Notes on Combinatorial Functors
(Draft)

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Abstract

Taking a combinatorial view of presheaves, we relate the Schanuel topos, species of structure, analytic functors, and the object classifier topos.

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1 Combinatorial presheaves in $\mathbf{Set}^{\mathbb{I}}$

The identity

$$k^n = \sum_{i=0}^{n} S(n,i)\ i! \binom{k}{i} \tag{1}$$

where $S(n,i)$, a Stirling number of the second kind, is the number of partitions of an n element set into i blocks is well-known (see, *e.g.*, [Sta97]), and expresses the fact that, up to isomorphism, functions have a unique surjection-injection factorisation. Indeed, writing $\mathbb{F}$, $\mathbb{S}$ and $\mathbb{I}$ respectively for the categories of functions, surjective functions and injective functions between *finite cardinals*, this unique factorisation property amounts to the combinatorial bijection

$$\mathbb{F}(n,k) \cong \sum_{i} \mathbb{S}(n,i) \underset{\mathfrak{S}_i}{\otimes} \mathbb{I}(i,k) \tag{2}$$

where $\mathbb{S}(n,i)$ is regarded as a right $\mathfrak{S}_i$-set (with action given by post-composition) and, dually, $\mathbb{I}(i,m)$ is a regarded as a left $\mathfrak{S}_i$-set (with action given by pre-composition), and their tensor product is the quotient

$$\_\otimes = : \mathbb{S}(n,i) \times \mathbb{I}(i,k) \twoheadrightarrow \mathbb{S}(n,i) \underset{\mathfrak{S}_i}{\otimes} \mathbb{I}(i,k)$$

under the equivalence relation that identifies $(\varepsilon \cdot \sigma, \imath)$ and $(\varepsilon, \sigma \cdot \imath)$ for all $\varepsilon \in \mathbb{S}(n,i)$, $\sigma \in \mathfrak{S}_i$ and $\imath \in \mathbb{I}(i,k)$. Hence, we have the identity

$$\#\ \mathbb{S}(n,i) \underset{\mathfrak{S}_i}{\otimes} \mathbb{I}(i,k) = \frac{1}{i!}\ \#\mathbb{S}(n,i)\ \#\mathbb{I}(i,k) \quad ,$$

and (1) follows from the further well-known (again see, *e.g.*, [Sta97]) identities

$$\#\mathbb{F}(n,k) = k^n \ , \quad \#\mathbb{S}(n,i) = i!\ S(n,i)\ , \quad \#\mathbb{I}(i,k) = i! \binom{k}{i} \quad .$$

The family of tensor products $\left\{ \mathbb{S}(n,i) \underset{\mathfrak{S}_i}{\otimes} \mathbb{I}(i,k) \right\}_k$ admits a covariant action along injections as follows

$$\begin{array}{rcl} \left(\mathbb{S}(n,i) \underset{\mathfrak{S}_i}{\otimes} \mathbb{I}(i,k) \right) \times \mathbb{I}(k,\ell) & \longrightarrow & \mathbb{S}(n,i) \underset{\mathfrak{S}_i}{\otimes} \mathbb{I}(k,\ell) \\ \varepsilon \otimes \imath\ ,\ \jmath & \longmapsto & \varepsilon \otimes (\imath \cdot \jmath) \end{array}$$

making the mapping $k \longmapsto \sum_i \mathbb{S}(n,i) \underset{\mathfrak{S}_i}{\otimes} \mathbb{I}(i,k)$ into a presheaf in $\mathbf{Set}^{\mathbb{I}}$. Letting $\mathrm{N} \in \mathbf{Set}^{\mathbb{I}}$ be the inclusion $\mathbb{I} \longrightarrow \mathbf{Set}$, the bijection (2) yields a natural isomorphism

$$\mathrm{N}^n \cong \textstyle\sum_i \mathbb{S}(n,i) \underset{\mathfrak{S}_i}{\otimes} \mathbf{y}_{\mathbb{I}}(i) \text{ in } \mathbf{Set}^{\mathbb{I}}$$

that provides a combinatorial representation of $\mathrm{N}^n \in \mathbf{Set}^{\mathbb{I}}$ in terms of representables.

More generally, we introduce the following notion of combinatorial presheaf.

Definition 1.1 *A presheaf in* $\mathbf{Set}^{\mathbb{I}}$ *is* combinatorial *if it has a representation*

$$A_{!\mathbb{I}} = \sum_i A_i \underset{\mathfrak{S}_i}{\otimes} \mathbf{y}_{\mathbb{I}}(i)$$

for a family $A = \{A_i \times \mathfrak{S}_i \longrightarrow A_i\}_i$ *of representations of the finite symmetric groups.*

As before, the elements of $A_!(k) = \sum_i A_i \underset{\mathfrak{S}_i}{\otimes} \mathbb{I}(i,k)$ are denoted $x \otimes \imath$ ($x \in A_i$, $\imath : i \rightarrowtail k$) and are subject to the identity $(x \cdot \sigma) \otimes \imath = x \otimes (\sigma \cdot \imath)$ for all $\sigma \in \mathfrak{S}_i$. Moreover, with this notation, the action $A_!(k) \times \mathbb{I}(k,\ell) \longrightarrow A_!(\ell)$ is given by $(x \otimes \imath) \cdot \jmath \longmapsto x \otimes (\imath \cdot \jmath)$.

For an example of combinatorial presheaf, note that for $\mathrm{X} = (0, 1, 0, \cdots, 0, \cdots)$ in $\mathbf{Set}^{\mathbb{B}}$, we have that $\mathrm{X}_! = \mathrm{N}$ in $\mathbf{Set}^{\mathbb{I}}$.

Proposition 1.2 *The series of coefficients of a combinatorial presheaf in* $\mathbf{Set}^{\mathbb{I}}$ *is unique (up to isomorphism).*

PROOF: Consider, for example, the following situation

$$\varphi : \textstyle\sum_i A_i \underset{\mathfrak{S}_i}{\otimes} \mathbf{y}_{\mathbb{I}}(i) \cong \sum_i B_i \underset{\mathfrak{S}_i}{\otimes} \mathbf{y}_{\mathbb{I}}(i) : \phi \quad \text{in } \mathbf{Set}^{\mathbb{I}} .$$

For $a \in A_n$ let $\varphi(a \otimes \mathrm{id}_n) = b \otimes \jmath$ ($b \in B_m$, $\jmath : m \rightarrowtail n$) and let $\phi(b \otimes \mathrm{id}_m) = a' \otimes \imath$ ($a' \in A_\ell$, $\imath : \ell \rightarrowtail m$).

We have the following identities

$$\begin{aligned} a \otimes \mathrm{id}_m &= \phi(b \otimes \jmath) &= \phi((b \otimes \mathrm{id}_m) \cdot \jmath) &= \phi(b \otimes \mathrm{id}_m) \cdot \jmath \\ &= (a' \otimes \imath) \cdot \jmath &= a' \otimes (\imath \cdot \jmath) & \end{aligned}$$

from which it follows that $\imath \cdot \jmath$ is a bijection. Thus, so are $\jmath$ and $\imath$, and $n = m = \ell$.

Finally, the assignment

$$A_i \longrightarrow B_i : a \longmapsto b \cdot \sigma, \text{ where } \varphi(a \otimes \mathrm{id}_i) = b \otimes \sigma \ (b \in B_i,\ \sigma \in \mathfrak{S}_i)$$

yields a $\mathfrak{S}_i$-equivariant bijection. □

2 The Schanuel topos

We investigate the structure of combinatorial presheaves.

Definition 2.1 *(c.f. [Par99]) With respect to a presheaf* $P \in \mathbf{Set}^{\mathbb{I}}$*, an element* $p \in P(n)$ *is* minimal *whenever, for all* $p' \in P(m)$ *and* $\imath : m \rightarrowtail n$*, if* $p = p' \cdot_P \imath$ *then* $\imath$ *is bijective.*

For an example, note that the minimal elements of a combinatorial presheaf are of the form $x \otimes \mathrm{id}$.

The subset of minimal elements in $P(m)$ is denoted $\langle P \rangle_m$. As minimal elements are invariant under the action of a bijection, we have a family $\langle P \rangle = \{\langle P \rangle_i \times \mathfrak{S}_i \longrightarrow \langle P \rangle_i\}_i$ of representations of the finite symmetric groups. Further, the action of P induces the natural transformation

$$\epsilon_P : \langle P \rangle_! \longrightarrow P : p \otimes \imath \longmapsto p \cdot \imath$$

in $\mathbf{Set}^{\mathbb{I}}$.

Proposition 2.2 *The map* ϵ_P *is an epimorphism.*

PROOF: We need show that, for every $p \in P(n)$ there exist a minimal $p_0 \in P(m)$ and an injection $\imath_0 : m \rightarrowtail n$ such that $p = p_0 \cdot \imath_0$.

Given $p \in P(n)$, consider the non-empty set of pairs $(p', \imath') \in P(i) \times \mathbb{I}(i,n)$ such that $p' \cdot \imath' = p$ and chose $(p_0, \imath_0) \in P(i_0) \times \mathbb{I}(i_0, n)$ with minimal i_0. □

This proposition establishes that every presheaf in $\mathbf{Set}^{\mathbb{I}}$ is *engendered* by its minimal elements.

Definition 2.3 *(c.f. [Joy86]) With respect to a presheaf $P \in \mathbf{Set}^{\mathbb{I}}$, an element $p \in P(i)$ is* generic *whenever, for all $\imath : i \rightarrowtail k$, $\kappa : j \rightarrowtail k$ and $q \in P(j)$, if $p \cdot \imath = q \cdot \kappa$ then $\jmath : \imath \subseteq \kappa$ and $q = p \cdot \jmath$ ($\jmath : i \rightarrowtail j$).*

Proposition 2.4 *For any presheaf, generic elements are minimal.*

Theorem 2.5 *For $P \in \mathbf{Set}^{\mathbb{I}}$, the following are equivalent:*

1. *The map ϵ_P is a monomorphism.*
2. *The presheaf P is combinatorial.*
3. *Minimal elements in P are generic.*
4. *The discrete op-fibration $\int P \rightarrowtail \mathbb{I}$ creates pullbacks.*
5. *The presheaf P preserves pullbacks.*

PROOF:

(1) $\Longrightarrow$ (2) Because, by Proposition 2.2, the map ϵ_P is an isomorphism.

(2) $\Longrightarrow$ (3) If $(x \otimes \mathrm{id}) \cdot \imath = (z \otimes \kappa) \cdot \jmath$ then $\imath = \sigma \cdot \kappa \cdot \jmath$ and $z = x \cdot \sigma$ for some bijection σ. Hence, $\sigma \cdot \kappa : \imath \subseteq \jmath$ and $(x \otimes \mathrm{id}) \cdot (\sigma \cdot \kappa) = z \otimes \kappa$.

(3) $\Longrightarrow$ (4) We will consider the co-span

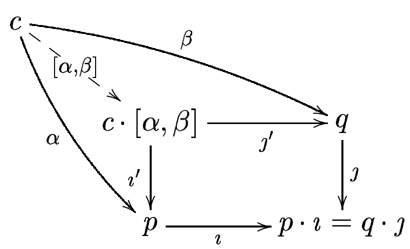
$$\begin{array}{ccc} & & q \\ & & \big\downarrow{\scriptstyle \jmath} \\ p & \xrightarrow[\imath]{} & p \cdot \imath = q \cdot \jmath \end{array} \qquad \text{in } \textstyle\int P \tag{3}$$

above the pullback square

$$\begin{array}{ccc} \ell & \xrightarrow{\jmath'} & j \\ {\scriptstyle \imath'}\big\downarrow & & \big\downarrow{\scriptstyle \jmath} \\ i & \xrightarrow[\imath]{} & k \end{array} \qquad \text{in } \mathbb{I} \tag{4}$$

(Note that if (3) has a pullback above (4) then it is unique.)

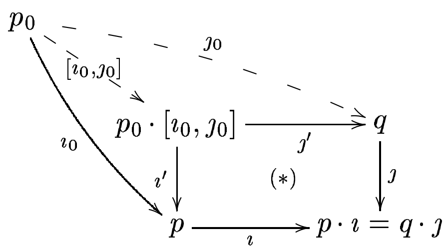
Every cone $p \xleftarrow{\alpha} c \xrightarrow{\beta} q$ in $\int P$ for (3) induces the following situation



(5)

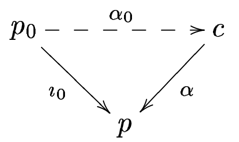
where $[\alpha, \beta]$ is given by the universal property of pullbacks.

In particular, as every element in P is engendered by a minimal element (Proposition 2.2) and minimal elements are assumed to be generic, we have a morphism $\imath_0 : p_0 \rightarrowtail p$ in $\int P$ with p_0 generic inducing the following situation



where $\jmath_0$ is given by the property of generic elements.

We show that $(*)$ is a pullback. Indeed, in the situation (5), as p_0 is generic, we have a factorisation



in $\int P$

and hence the identity $c \cdot [\alpha, \beta] = p_0 \cdot \alpha_0 \cdot [\alpha, \beta] = p_0 \cdot [\imath_0, \jmath_0]$.

(4) $\Longrightarrow$ (5) Easy.

(5) $\Longrightarrow$ (1) Consider the pullback square (4) and let $p \cdot \imath = q \cdot \jmath$ with $p \in \langle P \rangle_i$ and $q \in \langle P \rangle_j$.

As P preserves pullbacks, there exists a unique $o \in P(\ell)$ such that $o \cdot \imath' = p$ and $o \cdot \jmath' = q$. Then, since p and q are minimal, it follows that $\imath'$ and $\jmath'$ are bijections and hence that $p \otimes \imath = q \otimes \jmath$. □

Corollary 2.6 *The category of combinatorial presheaves in* $\mathbf{Set}^{\mathbb{I}}$ *and natural transformations is equivalent to the Schanuel topos* $\underline{Sch}$.

3 Species of structure

The category of representations of the finite symmetric groups and equivariant maps is isomorphic to the category $\mathbf{Set}^{\mathbb{B}}$, for $\mathbb{B}$ the category of finite cardinals and bijections, and equivalent to the category of *species of structure* $[\mathbf{B}, \mathbf{Set}]$, where $\mathbf{B}$ is the category of finite sets and bijections.

A natural transformation $f : A \longrightarrow B$ in $\mathbf{Set}^{\mathbb{B}}$ induces a natural transformation between the associated combinatorial presheaves as follows

$$f_! : A_! \longrightarrow B_! : a \otimes \imath \longmapsto f(a) \otimes \imath \quad .$$

This assignment $f \longmapsto f_!$ defines an *extension* functor $(\_)_! : \mathbf{Set}^{\mathbb{B}} \longrightarrow \mathbf{Set}^{\mathbb{I}}$ which is left adjoint to the forgetful functor $\mathbf{Set}^{\mathbb{I}} \longrightarrow \mathbf{Set}^{\mathbb{B}}$.

The extension functor $(\_)_! : \mathbf{Set}^{\mathbb{B}} \longrightarrow \mathbf{Set}^{\mathbb{I}}$ is faithful and creates isomorphisms (Proposition 1.2). Thus, there is a bijective correspondence between isomorphisms $A \cong B$ in $\mathbf{Set}^{\mathbb{B}}$ and isomorphisms $A_! \cong B_!$ in $\mathbf{Set}^{\mathbb{I}}$. This result generalises as follows.

Theorem 3.1 *The category of species of structure is equivalent to the category of combinatorial presheaves in* $\mathbf{Set}^{\mathbb{I}}$ *and cartesian natural transformations.*

This theorem is a corollary of the following proposition about (quasi-)cartesian natural transformations (*viz.*, natural transformations whose naturality squares are (quasi-)pullbacks).

Proposition 3.2 *Let $\varphi : P \longrightarrow Q$ in* $\mathbf{Set}^{\mathbb{I}}$.

1. *If φ maps minimal elements to generic ones then it is quasi-cartesian.*

2. *If φ is quasi-cartesian then it preserves minimal and generic elements.*

3. *For P combinatorial, if φ is quasi-cartesian then it is cartesian.*

PROOF:

(1) Let $p \in P(n)$ and $q \in Q(m)$ be such that $\varphi(p) = q \cdot \imath$ for $\imath : m \rightarrowtail n$ in $\mathbb{I}$. As the elements of P are engendered by its minimal elements (Proposition 2.2) there is a morphism $\imath_0 : p_0 \rightarrowtail p$ in $\int P$ with p_0 minimal. Moreover, as φ is assumed to map minimal elements to generic ones, we have a factorisation

$$\begin{array}{ccc} \varphi(p_0) & \overset{\imath'}{\dashrightarrow} & q \\ & {\scriptstyle \imath_0}\searrow \quad \swarrow{\scriptstyle \imath} & \\ & \varphi(p) & \end{array} \qquad \text{in } \int P$$

from which it follows that $(p_0 \cdot \imath') \cdot \imath = p$ and $\varphi(p_0 \cdot \imath') = q$. Thus φ is quasi-cartesian.

(2) Easy.

(3) Because the action of combinatorial presheaves is injective (Theorem 2.5). □

Corollary 3.3 *A natural transformation between combinatorial presheaves in* $\mathbf{Set}^{\mathbb{I}}$ *is cartesian iff it preserves minimal elements.*

4 Analytic functors

Definition 4.1 *([Joy86]) A functor F :* $\mathbf{Set} \longrightarrow \mathbf{Set}$ *is* analytic *if it has a Taylor series development*

$$F(X) = \sum_i F[i] \underset{\mathfrak{S}_i}{\otimes} X^i$$

for a family $F[\,] = \{F[i] \times \mathfrak{S}_i \longrightarrow F[i]\}_i$ of representations of the finite symmetric groups.

Every presheaf $P \in \mathbf{Set}^{\mathbb{I}}$ induces an analytic functor $\widetilde{P}$ with $\widetilde{P}[i] = \langle P \rangle_i$. Moreover, for P combinatorial, we have the following situation

$$\begin{array}{rcll} P(n) & \longrightarrow & \sum_i \langle P \rangle_i \underset{\mathfrak{S}_i}{\otimes} n^i & \\ p & \longmapsto & p_0 \otimes \imath & \text{where } p = p_0 \cdot \imath \\ & & & \text{with } p_0 \text{ minimal} \end{array} \qquad \begin{array}{ccc} \mathbb{I} & \xrightarrow{\ \mathrm{N}\ } & \mathbf{Set} \\ & {\scriptstyle P}\searrow \ \overset{\mathrm{Lan}}{\Longrightarrow} & \downarrow{\scriptstyle \widetilde{P}} \\ & & \mathbf{Set} \end{array}$$

and thus we have an extension functor

$$\widetilde{(\_)} : \underline{\mathit{Sch}} \longrightarrow \underline{\mathit{Ana}}$$

where $\underline{\mathit{Ana}}$ denotes the category of analytic functors and natural transformations. In elementary terms,

$$\widetilde{\varphi}(p \otimes x) = q_0 \otimes (\imath \cdot x) \text{ where } \varphi(p) = q_0 \cdot \imath \text{ with } q_0 \text{ minimal}$$

for all φ in $\underline{Sch}$.

Proposition 4.2 *The extension functor* $\widetilde{(\_)} : \underline{Sch} \longrightarrow \underline{Ana}$ *is essentially surjective and faithful.*

Definition 4.3 1. *(c.f. [Joy86]) With respect to a functor* $F : \mathbf{Set} \longrightarrow \mathbf{Set}$, *an element* $x \in F(X)$ *is* $\mathcal{E}$generic *whenever, for all* $f : X \longrightarrow Z$, *epimorphic* $\varepsilon : Y \twoheadrightarrow Z$ *and* $y \in F(Y)$, *if* $x \cdot_F f = y \cdot_F \varepsilon$ *then there exists* $f' : X \longrightarrow Y$ *such that* $f = f' \cdot \varepsilon$ *and* $x \cdot_F f' = y$.

2. *A natural transformation* $\phi : F \dot{\longrightarrow} G$ *is* $\mathcal{E}$(quasi-)cartesian *if for every epimorphism* $\varepsilon : X \twoheadrightarrow Y$ *the naturality square*

$$\begin{array}{ccc} F(X) & \xrightarrow{\phi_X} & G(X) \\ {\scriptstyle - \cdot_F \varepsilon} \downarrow & & \downarrow {\scriptstyle - \cdot_G \varepsilon} \\ F(Y) & \xrightarrow[\phi_Y]{} & G(Y) \end{array}$$

is a (quasi-)pullback.

Proposition 4.4 *The* $\mathcal{E}$*generic elements of an analytic functor are of the form* $x \otimes \imath$ *with* $\imath$ *injective.*

PROOF: If $x \otimes h$ $(h : n \longrightarrow X)$ is generic then, as $(x \otimes h) \cdot ! = (x \otimes \mathrm{id}_n) \cdot !$ $(! : X \longrightarrow 1)$, there exists $h' : X \longrightarrow n$ such that $h \cdot h'$ is a bijection; hence h is an injection. Conversely, if $(x \otimes \imath) \cdot f = (y \otimes g) \cdot \varepsilon$ ($\imath$ injective, ε surjective) then $x \cdot \sigma = y$ and $\imath \cdot f = \sigma \cdot g \cdot \varepsilon$ for some bijection σ, and a diagonal fill-in for the square

$$\begin{array}{ccc} \cdot & \overset{\imath}{\rightarrowtail} & \cdot \\ {\scriptstyle \sigma \cdot g} \downarrow & & \downarrow {\scriptstyle f} \\ \cdot & \underset{\varepsilon}{\twoheadrightarrow} & \cdot \end{array}$$

provides a map with the required factorisation property

$$\begin{array}{ccc} x \otimes \imath & \dashrightarrow & y \otimes g \\ & {\scriptstyle f} \searrow \quad \swarrow {\scriptstyle \varepsilon} & \\ & x \otimes (\imath \cdot f) = y \otimes (g \cdot \varepsilon) & \end{array}$$

in the category of elements. □

Corollary 4.5 *For every map* $\varphi \in \underline{Sch}$, *the induced map* $\widetilde{\varphi} \in \underline{Ana}$ *preserves* $\mathcal{E}$*generic elements.*

Proposition 4.6 *Let* $\phi : F \dot{\longrightarrow} G : \mathbf{Set} \longrightarrow \mathbf{Set}$.

1. *For* F *analytic, if* ϕ *preserves* $\mathcal{E}$*generic elements then it is* $\mathcal{E}$*quasi-cartesian.*

2. *If* ϕ *is* $\mathcal{E}$*quasi-cartesian then it preserves* $\mathcal{E}$*generic elements.*

Theorem 4.7 *The Schanuel topos is equivalent to the category of analytic functors and* $\mathcal{E}$*quasi-cartesian natural tranformations.*

PROOF: Follows from Proposition 4.2, Corollary 4.5, Proposition 4.6 (1) and the fact that for every $\mathcal{E}$quasi-cartesian natural transformation $\phi : F \longrightarrow G$ in $\underline{Ana}$, the natural transformation $\varphi : F[\,]_! \longrightarrow G[\,]_!$ in $\underline{Sch}$ defined as $\varphi(a \otimes \imath) = b \otimes (\jmath \cdot \imath)$ where $\phi(a \otimes \mathrm{id}) = b \otimes \jmath$ (see Propositions 4.6 (2) and 4.4) is such that $\widetilde{\varphi} = \phi$. □

5 Combinatorial presheaves in $\mathbf{Set}^{\mathbb{F}}$

Definition 5.1 *A presheaf in* $\mathbf{Set}^{\mathbb{F}}$ *is* combinatorial *if it has a representation*

$$\overline{S}(n) = \sum_i S_i \underset{\mathfrak{S}_i}{\otimes} \mathbb{I}(i,n)$$

with action

$$\begin{array}{rcl} \overline{S}(n) \times \mathbb{F}(n,m) & \longrightarrow & \overline{S}(m) \\ s \otimes \imath \ , \ f & \longmapsto & (s \cdot \varepsilon) \otimes \jmath \ \textit{ where } \varepsilon \cdot \jmath \textit{ is an epi-mono} \\ & & \qquad \textit{factorisation of } \imath \cdot f \end{array}$$

for some $S \in \mathbf{Set}^{\mathbb{S}}$.

For examples, note that every $A_{!\mathbb{F}} = \sum_i A_i \underset{\mathfrak{S}_i}{\otimes} \mathbf{y}_{\mathbb{F}}(i) \in \mathbf{Set}^{\mathbb{F}}$ ($A \in \mathbf{Set}^{\mathbb{B}}$) is combinatorial. Indeed, $A_{!\mathbb{F}} \cong \overline{A_{!\mathbb{S}}}$, where $A_{!\mathbb{S}} = \sum_i A_i \underset{\mathfrak{S}_i}{\otimes} \mathbf{y}_{\mathbb{S}}(i) \in \mathbf{Set}^{\mathbb{S}}$, as can be easily seen from the following calculation

$$\sum_i A_i \underset{\mathfrak{S}_i}{\otimes} \mathbb{F}(i,n) \cong \sum_i A_i \underset{\mathfrak{S}_i}{\otimes} \left(\sum_j \mathbb{S}(i,j) \underset{\mathfrak{S}_j}{\otimes} \mathbb{I}(j,n) \right) \cong \sum_j \left(\sum_i A_i \underset{\mathfrak{S}_i}{\otimes} \mathbb{S}(i,j) \right) \underset{\mathfrak{S}_j}{\otimes} \mathbb{I}(j,n)$$

using (2). In particular, for $\mathtt{X} = (0,1,0,\ldots,0,\ldots)$ in $\mathbf{Set}^{\mathbb{B}}$, we have that $\mathtt{X}_{!\mathbb{F}}$ is the universal object in $\mathbf{Set}^{\mathbb{F}}$.

6 An algebraic view

Proposition 6.1 *For every bijective on objects inclusion functor* $\mathsf{A} \longrightarrow \mathsf{B}$ *between small categories, the induced adjunction* $\mathbf{Set}^{\mathsf{A}} \underset{\longrightarrow}{\overset{\longleftarrow}{\top}} \mathbf{Set}^{\mathsf{B}}$ *is monadic.*

The inclusions

$$\begin{array}{ccc} \mathbb{B} & \longrightarrow & \mathbb{S} \\ \downarrow & & \downarrow \\ \mathbb{I} & \longrightarrow & \mathbb{F} \end{array}$$

induce the monadic adjunctions

$$\begin{array}{ccc} \mathbf{Set}^{\mathbb{B}} & \underset{\longrightarrow}{\overset{\longleftarrow}{\top}} & \mathbf{Set}^{\mathbb{S}} \\ \downarrow \dashv \uparrow & & \downarrow \dashv \uparrow \\ \mathbf{Set}^{\mathbb{I}} & \underset{\longrightarrow}{\overset{\longleftarrow}{\top}} & \mathbf{Set}^{\mathbb{F}} \end{array}$$

Theorem 6.2 *The Schanuel topos is equivalent to the Kleisli category of the monad on* $\mathbf{Set}^{\mathbb{B}}$ *induced by the adjunction* $\mathbf{Set}^{\mathbb{B}} \underset{\longrightarrow}{\overset{\longleftarrow}{\top}} \mathbf{Set}^{\mathbb{I}}$.

Write $\mathcal{I}$ and $\mathcal{S}$, respectively, for the monads on $\mathbf{Set}^{\mathbb{B}}$ induced by the adjunctions $\mathbf{Set}^{\mathbb{B}} \underset{\longrightarrow}{\overset{\longleftarrow}{\top}} \mathbf{Set}^{\mathbb{I}}$ and $\mathbf{Set}^{\mathbb{B}} \underset{\longrightarrow}{\overset{\longleftarrow}{\top}} \mathbf{Set}^{\mathbb{S}}$. We have a distributive law

$$\mathcal{S}\mathcal{I} \stackrel{\cdot}{\longrightarrow} \mathcal{I}\mathcal{S}$$

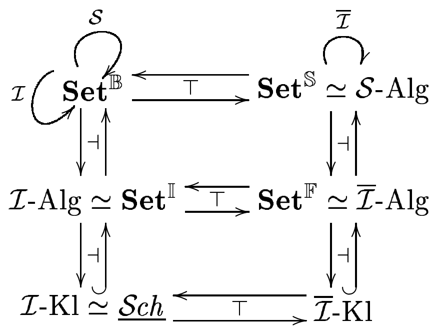
given as follows

$$\begin{array}{rcll} \sum_j \left(\sum_i A_i \underset{\mathfrak{S}_i}{\otimes} \mathbb{I}(i,j) \right) \underset{\mathfrak{S}_j}{\otimes} \mathbb{S}(j,n) & \longrightarrow & \sum_\ell \left(\sum_k A_k \underset{\mathfrak{S}_k}{\otimes} \mathbb{S}(k,\ell) \right) \underset{\mathfrak{S}_\ell}{\otimes} \mathbb{I}(\ell,n) & \\ (x \otimes \imath) \otimes \varepsilon & \longmapsto & (x \otimes \varepsilon') \otimes \imath' & \text{where } \varepsilon' \cdot \imath' \text{ is an epi-mono} \\ & & & \text{factorisation of } \imath \cdot \varepsilon \end{array}$$

Thus, the monad $\mathcal{I}$ on $\mathbf{Set}^{\mathbb{B}}$ lifts to a monad $\overline{\mathcal{I}}$ on $\mathbf{Set}^{\mathbb{S}}$.

Proposition 6.3 *The topos* $\mathbf{Set}^{\mathbb{F}}$ *is isomorphic to the category of Eilenberg-Moore algebras of the monad* $\overline{\mathcal{I}}$ *on* $\mathbf{Set}^{\mathbb{S}}$.

Hence we have the following situation



and when I resume this work I will complete the picture.

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