\overline{m}$. A mono is *j-closed* if it is (isomorphic to) its j-closure. Once the notion of closure is available, one can introduce the notion of density: a mono is *j-dense* if its j-closure is (isomorphic to) the identity. And *j-sheaves* are those objects which are orthogonal to all j-dense monos. Also an object A is said to be *j-separated* if, for every j-dense mono $D \rightarrowtail C$ and every map $D \longrightarrow A$, there exists at

most one map $D \longrightarrow A$ such that $(D \rightarrowtail C \longrightarrow A) = (D \longrightarrow A)$. Equivalently, j-separated objects are those A for which the diagonal mono $A \rightarrowtail A \times A$ is j-closed.

We present some basic properties about the above notions, *cf.* [14, 17], which we shall then instantiate to the $\neg\neg$-topology. We write a_j for the associated j-sheaf functor, and ρ for the unit of the reflection determined by it.

Proposition 1.3 (i) *A mono $m\colon R \rightarrowtail A$ is j-closed if and only if the diagram*

$$\begin{array}{ccc} R & \xrightarrow{\rho_R} & \mathrm{a}_j(R) \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle \mathrm{a}_j(m)} \\ A & \xrightarrow[\rho_A]{} & \mathrm{a}_j(A) \end{array}$$

is a pullback (i.e. *if and only if* $\rho_A{}^{-1}(\mathrm{a}_j(R)) \subseteq R$).

(ii) *A mono $m\colon R \rightarrowtail A$ is j-dense if and only if $\mathrm{a}_j(m)\colon \mathrm{a}_j(R) \rightarrowtail \mathrm{a}_j(A)$ is an iso.*

(iii) *An object A is j-separated if and only if $\rho_A\colon A \longrightarrow \mathrm{a}_j(A)$ is a mono.*

Proof: Follows from the characterization of the j-closure $\overline{m}\colon \overline{R} \rightarrowtail A$ of a mono $m\colon R \rightarrowtail A$ as the pullback of $\mathrm{a}_j(m)$ along ρ_A as indicated in the following diagram

$$\begin{array}{ccc} \overline{R} & \longrightarrow & \mathrm{a}_j(R) \\ {\scriptstyle \overline{m}}\downarrow & \mathrm{pb} & \downarrow{\scriptstyle \mathrm{a}_j(m)} \\ A & \xrightarrow[\rho_A]{} & \mathrm{a}_j(A) \end{array} \quad . \qquad \square$$

We write $\widehat{\mathbb{C}}$ for the topos of presheaves on a small category $\mathbb{C}$, and we typically identify $\mathbb{C}$ with its image under the Yoneda embedding $\mathrm{Y}_{\mathbb{C}}\colon \mathbb{C} \hookrightarrow \widehat{\mathbb{C}}$. Further, we let $\gamma^*\colon \widehat{\mathbb{C}} \longrightarrow \mathbf{Set}$ be the *global-sections functor* $\widehat{\mathbb{C}}(1, -)$ and we define $\gamma_*\colon \mathbf{Set} \longrightarrow \widehat{\mathbb{C}}$ by $\gamma_*(S) \stackrel{\mathrm{def}}{=} \mathbf{Set}(\widehat{\mathbb{C}}(1, -), S)$.

Proposition 1.4 *Let $\mathbb{C}$ be a small category whose idempotent splitting has a terminal object.*

(i) *The functor γ_* is right adjoint to the global-sections functor γ^* yielding an essential geometric morphism $\gamma^* \dashv \gamma_*\colon \mathbf{Set} \longrightarrow \widehat{\mathbb{C}}$.*

(ii) *Let $\mathbb{C}$ be equipped with a Grothendieck topology J such that the functor $\widehat{\mathbb{C}}(1, -)\colon \mathbb{C} \longrightarrow \mathbf{Set}$ maps covers in J to epimorphic families in $\mathbf{Set}$. Then, γ_* factors through the embedding $\mathrm{sh}(\mathbb{C}, J) \hookrightarrow \widehat{\mathbb{C}}$ yielding a geometric morphism $\gamma^* \dashv \gamma_*\colon \mathbf{Set} \longrightarrow \mathrm{sh}(\mathbb{C}, J)$.*

In both cases, the geometric morphism is an inclusion. Moreover, it is (equivalent to) the inclusion of $\neg\neg$-sheaves if and only if γ_* preserves the initial object.

In terms of the site $\mathbb{C}$, the property that γ_* preserves the initial object in Proposition 1.4 reads as follows:

(i) for the case $\gamma^* \dashv \gamma_*\colon \mathbf{Set} \hookrightarrow \widehat{\mathbb{C}}$, it means that 1 is weakly initial in $\mathbb{C}$, and

(ii) for the case $\gamma^* \dashv \gamma_*\colon \mathbf{Set} \hookrightarrow \mathrm{sh}(\mathbb{C}, J)$, it means that 1 is weakly initial in the full subcategory of those objects of $\mathbb{C}$ which are not covered by the empty family.

Thus, in the cases of Proposition 1.4, we have an explicit description of the associated sheaf functor with respect to the double-negation topology (*viz.* as γ^*). The unit of the reflection $\gamma^* \dashv \gamma_*$ is the composite

$$A(C) \xrightarrow{\cong} \widehat{\mathbb{C}}(C, A) \xrightarrow{\gamma^*_{C,A}} \mathbf{Set}(\widehat{\mathbb{C}}(1, C), \widehat{\mathbb{C}}(1, A)) \quad . \tag{2}$$

Moreover, by Proposition 1.3, we have that

(i) a mono $R \rightarrowtail A$ is $\neg\neg$-closed if and only if, for every C in $\mathbb{C}$ and every $h: C \longrightarrow A$ in $\widehat{\mathbb{C}}$, if $hx: 1 \longrightarrow A$ factors through $R \rightarrowtail A$ for all global elements $x: 1 \longrightarrow C$, then so does h;

(ii) a mono $R \rightarrowtail A$ is $\neg\neg$-dense if and only if every global element $1 \longrightarrow A$ factors through it;

(iii) an object A is $\neg\neg$-separated if and only if, for every C in $\mathbb{C}$ and every $f, g: C \longrightarrow A$ in $\widehat{\mathbb{C}}$, if $fx = gx$ for all global elements $x: 1 \longrightarrow C$, then $f = g$.

2 Notions of domain in SDT

A model of Synthetic Domain Theory is an elementary topos with a natural numbers object equipped with a chosen object Σ, whose rôle is to determine the "computational properties" of other objects in the topos, and with a lifting monad (L, η, μ) which provides the notion of "computation". In all the studied models, the computational properties *are* the possible computations at the terminal object; *i.e.* $\Sigma = L1$. Henceforth we will restrict attention to this situation.

The intuition is that the ambient topos embodies some internal computational universe and, as a consequence, all functions between domain-like objects are intrinsically *continuous*. In the literature there are at least four related suggestions for selecting the objects in the topos which are to be thought of as domains, see [10, 11, 12, 15, 16, 22, 23, 26]. They all concentrate on abstracting peculiar properties of Σ: they range from asking as many as possible (repleteness) to as few as acceptable (well-completeness). In all the various cases, it is guaranteed that the chosen domains are closed under the action of the monad of computations. We refer the reader to the literature cited above and to [24] for more on the subject of SDT. Here we recall some of the basic definitions and the facts we need in the paper.

The underlying functor L of a lifting monad always has an initial algebra and a final coalgebra [11, 13]; call them $\iota: LI \longrightarrow I$ and $\varphi: F \longrightarrow LF$ respectively. By a lemma noticed by Lambek, ι and φ are isomorphisms. Let $c: I \longrightarrow F$ be the canonical map determined by inverting one of the two and recalling universality of the other; *i.e.* the unique map making the following diagram commute

$$\begin{array}{ccc} LI & \xrightarrow{Lc} & LF \\ {\scriptstyle\iota}\downarrow{\scriptstyle\cong} & & {\scriptstyle\cong}\uparrow{\scriptstyle\varphi} \\ I & \xrightarrow[c]{} & F \end{array} \quad .$$

The main infinitary axiom of a model of SDT is that Σ *thinks* that the initial L-algebra and the final L-coalgebra coincide. By this we mean the Completeness Axiom of [15, 16]: Σ is internally orthogonal to $c: I \longrightarrow F$; that is,

$$\Sigma^c: \Sigma^F \longrightarrow \Sigma^I \text{ is an iso} \quad .$$

We shall refer to a topos with a lifting monad in which the completeness axiom holds as a *model of SDT*. In models of SDT, various intrinsic notions of "domain" are available; we review three of them below.

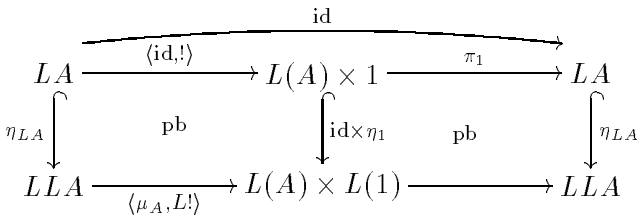
Completeness and well-completeness. Recall from [15, 16] that an object A is *complete* if it is internally orthogonal to $c\colon I \longrightarrow F$; it is *well-complete* if LA is complete. As in a model of SDT the terminal object is well-complete, we have the following proposition, *cf.* [16, 7].

Proposition 2.1 (i) *Well-complete objects are closed under lifting,*

(ii) *a well-complete object is complete,*

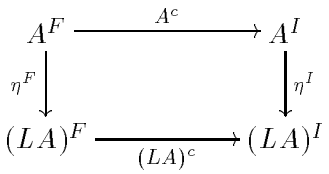
(iii) *an object is well-complete if and only if it is orthogonal to all pullbacks of* $c\times\mathrm{id}\colon I\times X \longrightarrow F\times X$ *along an admissible mono.*

Proof: (i) The pullback diagram

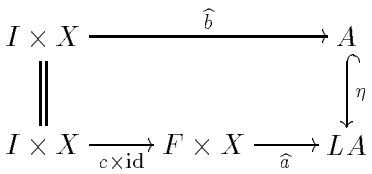


shows that LLA is a retract of the product of the complete objects LA and $L1$. Hence LLA is also complete; see [24].

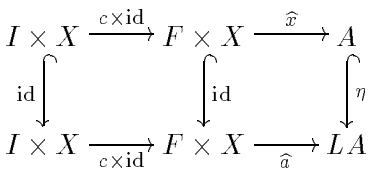
(ii) We show that the commutative diagram



is a pullback. Hence, if the bottom arrow is iso then so is the top one. Indeed, consider a span $(LA)^F \xleftarrow{a} X \xrightarrow{b} A^I$. Using the classifying property of η one sees that the condition of commutativity $\eta^I \circ b = (LA)^c \circ a$ translates as the commutativity of the following diagram

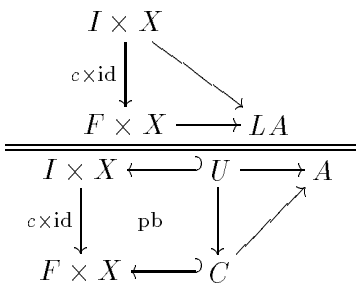


where $\widehat{(-)}$ denotes the exponential transpose. We are interested in determining the partial map represented by $\widehat{a}$; the one determined by $\widehat{a}\circ(c\times\mathrm{id})$ is $\widehat{b}\colon I\times X \longrightarrow A$ defined on $\mathrm{id}_{I\times X}$. The fact that $L1$ is complete means that there is a bijective correspondence between monos classified by $L1$ on $I\times X$ and those on $F\times X$ given by pulling back. Hence the only mono classified by $L1$ which pulls back to the identity on $I\times X$ is the identity on $F\times X$. Thus the diagram



commutes for a unique $x : X \longrightarrow A^F$ which is what we wanted.

(iii) By Proposition 1.1 and the universal property of lifting monads, one has only to check that there is a bijective correspondence between commutative triangles into LA and commutative triangles on pullbacks along an admissible mono as follows



This follows again from the completeness of $L1$. □

Extensional Σ-spaces. A useful condition to require on a (well-)complete object is that of being an *extensional Σ-space*. Indeed, in a model of SDT, complete Σ-spaces form a full reflective exponential ideal of the ambient topos closely resembling standard domains, *cf.* [20, 10].

We recall the relevant definitions. An object A is a *Σ-space* (with respect to an object Σ in a cartesian closed category) if the map

$$\tau_A \stackrel{\mathrm{def}}{=} \lambda x : A.\, \lambda u : \Sigma^A.\, u(x) : A \longrightarrow \Sigma^{(\Sigma^A)}$$

is a mono. Further, in a topos, a Σ-space A is said to be *extensional* if τ_A is $\neg\neg$-closed.

An important example of Σ-spaces, encompassing T_0 topological spaces, is obtained in (cartesian closed extensions of) the category of topological spaces and continuous functions by choosing Σ to be the Sierpinski space, see [25].

Repleteness. The definition of repleteness fits into a general scheme to produce reflective subcategories [9, 12, 1]. In a model of SDT, an object is *replete* [11, 26] (with respect to Σ) if it is orthogonal to all those maps $f : D \longrightarrow C$ such that $\Sigma^f : \Sigma^C \longrightarrow \Sigma^D$ is an iso. Roughly speaking, the replete objects are those objects which are completely recoverable from their Σ-definable properties.

Replete objects are closed under lifting (see [11]) and hence, by the completeness axiom, are well-complete.

3 Domains in $\mathcal{H}$

Two models of SDT have been introduced in [8]; the model $\boldsymbol{\mathcal{H}}$ was shown to embed the category ω-**Cpo** of ω-complete posets and ω-continuous functions.

The model $\boldsymbol{\mathcal{H}}$. The model $\boldsymbol{\mathcal{H}}$ of SDT is the topos of sheaves for the canonical topology on the monoid $\mathbf{L}$ of ω-continuous endomorphisms on $\overline{\omega}$ (the infinite countable chain with a top: $0 \leq 1 \leq \cdots \leq n \leq \cdots \leq \infty$). Though this is a complete definition, it is useful to give some more intuitions about it.

A basis for the *canonical topology* on $\mathbf{L}$ (see [8]) is given by the families R of endomorphisms on $\overline{\omega}$ such that

(i) R is non-empty,

(ii) every pair of elements in $\overline{\omega}$ is in the image of some function in R, and

(iii) for every countably infinite chain in $\overline{\omega}$ there is a function $f \in R$ such that $\mathrm{im}(f)$ contains an infinite subchain of it.

More generally, we have the following result.

Lemma 3.1 *Let G be a dense generating set in ω-**Cpo**, and write **G** for the full subcategory of ω-**Cpo** determined by G. A basis for the canonical topology $\mathrm{Can}_{\mathbf{G}}$ on **G** is given by the families $R \in \mathrm{Can}_{\mathbf{G}}(D)$ such that*

(i) *$R = \emptyset$ implies $D = 0$,*

(ii) *for every $x \leq y$ in D there exists an ω-continuous function $f\colon D' \longrightarrow D$ in R and $x' \leq y'$ in D' such that $f(x') = x$ and $f(y') = y$,*

(iii) *for every countably infinite chain $\langle x_n \rangle$ in D there exists an ω-continuous function $f\colon D' \longrightarrow D$ in R and a chain $\langle x'_n \rangle$ in D' such that the chain $\langle f(x'_n) \rangle$ contains an infinite subchain of $\langle x_n \rangle$.*

There is a wide collection of sites of definition for $\boldsymbol{\mathcal{H}}$.

Theorem 3.2 *Let G be a dense generating set in ω-**Cpo**, and write **G** for the full subcategory of ω-**Cpo** determined by G. Then $\boldsymbol{\mathcal{H}}$ is equivalent to the topos of sheaves on **G** for the canonical topology $\mathrm{Can}_{\mathbf{G}}$.*

PROOF: Given a site $(\mathbb{C}, J)$ and a full subcategory $\mathbb{A}$ of $\mathbb{C}$, write $J^{(\mathbb{A})}$ for the topology on $\mathbb{A}$ specified by declaring a sieve S to be in $J^{(\mathbb{A})}$ if and only if the sieve generated by S in $\mathbb{C}$ is in J.

Using Lemma 3.1, and writing $\overline{\mathbf{G}}$ for the full subcategory of ω-**Cpo** determined by $G \cup \{\overline{\omega}\}$, we have that

$$\mathrm{Can}_{\overline{\mathbf{G}}}^{(\mathbf{L})} = \mathrm{Can}_{\mathbf{L}} \quad \text{and} \quad \mathrm{Can}_{\overline{\mathbf{G}}}^{(\mathbf{G})} = \mathrm{Can}_{\mathbf{G}} \quad .$$

And a direct application of the "Comparison Lemma" in [17, page 588] yields the following equivalence of categories

$$\mathrm{sh}(\mathbf{L}, \mathrm{Can}_{\mathbf{L}}) \simeq \mathrm{sh}(\overline{\mathbf{G}}, \mathrm{Can}_{\overline{\mathbf{G}}}) \simeq \mathrm{sh}(\mathbf{G}, \mathrm{Can}_{\mathbf{G}}) \quad . \qquad \square$$

It follows that other sites of definition for $\boldsymbol{\mathcal{H}}$ are the category **F** of ω-continuous functions between finite products of $\overline{\omega}$ endowed with its canonical topology and those obtained by splitting the idempotents of **L** and **F**; the first splitting $\mathrm{S}(\mathbf{L})$ gives the full subcategory of ω-**Cpo** consisting of the finite non-empty ordinals and $\overline{\omega}$. The different sites of definition for $\boldsymbol{\mathcal{H}}$ give us various presentations of the sheaves which are useful, for example, to give succinct definitions (in the case of **L**) or to grasp the intrinsic structure of the sheaves (in the case of $\mathrm{S}(\mathbf{F})$).

Cpos. As the monoid $\mathbf{L}$ is a dense generator in ω-$\mathbf{Cpo}$, we have a "Yoneda embedding" ω-$\mathbf{Cpo} \hookrightarrow \widehat{\mathbf{L}}$. This embedding cuts down to an embedding $\mathrm{Y}{:}\omega$-$\mathbf{Cpo} \hookrightarrow \boldsymbol{\mathcal{H}}$ mapping an ω-cpo D to the *sheaf of $\overline{\omega}$-continuous paths* $\mathrm{Y}(D)$ defined as ω-$\mathbf{Cpo}(J\,-, D)$, where J denotes the inclusion of $\mathbf{L}$ into ω-$\mathbf{Cpo}$. The (closure under isomorphisms of the) image of the embedding Y produces a reflective exponential ideal of $\boldsymbol{\mathcal{H}}$ (consult [8] for details). We shall refrain from writing Y when this causes no confusion.

Many of the investigations to follow may be carried out in the topos of presheaves on any small dense generator in ω-$\mathbf{Cpo}$ rather than in $\boldsymbol{\mathcal{H}}$. However, the topos $\boldsymbol{\mathcal{H}}$ provides a more accurate view of ω-$\mathbf{Cpo}$. Indeed, the embedding $\mathrm{Y}{:}\omega$-$\mathbf{Cpo} \hookrightarrow \boldsymbol{\mathcal{H}}$ preserves limits and as many colimits as possible, as follows from Theorem 3.2, including coproducts, stable filtered unions (*i.e.* colimiting cones of monos), and stable coequalisers of kernel pairs. Moreover, the characterizations of ω-$\mathbf{Cpo}$ in $\boldsymbol{\mathcal{H}}$ given in Section 4 does not hold in the presheaf toposes.

The lifting monad. The model $\boldsymbol{\mathcal{H}}$ admits a lifting monad extending that of ω-$\mathbf{Cpo}$ and $\widehat{\boldsymbol{\Delta}}$. The construction is by left Kan extension:

$$\begin{array}{ccc} \omega\text{-}\mathbf{Cpo} & \xhookrightarrow{\mathrm{Y}} & \boldsymbol{\mathcal{H}} \\ {\scriptstyle (-)_\perp}\downarrow & \overset{\mathrm{Lan}}{\cong} & \downarrow{\scriptstyle L} \\ \omega\text{-}\mathbf{Cpo} & \xhookrightarrow[\mathrm{Y}]{} & \boldsymbol{\mathcal{H}} \end{array} .$$

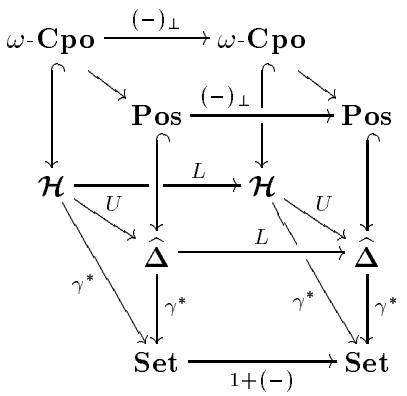
Explicitly, one can describe L as follows: for a sheaf A,

- $(LA)(\overline{\omega}) = \{\perp\} \uplus (\mathbb{N} \times A(\overline{\omega}))$; and
- for f in $\mathbf{L}$, the action $(LA)(f)$ in $\mathbf{Set}$ is given by the map

$$\begin{array}{rcll} \perp & \longmapsto & \perp & \\ (n,x) & \longmapsto & \perp & , \text{if } f^{-1}(\uparrow(n)) = \emptyset \\ (n,x) & \longmapsto & (m, A(\phi)(x)) & , \text{if } \begin{array}{ccc} \overline{\omega} & \xrightarrow{\phi} & \overline{\omega} \\ {\scriptstyle \uparrow(m)}\downarrow & \mathrm{pb} & \downarrow{\scriptstyle \uparrow(n)} \\ \overline{\omega} & \xrightarrow[f]{} & \overline{\omega} \end{array} \end{array}$$

where, for $\ell \in \mathbb{N}$, $\uparrow(\ell)$ denotes the admissible mono $\overline{\omega} \hookrightarrow \overline{\omega}{:}\, i \longmapsto i + \ell$ with image the basic Scott-open $\{\, i \in \overline{\omega} \mid \ell \leq i \,\}$.

The diagram below, where $U{:}\boldsymbol{\mathcal{H}} \longrightarrow \widehat{\boldsymbol{\Delta}}$ is the forgetful functor induced by the embedding $\boldsymbol{\Delta} \hookrightarrow \mathrm{S}(\mathbf{L})$, summarizes the relationship between the various lifting monads introduced so far.



Final L-coalgebra and initial L-algebra. The cone $\overline{\omega} \dashrightarrow \langle L^n 1 \leftarrow L^n \Sigma \rangle_n$ given by the diagram

$$\begin{array}{ccccccccccc}
\overline{\omega} & \xrightarrow[\cong]{\varphi} & L\overline{\omega} & \xrightarrow[\cong]{L\varphi} & L^2\overline{\omega} & \longrightarrow \cdots \longrightarrow & L^n\overline{\omega} & \xrightarrow[\cong]{L^n\varphi} & \cdots \\
{\scriptstyle !}\downarrow & & {\scriptstyle L!}\downarrow & & {\scriptstyle L^2!}\downarrow & & {\scriptstyle L^n!}\downarrow & & \\
1 & \xleftarrow[!]{} & \Sigma & \xleftarrow[L!]{} & \Sigma_2 & \longleftarrow \cdots \longleftarrow & \Sigma_n & \xleftarrow[L^n!]{} & \cdots
\end{array}$$

is limiting in ω-**Cpo**, and hence also in $\boldsymbol{\mathcal{H}}$. Thus, $\overline{\omega} \xrightarrow[\cong]{\varphi} L\overline{\omega}$ is a final L-coalgebra. We remark that the final L-coalgebra can be described as a retract of Σ^{N} (the object of admissible subsets of the natural numbers object N) as follows

$$\{p \in \Sigma^{\mathsf{N}} \mid \forall n \in \mathsf{N}.\, p_n \Longrightarrow p_{n+1}\} \rightarrowtail \Sigma^{\mathsf{N}} \quad ,$$

see [11].

The initial L-algebra $LI \xrightarrow{\cong} I$ is the smallest L-subalgebra of the (inverse of the) final L-coalgebra, see [13] for an explicit description. In our situation, it is obtained from the colimit of the diagram

$$0 \xrightarrow{!} 1 \xrightarrow{L!} \Sigma \longrightarrow \cdots \longrightarrow \Sigma_n \xrightarrow{L^{n+1}!} \cdots \tag{3}$$

(which coincides in $\boldsymbol{\mathcal{H}}$ and $\widehat{\mathbf{L}}$) because $L\colon \boldsymbol{\mathcal{H}} \longrightarrow \boldsymbol{\mathcal{H}}$ preserves non-empty connected colimits that are computed as in $\widehat{\mathbf{L}}$. Thus, the sheaf I can be explicitly described as the subobject of $\overline{\omega}$ with underlying set

$$\{\, f \in \omega\text{-}\mathbf{Cpo}(\overline{\omega}, \overline{\omega}) \mid f(\infty) < \infty \,\} \quad ,$$

see [8].

It follows that the canonical map $c\colon I \longrightarrow \overline{\omega}$ is the unique mediating morphism from the colimit of (3) to the cone $\langle L^n 0 \to L^n 1 \rangle_n \dashrightarrow \overline{\omega}$ given by the diagram

$$\begin{array}{ccccccccccc}
0 & \xrightarrow{!} & 1 & \xrightarrow{L!} & \Sigma & \longrightarrow \cdots \longrightarrow & \Sigma_n & \xrightarrow{L^{n+1}!} & \cdots \\
{\scriptstyle !}\downarrow & & {\scriptstyle L!}\downarrow & & {\scriptstyle L^2!}\downarrow & & {\scriptstyle L^{n+1}!}\downarrow & & \\
\overline{\omega} & \xleftarrow[\varphi^{-1}]{\cong} & L\overline{\omega} & \xleftarrow[L\varphi^{-1}]{\cong} & L^2\overline{\omega} & \longleftarrow \cdots \longleftarrow & L^{n+1}\overline{\omega} & \xleftarrow[L^{n+1}\varphi^{-1}]{\cong} & \cdots
\end{array} \quad . \tag{4}$$

Completeness and well-completeness. The completeness axiom is satisfied in $\boldsymbol{\mathcal{H}}$ because of the following general result.

Proposition 3.3 *Consider a diagram Γ in ω-**Cpo**. For $\alpha\colon \Gamma \dashrightarrow A$ colimiting in $\boldsymbol{\mathcal{H}}$ and $\beta\colon \Gamma \dashrightarrow B$ colimiting in ω-**Cpo**, let $f\colon A \longrightarrow B$ be the unique mediating morphism in $\boldsymbol{\mathcal{H}}$ from α to β. In $\boldsymbol{\mathcal{H}}$, every ω-cpo is internally orthogonal to f.*

Proof: For an ω-cpo D, the cone $D^\alpha\colon D^A \dashrightarrow D^\Gamma$ is limiting in $\boldsymbol{\mathcal{H}}$; and the cone $D^\beta\colon D^B \dashrightarrow D^\Gamma$ is limiting in ω-**Cpo**, and hence also in $\boldsymbol{\mathcal{H}}$. Since f mediates between α and β, it follows that D^f mediates between D^β and D^α; and so it is an iso. □

It is furthermore interesting to note that in $\boldsymbol{\mathcal{H}}$ the notions of completeness and well-completeness coincide.

Theorem 3.4 *In $\boldsymbol{\mathcal{H}}$, complete objects are well-complete.*

PROOF: Since the monoid $\mathbf{L}$ is a dense generator in $\boldsymbol{\mathcal{H}}$ (and, in a topos, colimits are stable under pullbacks), by Proposition 2.1 (iii), an object is well-complete if and only if it is orthogonal to all pullbacks of $c \times \mathrm{id}\colon I \times \overline{\omega} \longrightarrow \overline{\omega} \times \overline{\omega}$ along an admissible mono. This result follows from the characterization of these pullbacks in the lemma below (using again that colimits are stable under pullbacks). □

Lemma 3.5 (i) *In $\boldsymbol{\mathcal{H}}$, every admissible mono $U \hookrightarrow \overline{\omega} \times \overline{\omega}^n$ ($n \in \mathbb{N}$) is a (finite) colimit of admissible monos of the form $\overline{\omega} \times \overline{\omega}^n \hookrightarrow \overline{\omega} \times \overline{\omega}^n$.*

(ii) *Let*

$$\begin{array}{ccc} V & \xrightarrow{z} & \overline{\omega} \times \overline{\omega}^n \\ \downarrow & & \downarrow \\ I \times \overline{\omega}^n & \xrightarrow[c \times \mathrm{id}]{} & \overline{\omega} \times \overline{\omega}^n \end{array}$$

be a pullback in $\boldsymbol{\mathcal{H}}$. Then, we have a canonical isomorphism $V \cong I \times \overline{\omega}^n$ such that the composite $I \times \overline{\omega}^n \cong V \xrightarrow{z} \overline{\omega} \times \overline{\omega}^n$ equals $c \times \mathrm{id}$.

PROOF: We only consider the case $n = 1$, which is the one needed in the theorem above. The general case is proved similarly.

(i) Admissible subsets of $\overline{\omega} \times \overline{\omega}$ are pullbacks of $\top\colon 1 \hookrightarrow \Sigma$ and, since ω-**Cpo** is closed under limits in $\boldsymbol{\mathcal{H}}$, these can be computed in ω-**Cpo**. Thus an admissible subset U of $\overline{\omega} \times \overline{\omega}$ is a Scott-open subset. The family

$$\{\, \overline{\omega} \times \overline{\omega} \xhookrightarrow{\uparrow(\bigvee S)} U \;\mid\; \emptyset \neq S \subseteq_{\mathrm{fin}} M \,\} \quad ,$$

where M is the set of minimal elements of U, is a cover in the canonical topology (on a dense generator containing U) and presents $U = \bigcup_{p \in M} \uparrow(p)$ as a (finite) colimit in $\boldsymbol{\mathcal{H}}$.

(ii) An admissible subset V of $I \times \overline{\omega}$ in $\boldsymbol{\mathcal{H}}$ is an increasing chain $\langle V_i \rangle$ of Scott-open subsets as in the following diagram.

$$\begin{array}{ccccccc} 0 = V_0 & \rightarrowtail & V_1 & \rightarrowtail & V_2 & \rightarrowtail & \cdots \\ \downarrow & \mathrm{pb} & \downarrow & \mathrm{pb} & \downarrow & & \\ 0 \times \overline{\omega} & \xrightarrow[!\times\mathrm{id}]{} & 1 \times \overline{\omega} & \xrightarrow[L!\times\mathrm{id}]{} & \Sigma \times \overline{\omega} & \rightarrowtail & \cdots \end{array}$$

If V is obtained by pullback along $\overline{\omega} \times \overline{\omega} \xhookrightarrow{\uparrow(i,j)} \overline{\omega} \times \overline{\omega}$ ($i, j \in \mathbb{N}$), then $V_k = 0$, for all $0 \leq k \leq i$, and $V_{k+i+1} = \uparrow(i,j) \cong \Sigma_k \times \overline{\omega}$, for all $k \geq 0$. Hence the claim follows. □

Double-negation topology. We instantiate the descriptions of $\neg\neg$-closure, $\neg\neg$-density, and $\neg\neg$-separation from Section 1 '§ Double-negation topology' in the topos $\boldsymbol{\mathcal{H}}$.

Corollary 3.6 (i) *A mono $R \rightarrowtail A$ in $\boldsymbol{\mathcal{H}}$ is $\neg\neg$-closed if and only if, for every $h\colon \overline{\omega} \longrightarrow A$ in $\boldsymbol{\mathcal{H}}$, if $hx\colon 1 \longrightarrow A$ factors through $R \rightarrowtail A$ for all global elements $x\colon 1 \longrightarrow C$, then so does h.*

(ii) *A mono $R \rightarrowtail A$ in $\boldsymbol{\mathcal{H}}$ is $\neg\neg$-dense if and only if every global element $1 \longrightarrow A$ factors through it.*

(iii) *A sheaf A in $\boldsymbol{\mathcal{H}}$ is $\neg\neg$-separated if and only if, for all $f, g\colon \overline{\omega} \longrightarrow A$ in $\boldsymbol{\mathcal{H}}$, if $fx = gx$ for all global elements $x\colon 1 \longrightarrow \overline{\omega}$, then $f = g$.*

Lemma 3.7 *For a complete object in $\boldsymbol{\mathcal{H}}$, the following properties are equivalent:*

(i) *$\neg\neg$-separation.*

(ii) *Σ-separation.*

PROOF: (i) $\Longrightarrow$ (ii) Because, by Corollary 3.6 (ii), the mono $d\colon 2 \longrightarrow \Sigma$ is $\neg\neg$-dense.

(ii) $\Longrightarrow$ (i) For a Σ-separated sheaf A, first observe that, for $f, g\colon \Sigma_n \longrightarrow A$, we have that

$$\forall\, x\colon 1 \longrightarrow \Sigma_n.\ fx = gx\colon 1 \longrightarrow A \quad \Longrightarrow \quad f = g\colon \Sigma_n \longrightarrow A \quad . \tag{5}$$

Indeed, assuming the left hand side of (5), the identities

$$f\alpha = g\alpha\colon \Sigma \longrightarrow A,$$

hold for all $\alpha\colon \Sigma \longrightarrow \Sigma_n$ (because $f\alpha\bot = g\alpha\bot$ and $f\alpha\top = g\alpha\top$); and, since the family of all maps from Σ to Σ_n is a cover in the canonical topology (on $\mathrm{S}(\mathbf{L})$), the right hand side of (5) follows.

Finally, note that if in addition A is complete then, for $f, g\colon \overline{\omega} \longrightarrow A$, we have that

$$\begin{aligned}
&\forall\, x\colon 1 \longrightarrow \overline{\omega}.\ fx = gx\colon 1 \longrightarrow A \\
&\quad \Longrightarrow\ \forall\, n \in \mathbb{N}.\ \forall\, x\colon 1 \longrightarrow \Sigma_n.\ f\iota_n x = g\iota_n x\colon 1 \longrightarrow A \\
&\qquad\quad \text{, where } \langle \iota_n\colon \Sigma_n \longrightarrow \overline{\omega} \rangle \text{ denotes the cone in (4)} \\
&\quad \Longrightarrow\ \forall\, n \in \mathbb{N}.\ f\iota_n = g\iota_n\colon \Sigma_n \longrightarrow A \\
&\qquad\quad \text{, by (5)} \\
&\quad \Longrightarrow\ f = g\colon \overline{\omega} \longrightarrow A \\
&\qquad\quad \text{, by completeness} \quad .
\end{aligned}$$

□

Cpos revisited. Let $\mathbf{D}$ be the full subcategory of $\boldsymbol{\mathcal{H}}$ consisting of the complete, path-transitive, Σ-separated sheaves; and let $\mathrm{CE}\Sigma\mathrm{S}(\boldsymbol{\mathcal{H}})$ be the full subcategory of complete extensional Σ-spaces in $\boldsymbol{\mathcal{H}}$.

Corollary 3.8 *The embedding ω-$\mathbf{Cpo} \hookrightarrow \boldsymbol{\mathcal{H}}$ factors through the embeddings*

(i) *$\mathbf{D} \hookrightarrow \boldsymbol{\mathcal{H}}$, and*

(ii) *$\mathrm{CE}\Sigma\mathrm{S}(\boldsymbol{\mathcal{H}}) \hookrightarrow \boldsymbol{\mathcal{H}}$.*

PROOF: To check completeness, note that the cone (4) is colimiting in ω-$\mathbf{Cpo}$, and apply Proposition 3.3.

(i) To check path-transitivity and Σ-separation, note that the diagrams

$$\begin{array}{ccc} 1 & \xrightarrow{\bot} & \Sigma \\ {\scriptstyle\top}\downarrow & & \downarrow{\scriptstyle\eta_\Sigma} \\ \Sigma & \xrightarrow[L\bot]{} & \Sigma_2 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} 2 & \xrightarrow{d} & \Sigma \\ {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle \mathrm{id}} \\ \Sigma & \xrightarrow[\mathrm{id}]{} & \Sigma \end{array}$$

are pushouts in ω-$\mathbf{Cpo}$, and apply Proposition 3.3.

(ii) For an ω-cpo D, since elements in D are separated by open subsets, the ω-continuous function $\tau_D\colon D \longrightarrow \Sigma^{(\Sigma^D)}$ is a mono. Moreover, by Corollary 3.6 (i), we have that τ_D is $\neg\neg$-closed if and only if every ω-continuous function $\overline{\omega} \longrightarrow \Sigma^{(\Sigma^D)}$ with image in (the image of) D (under τ_D), restricts to an ω-continuous function $\overline{\omega} \longrightarrow D$. This is easily seen using the fact that τ_D is order reflecting. □

4 Internal characterizations of ω-Cpo

We show that the embedding $\mathrm{Y}\colon \omega\text{-}\mathbf{Cpo} \hookrightarrow \mathbf{D}$ is an equivalence of categories.

Let A be a sheaf in $\mathbf{D}$. Henceforth we will consider sheaves defined on the splitting $\mathrm{S}(\mathbf{F})$ (because, for instance, it will be useful to compute the values of A at objects of the form $\Sigma_m \times \Sigma_n$). Since, by Lemma 3.7, A is $\neg\neg$-separated, the natural family (2)

$$A(C) \xrightarrow{\cong} \boldsymbol{\mathcal{H}}(C, A) \xrightarrow{\gamma^*_{C,A}} \mathbf{Set}(\boldsymbol{\mathcal{H}}(1, C), \boldsymbol{\mathcal{H}}(1, A)) \xrightarrow{\cong} \mathbf{Set}(|C|, |A|) \quad ,$$

where $|C|$ denotes the underlying set of C and $|A| \stackrel{\text{def}}{=} A(1)$, is injective. We proceed to characterize the images of these monos for $C = \overline{\omega} \times \Sigma_n$.

As A is Σ-separated, we have that A^Σ is the path relation P_A on A. The path relation is always reflexive, and it is transitive because A is path-transitive. Hence

$$O_A = (|A|, \sqsubseteq_A) \stackrel{\text{def}}{=} (A(1), A^\Sigma(1))$$

is a preorder and we have the bijective correspondence

$$A(\Sigma) \;\cong\; \boldsymbol{\mathcal{H}}(\Sigma, A) \;\cong\; \boldsymbol{\mathcal{H}}(1, A^\Sigma) \;\cong\; \sqsubseteq_A \;\cong\; \mathbf{Preo}(\Sigma, O_A) \quad ,$$

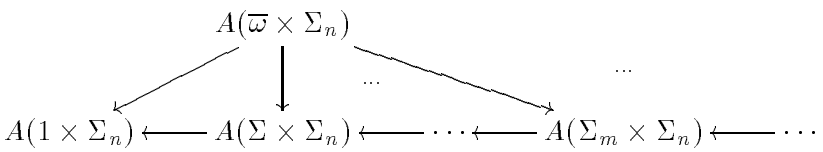
where $\mathbf{Preo}$ is the category of preorders and monotone functions, exhibiting the preorder O_A as an *intrinsic notion of approximation* on the sheaf A.

Consider the monoid $\mathbf{S}$ of endomorphisms of Σ as a full subcategory of $\mathrm{S}(\mathbf{F})$, and let $D_{m,n}\colon \mathbf{S} \downarrow (\Sigma_m \times \Sigma_n) \longrightarrow \boldsymbol{\mathcal{H}}$ be the domain-projection functor from the comma category; *i.e.* the diagram of all paths of $\Sigma_m \times \Sigma_n$. Using the fact that the family of all maps in $\mathrm{S}(\mathbf{F})$ from Σ to $\Sigma_m \times \Sigma_n$ is a cover in the canonical topology, one sees that

$$\Sigma_m \times \Sigma_n \cong \operatorname{colim} D_{m,n}$$

in $\boldsymbol{\mathcal{H}}$. Hence, an element of $A(\Sigma_m \times \Sigma_n)$ is a function $|\Sigma_m \times \Sigma_n| \longrightarrow |A|$ mapping paths to paths, in other words a map in $\mathbf{Preo}(\Sigma_m \times \Sigma_n, O_A)$.

Since A is complete it follows that, for $n \in \mathbb{N}$, the following cone



is limiting in $\mathbf{Set}$ (*cf.* [8]). Hence, the function

$$\begin{array}{rcl} A(\overline{\omega} \times \Sigma_n) & \longrightarrow & \mathbf{Preo}(\omega \times \Sigma_n, O_A) \\ x & \longmapsto & \langle x \circ \langle i, j \rangle \rangle_{(i,j)} \end{array} \tag{6}$$

is a bijection.

From now on, we identify $A(\overline{\omega} \times \Sigma_n)$ with $\mathbf{Preo}(\omega \times \Sigma_n, O_A)$. For instance, to describe a map $\overline{\omega} \longrightarrow A$ in $\boldsymbol{\mathcal{H}}$, or equivalently, by Yoneda, an element of $A(\overline{\omega})$, we give a chain in O_A.

We claim that $A(\infty)\colon A(\overline{\omega}) \longrightarrow A(1)$ yields a lub operator for ω-chains in O_A. Note that with the identifications just introduced, for every ω-chain $\langle x_i \rangle$ in O_A, we have that

$$\bigsqcup_A \langle x_i \rangle \;\stackrel{\text{def}}{=}\; A(\infty)\langle x_i \rangle \;=\; \langle x_i \rangle \circ \infty \quad . \tag{7}$$

Proposition 4.1 *For every ω-chain $\langle x_i \rangle$ in O_A, $\bigsqcup_A \langle x_i \rangle$ is an upper bound.*

PROOF: For $k \in \mathbb{N}$, consider the path $\langle k, \infty \rangle : \Sigma \longrightarrow \overline{\omega}$. Then,

$$\begin{array}{rcccl} x_k & = & \langle x_i \rangle \circ k & = & \langle x_i \rangle \circ \langle k, \infty \rangle \circ \bot \\ & \sqsubseteq_A & \langle x_i \rangle \circ \langle k, \infty \rangle \circ \top & = & \langle x_i \rangle \circ \infty \\ & = & \bigsqcup_A \langle x_i \rangle & & \end{array}$$

□

Moreover, the operator $\bigsqcup_A$ satisfies the following properties, *cf.* [5, 6].

Proposition 4.2

(Constant) *For $x \in |A|$, we have that $\bigsqcup_A \langle x \sqsubseteq_A \ldots \sqsubseteq_A x \sqsubseteq_A \ldots \rangle = x$.*

(Monotonicity) *If $\langle x_i \rangle$ and $\langle y_j \rangle$ are ω-chains in O_A such that $x_k \sqsubseteq_A y_k$ for all $k \in \mathbb{N}$, then $\bigsqcup_A \langle x_i \rangle \sqsubseteq_A \bigsqcup_A \langle y_j \rangle$.*

(Finality) *If $\langle x_i \rangle$ is an ω-chain in O_A and $f : \omega \longrightarrow \omega$ is monotone and unbounded, then $\bigsqcup_A \langle x_i \rangle = \bigsqcup_A \langle x_{f(i)} \rangle$.*

PROOF:

(Constant) By (7), one has

$$\bigsqcup_A \langle x \sqsubseteq_A \ldots \sqsubseteq_A x \sqsubseteq_A \ldots \rangle \;=\; \bigsqcup_A x \circ !_{\overline{\omega}} \;=\; x \circ !_{\overline{\omega}} \circ \infty \;=\; x \quad .$$

(Monotonicity) Let $\alpha : \overline{\omega} \times \Sigma \longrightarrow A$ correspond, by the bijection (6), to the monotone function $\omega \times \Sigma \longrightarrow O_A$ defined as

$$\langle i, j \rangle \longmapsto \begin{cases} x_i & \text{, if } j = \bot \\ y_i & \text{, if } j = \top \end{cases}$$

Then,

$$\begin{array}{rcccl} \bigsqcup_A \langle x_i \rangle & = & \alpha \circ \langle \mathrm{id}_{\overline{\omega}}, \bot \circ !_{\overline{\omega}} \rangle \circ \infty & = & \alpha \circ \langle \infty \circ !_\Sigma, \mathrm{id}_\Sigma \rangle \circ \bot \\ & \sqsubseteq_A & \alpha \circ \langle \infty \circ !_\Sigma, \mathrm{id}_\Sigma \rangle \circ \top & = & \alpha \circ \langle \mathrm{id}_{\overline{\omega}}, \top \circ !_{\overline{\omega}} \rangle \circ \infty \\ & = & \bigsqcup_A \langle y_i \rangle \quad . & & \end{array}$$

(Finality) Let $\overline{f} : \overline{\omega} \longrightarrow \overline{\omega}$ be f extended continuously by $\overline{f}(\infty) = \infty$. Then

$$\bigsqcup_A \langle x_i \rangle \;=\; \langle x_i \rangle \circ \infty \;=\; \langle x_i \rangle \circ \overline{f} \circ \infty \;=\; \bigsqcup_A \langle x_{f(i)} \rangle \quad .$$

□

Proposition 4.3 *For every ω-chain $\langle x_i \rangle$ in O_A, $\bigsqcup_A \langle x_i \rangle$ is a least upper bound.*

PROOF: Given an ω-chain $\langle x_i \rangle$ in O_A and x in $|A|$ such that $x_i \sqsubseteq_A x$, for all $i \in \mathbb{N}$, by (Monotonicity) and (Constant) we have that

$$\bigsqcup_A \langle x_i \rangle \;\sqsubseteq_A\; \bigsqcup_A \langle x \sqsubseteq_A \ldots \sqsubseteq_A x \sqsubseteq_A \ldots \rangle \;=\; x \quad .$$

□

Theorem 4.4 *The embedding ω-**Cpo** $\hookrightarrow$ **D** is an equivalence of categories.*

PROOF: The relation $\sqsubseteq_A$ is antisymmetric because, for $x \sqsubseteq_A y \sqsubseteq_A x$ in O_A,

$$\begin{array}{rcl} x & = & \bigsqcup_A \langle x \sqsubseteq_A \ldots \sqsubseteq_A x \sqsubseteq_A \ldots \rangle \\ & = & \bigsqcup_A \langle x \sqsubseteq_A y \sqsubseteq_A \ldots \sqsubseteq_A x \sqsubseteq_A y \sqsubseteq_A \ldots \rangle \\ & = & \bigsqcup_A \langle y \sqsubseteq_A \ldots \sqsubseteq_A y \sqsubseteq_A \ldots \rangle \\ & = & y \end{array}$$

using (Constant) and (Finality) twice. Thus, O_A is an ω-cpo.

We show that the bijection

$$\begin{array}{rcl} A(\overline{\omega}) & \xrightarrow{\cong} & \omega\text{-}\mathbf{Cpo}(\overline{\omega}, O_A) \\ \langle x_i \rangle & \longmapsto & \overline{\langle x_i \rangle} \end{array} \quad ,$$

where $\overline{\langle x_i \rangle}$ denotes the continuous extension of the ω-chain $\langle x_i \rangle$, is natural. This amounts to showing that, for every ω-chain $\langle x_i \rangle$ in O_A and every ω-continuous endomorphism f on $\overline{\omega}$,

$$\overline{\langle x_{f(i)} \rangle_i}\ (n) = \overline{\langle x_i \rangle_i}\ (f(n)) \tag{8}$$

for all $n \in \overline{\omega}$. For $n < \infty$, (8) holds trivially; thus, consider the case $n = \infty$:

(1) if $f(\infty) < \infty$ then

$$\overline{\langle x_{f(i)} \rangle}\ (\infty) = \bigsqcup_A \langle x_{f(i)} \rangle = x_{f(\infty)} = \overline{\langle x_i \rangle}\ (f(\infty)) \quad ;$$

(2) if $f(\infty) = \infty$ then

$$\overline{\langle x_{f(i)} \rangle}\ (\infty) = \bigsqcup_A \langle x_{f(i)} \rangle = \bigsqcup_A \langle x_i \rangle = \overline{\langle x_i \rangle}\ (\infty) = \overline{\langle x_i \rangle}\ (f(\infty)) \quad .$$

It follows that, $A \cong \mathrm{Y}(O_A)$ in $\boldsymbol{\mathcal{H}}$. □

Next we show that also the embedding $\omega\text{-}\mathbf{Cpo} \hookrightarrow \mathrm{CE\Sigma S}(\boldsymbol{\mathcal{H}})$ is an equivalence of categories.

Theorem 4.5 *The embedding $\omega\text{-}\mathbf{Cpo} \hookrightarrow \mathrm{CE\Sigma S}(\boldsymbol{\mathcal{H}})$ is an equivalence of categories.*

PROOF: By Theorem 4.4 and Lemma 3.7, we need only show that a complete extensional Σ-space A in $\boldsymbol{\mathcal{H}}$ is path-transitive and $\neg\neg$-separated.

It is obviously $\neg\neg$-separated since $\tau_A \colon A \longrightarrow \Sigma^{(\Sigma^A)}$ is a mono and $\Sigma^{(\Sigma^A)}$ is $\neg\neg$-separated (as so is Σ, by Corollary 3.6 (iii)).

Moreover, since the mono $t \colon \Sigma \vee \Sigma \longrightarrow \Sigma_2$ is $\neg\neg$-dense (as follows from Corollary 3.6 (ii)) and, in a topos, dense monos are internally orthogonal (in the sense of [9]) to closed monos, we have that the diagram

$$\begin{array}{ccc} A^{\Sigma_2} & \xrightarrow{\tau_A{}^{\Sigma_2}} & (\Sigma^{(\Sigma^A)})^{\Sigma_2} \\ {\scriptstyle A^t}\downarrow & & \downarrow{\scriptstyle (\Sigma^{(\Sigma^A)})^t} \\ A^{\Sigma \vee \Sigma} & \xrightarrow[\tau_A{}^{\Sigma \vee \Sigma}]{} & (\Sigma^{(\Sigma^A)})^{\Sigma \vee \Sigma} \end{array}$$

is a pullback in $\boldsymbol{\mathcal{H}}$. It follows that A is path-transitive, as so is $\Sigma^{(\Sigma^A)}$ because $\omega\text{-}\mathbf{Cpo}$ is an exponential ideal of $\boldsymbol{\mathcal{H}}$ (see [8]). □

Corollary 4.6 *The category $\omega\text{-}\mathbf{Cpo}$ is equivalent to the full subcategory of $\boldsymbol{\mathcal{H}}$ consisting of the complete and path-transitive Σ-spaces.*

5 ω-Cpo in $\mathcal{H}$

The logical interpretation of Theorem 4.4 gives the following description: the full subcategory ω-**Cpo** $\hookrightarrow \mathcal{H}$ consists of those sheaves A such that

(A is complete)
$$\Vdash \forall\, \sigma \in A^I.\ \exists!\, \overline{\sigma} \in A^{\overline{\omega}}.\ \sigma = \overline{\sigma} \circ c$$

(A is path-transitive)
$$\Vdash \forall\, \alpha, \beta \in A^{\Sigma}.\ \alpha\top = \beta\bot \Longrightarrow \exists!\, \gamma \in A^{\Sigma_2}.\ \alpha = \gamma \circ L\bot \ \wedge\ \beta = \gamma \circ \eta_{\Sigma}$$

(A is Σ-separated)
$$\Vdash \forall\, \alpha, \beta \in A^{\Sigma}.\ (\alpha\bot = \beta\bot \ \wedge\ \alpha\top = \beta\top) \Longrightarrow \alpha = \beta$$

We now observe that every ω-cpo $D = (|D|, \sqsubseteq_D)$ comes with an intrinsic notion of approximation $\leq_D$ and with a passage-to-the-limit operator $\bigvee_D$, making $(D, \leq_D, \bigvee_D)$ an *internal* ω-cpo in $\mathcal{H}$, with respect to which all maps are continuous.

Let $\leq_D \subseteq D \times D$ be the ω-cpo, D^{Σ}, with underlying set $\sqsubseteq_D$ ordered pointwise. As an object in $\mathcal{H}$, the sheaf $\leq_D$ has two internal descriptions; namely, as the *specialization order*

$$\Vdash \forall\, x, y \in D.\ x \leq_D y \Longleftrightarrow \forall\, u \in \Sigma^D.\ u(x) \Longrightarrow u(y)$$

which is the intrinsic order defined by the admissible subsets of D, and as the *path relation*

$$\Vdash \forall\, x, y \in D.\ x \leq_D y \Longleftrightarrow \exists\, \alpha \in D^{\Sigma}.\ \alpha(\bot) = x \wedge \alpha(\top) = y \quad .$$

Thus, $(D, \leq_D)$ is an internal poset in $\mathcal{H}$.

Now let $\mathrm{Chain}(D, \leq_D)$ be the object of countable ascending chains in $(D, \leq_D)$ defined internally as $\{\, \sigma \in D^{\mathsf{N}} \mid \forall\, n \in \mathsf{N}.\ \sigma_n \leq_D \sigma_{n+1} \,\}$. We have that

$$\mathrm{Chain}(D, \leq_D) \cong D^I$$

and so the lub-operator $\bigvee_D \colon \mathrm{Chain}(D, \leq_D) \longrightarrow D$ may be obtained as the composite

$$\mathrm{Chain}(D, \leq_D) \xrightarrow{\cong} D^I \xrightarrow{D^c} D^{\overline{\omega}} \xrightarrow{D^{\infty}} D \quad .$$

One can show that $(D, \leq_D, \bigvee_D)$ is an internal ω-cpo in $\mathcal{H}$.

Furthermore, for ω-cpos P and Q we have that

$$\Vdash \forall\, \varphi \in Q^P.\ \forall\, \sigma \in \mathrm{Chain}(P, \leq_P).\ \varphi(\textstyle\bigvee_P \sigma) = \bigvee_Q (\varphi \circ \sigma) \quad .$$

Thus, every map $P \longrightarrow Q$ in $\mathcal{H}$ is ω-continuous with respect to the intrinsic order.

6 Conclusions

We have obtained full embeddings

$$\mathrm{Rep}(\mathcal{H}) \hookrightarrow \omega\text{-}\mathbf{Cpo} \hookrightarrow \mathrm{WC}(\mathcal{H}) = \mathrm{C}(\mathcal{H}) \hookrightarrow \mathcal{H} \quad ,$$

where $\mathrm{Rep}(\mathcal{H})$, $\mathrm{WC}(\mathcal{H})$, and $\mathrm{C}(\mathcal{H})$ respectively denote the full subcategories of $\mathcal{H}$ consisting of the replete, the well-complete, and the complete objects. Further, we have given various characterizations of ω-**Cpo** in $\mathcal{H}$.

Thanks to the universal characterization of the category of replete objects as the smallest reflective exponential ideal containing Σ (see [11, 12]), we know that $\mathrm{Rep}(\boldsymbol{\mathcal{H}}) \simeq \mathrm{Rep}(\omega\text{-}\mathbf{Cpo})$. Thus, for the topos $\boldsymbol{\mathcal{H}}$ the internal notion of repleteness determines a good class of "standard" domains. All constructions on ω-cpos used in denotational semantics actually take place within the replete objects, see [24]. It is not known whether the notion of repleteness in $\boldsymbol{\mathcal{H}}$ (or equivalently in $\omega\text{-}\mathbf{Cpo}$) coincides with some existent notion of domain: we only know that there are non-replete ω-cpos because of an example worked out by Michael Makkai and the second author [18].

The inclusion $\omega\text{-}\mathbf{Cpo} \hookrightarrow \mathrm{WC}(\boldsymbol{\mathcal{H}})$ is strict, as it is easy to imagine examples of complete objects which are not path-transitive and/or Σ-separated. For instance consider the following.

(1) The subsheaf of Σ_2 with underlying set

$$\{\, f \in \omega\text{-}\mathbf{Cpo}(\overline{\omega}, \underline{3}) \mid \mathrm{im}(f) \subseteq \{i, i+1\} \text{ for some } 0 \leq i \leq 1 \,\}$$

is complete and Σ-separated but it is not path-transitive (neither does it has a transitive path relation).

(2) The sheaf with underlying set given by *ω-continuous graph homomorphisms* from $\overline{\omega}$ to the graph

,

with action given by precomposition, is complete and path-transitive but it is not Σ-separated.

(3) The sum of the sheaves in (1) and (2) above yields a complete object which is neither path-transitive nor Σ-separated.

Our characterizations of $\omega\text{-}\mathbf{Cpo}$ in $\boldsymbol{\mathcal{H}}$ (Theorems 4.4 and 4.5, and Corollary 4.6) shed light on the subtle differences between notions of domains in purely logical terms. The notions of repleteness and well-completeness mark two extremes: the latter is just the bare essential in order to get the internal logic working to yield the basic results about fixed-points, the former imposes the further requirement that *all* the properties of Σ true in the universe hold for a domain. Theorem 4.4 shows that in $\boldsymbol{\mathcal{H}}$, besides completeness, the only two extra properties of Σ required to characterize ω-cpos are path-transitivity and Σ-separation.

Our results establish a direct link between traditional methods in denotational semantics and the synthetic approach. It is important to note that the synthetic approach suggests different kinds of categories of domains. In the case of repleteness, it suggest a different (probably new), intrinsic notion of a good category of ω-cpos; in the case of well-completeness, it presents a natural extension of the traditional interpretation of fixed-point operators by lubs of ω-chains.

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References

[1] A. Bucalo and G. Rosolini. Repleteness and the associated sheaf. *J. Pure Appl. Alg.*, 116:147–151, 1998.

[2] A. Carboni, M.C. Pedicchio, and G. Rosolini, editors. *Category Theory '90*, volume 1488 of *Lectures Notes in Mathematics*, Como, 1991. Springer-Verlag.

[3] M. Coste. Logique d'ordre supérieur dans les topos élémentaires, Seminaire Bénabou Paris, 1974.

[4] M.P. Fiore. *Axiomatic Domain Theory in Categories of Partial Maps.* Cambridge University Press Distinguished Dissertations in Computer Science, 1996.

[5] M.P. Fiore. Enrichment and representation theorems for categories of domains and continuous functions. Manuscript available from `http://www.cogs.susx.ac.uk/users/marcelo/`, 1996.

[6] M. P. Fiore. An enrichment theorem for an axiomatisation of categories of domains and continuous functions. *Math. Structures Comput. Sci.*, 7:591–618, 1997

[7] M.P. Fiore and G.D. Plotkin. An extension of models of axiomatic domain theory to models of synthetic domain theory. In *Computer Science Logic*, volume 1258 of *Lectures Notes in Computer Science*, pages 129–149, Utrecht, 1996. Springer-Verlag.

[8] M. Fiore and G. Rosolini. Two models of Synthetic Domain Theory. *J. Pure Appl. Alg.*, 116:151–162, 1997.

[9] P.J. Freyd and G.M. Kelly. Categories of continuous functors I. *J. Pure Appl. Alg.*, 2:169–191, 1972.

[10] P.J. Freyd, P. Mulry, G. Rosolini, and D.S. Scott. Extensional PERs. *Inform. and Comput.*, 98:211–227, 1992.

[11] J.M.E. Hyland. First steps in Synthetic Domain Theory. In [2], pages 95–104.

[12] J.M.E. Hyland and E. Moggi. The S-replete construction. In [19], pages 96–116.

[13] M. Jibladze. A presentation of the initial lift-algebra. *J. Pure Appl. Alg.*, 116:185–198, 1997.

[14] P.T. Johnstone. *Topos Theory.* Academic Press, 1977.

[15] J.R. Longley. *Realizability Toposes and Language Semantics.* PhD thesis, Department of Computer Science, University of Edinburgh, 1995.

[16] J. Longley and A. Simpson. A uniform approach to domain theory in realizability models. *Math. Structures Comput. Sci.*, 7:469–505, 1997.

[17] S. Mac Lane and I. Moerdijk. *Sheaves in geometry and logic.* Springer-Verlag, 1992.

[18] M. Makkai and G. Rosolini. Studying repleteness in the category of cpos. *Electron. Notes Theor. Comput. Sci.*, 6:6 pages (electronic), 1997.

[19] D.H. Pitt, P.T. Johnstone, and A.M. Pitts, editors. *Category theory and Computer Science*, Lectures Notes in Computer Science, Cambridge, 1995. Springer-Verlag.

[20] B. Reus. *Program Verification in Synthetic Domain Theory.* Dokt.Diss., Lüdwig-Maximilians-Universität München, 1995. Shaker Verlag, Aachen.

[21] G. Rosolini. *Continuity and effectiveness in topoi.* PhD thesis, University of Oxford, 1986.

[22] G. Rosolini. Categories and effective computations. In D.H. Pitt, A. Poigné, and D.E. Rydeheard, editors, *Category theory and Computer Science*, volume 283 of *Lectures Notes in Computer Science*, pages 1–11, Edinburgh, 1987. Springer-Verlag.

[23] G. Rosolini. An ExPER model for QUEST. In S. Brookes, editor, *Mathematical Foundations of Programming Semantics*, volume 598 of *Lectures Notes in Computer Science*, Pittsburgh, 1992. Springer-Verlag.

[24] G. Rosolini. Notes on Synthetic Domain Theory, 1996. Manuscript available from `ftp://ftp.disi.unige.it`.

[25] G. Rosolini. Equilogical spaces and filter spaces, 1999. To appear.

[26] P. Taylor. The fixed point property in synthetic domain theory. In G. Kahn, editor, *Logic in Computer Science 6*, pages 152–160, Amsterdam, 1991. IEEE.