Fibred Models of Processes: Discrete, Continuous, and Hybrid Systems

(Extended Abstract)

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Abstract. We present the rudiments of a unifying theory of general processes encompassing discrete, continuous, and hybrid systems. The main focus is on the study of process behaviour, but constructions on processes are also considered in some detail. In particular, we show that processes admit an abstract, conceptual treatment of bisimilarity via the notion of open map (as advocated by Winskel *et al.*). Furthermore, we present a tool-kit of categorical constructions on processes that can be regarded as the basis of a process description language. Within the general theory, typical operations of process calculi on discrete and hybrid systems are discussed.

Introduction

We advocate a fibred view of general processes; in which, roughly, a process

$$\begin{array}{c} \mathbb{X} \\ \downarrow \\ \mathbb{C} \end{array}$$

consists of a *state space* $\mathbb{X}$ (of states and their evolutions) varying according to a *control* $\mathbb{C}$ (of observations, time, *etc.*). By choosing the control $\mathbb{C}$ to consist of discrete observations or of continuous time we respectively obtain models of discrete and continuous processes (Section 1); hybrid systems arise from mixed discrete-continuous controls (Section 4). Further, we consider a *simulation*

$$\begin{array}{ccc} \mathbb{X} & & \mathbb{X}' \\ \downarrow & \longrightarrow & \downarrow \\ \mathbb{C} & & \mathbb{C} \end{array}$$

between processes to be a map $\mathbb{X} \longrightarrow \mathbb{X}'$ between the corresponding state spaces that respects the variation.

We will see that the category $\mathcal{U}$ of processes and simulations comes equipped with a canonical choice of *basic* processes (or paths) $\mathcal{P}$ from which arbitrary processes can be constructed by appropriately glueing basic ones. This situation yields a *model for concurrency*

$$\mathcal{P} \longrightarrow \mathcal{U}$$

in the sense of Winskel *et al.* [JNW96, CW97], and is important for one can study process behaviour via the notion of open map (Section 2). Moreover, we show that the category of processes $\mathcal{U}$ supports categorical constructions modelling typical operations of process calculi (Section 3).

We hope that this work will contribute towards the understanding of (and reasoning about) general processes and thereafter towards the development of a theory to aid the design of languages (and logics) for them.

1 Fibred Models of Discrete and Continuous Systems

We consider discrete and continuous systems from a fibred viewpoint. These systems are examples of *linearly-controlled* processes (or *interleaving models*, in the jargon of concurrency theory) in the sense of [BF99].

1.1 Discrete Systems

We study the familiar model of concurrent computation given by transition graphs (or automata) as an instance of the general theory to be developed in Section 2.

Transition graphs. For a set A, the category $\mathbf{TG}_A$ of *A-labelled transition graphs (or automata)* has objects given by graphs $\mathsf{dom}, \mathsf{cod} : E \longrightarrow N$ equipped with a labelling function $\mathsf{lab} : E \longrightarrow A$, and morphisms given by pairs of functions between nodes and edges that preserve the domain, codomain and labelling of edges. *Transition systems* are extensional transition graphs, for which the function $E \longrightarrow N \times A \times N : e \longmapsto (\mathsf{dom}(e), \mathsf{lab}(e), \mathsf{cod}(e))$ is injective.

Transition categories. Let $\mathbf{M}(A) = (A^*, \varepsilon, \cdot)$ be the free monoid on a set A. An *A-labelled transition category* is a functor $\ell\downarrow^{\mathbb{G}}_{\mathbf{M}(A)}$ such that, for every $e \in \mathbb{G}$ and $\alpha_0, \alpha_1 \in \mathbf{M}(A)$, if $\ell(e) = \alpha_0 \cdot \alpha_1$ in $\mathbf{M}(A)$ then there exists a unique factorisation $e = e_0 \cdot e_1$ in $\mathbb{G}$ for which $\ell(e_0) = \alpha_0$ and $\ell(e_1) = \alpha_1$. The category $\mathbf{TC}_A$ has A-labelled transition categories as objects and morphisms $\ell\downarrow^{\mathbb{G}}_{\mathbf{M}(A)} \xrightarrow{h} \ell'\downarrow^{\mathbb{G}'}_{\mathbf{M}(A)}$ given by functors $\mathbb{G} \xrightarrow{h} \mathbb{G}'$ that preserve the labelling; that is, such that $\ell = \ell' h$.

Note that there is a bijective correspondence between transition graphs and transition categories as follows. A transition graph $G = (\ N \underset{\mathsf{cod}}{\overset{\mathsf{dom}}{\leftleftarrows}} E \xrightarrow{\mathsf{lab}} A\ )$ corresponds to the transition category $\ell\downarrow^{\mathcal{F}G}_{\mathbf{M}(A)}$, where $\mathcal{F}G$ is the free category on the graph G (with set of objects given by the nodes of G and morphisms given

by paths in G) and where $\ell(e_0 \cdots e_n) \stackrel{\text{def}}{=} \mathsf{lab}(e_0) \cdots \mathsf{lab}(e_n)$. Conversely, a transition category $\ell \downarrow^{\mathbb{G}}_{\mathbf{M}(A)}$ corresponds to the transition graph $|\,\mathbb{G}\,| \underset{\mathsf{cod}}{\overset{\mathsf{dom}}{\leftleftarrows}} E \xrightarrow{\ell} A$ where $E \stackrel{\text{def}}{=} \{\, e \in \mathbb{G} \;\big|\; \ell(e) \in A \,\}$.

Proposition 1.1. *For any set A, the categories $\mathbf{TG}_A$ and $\mathbf{TC}_A$ are equivalent.*

Thus, transition graphs and transition categories provide the same model of parallel computation. However, the fibred representation provided by transition categories embodies the dynamics of the systems. Indeed, as we show below, $\mathbf{TC}_A$ comes equipped with canonical categories of *computation paths* which induce behavioural notions of equivalence.

Discrete path categories. The category $\mathbf{M}(A)_{\lessapprox}$ has A^* as set of objects and morphisms $\alpha \longrightarrow \beta$ given by pairs (φ, γ) in A^* such that $\varphi \cdot \alpha \cdot \gamma = \beta$. One thinks of a morphism $\alpha \longrightarrow \beta$ as a pre- and post- extension of α yielding β. The category $\mathbf{M}(A)_{>}$ is the subcategory of $\mathbf{M}(A)_{\lessapprox}$ consisting of post-extensions; *viz.*, morphisms of the form (ε, γ), where ε denotes the empty string. Note that $\mathbf{M}(A)_{>}$ is the poset $\mathbf{P}(A) = (A^*, \leq)$ of sequences of elements of A ordered by prefix. We have faithful functors

$$\mathbf{P}(A) \longrightarrow \mathbf{M}(A)_{\lessapprox} \hookrightarrow \mathbf{TC}_A$$

where the embedding $\mathbf{M}(A)_{\lessapprox} \hookrightarrow \mathbf{TC}_A$ maps a sequence α to the transition category $\downarrow^{[\![\alpha]\!]}_{\mathbf{M}(A)} \quad \downarrow^{\alpha_0 \leq \alpha_0 \cdot \alpha_1}_{\alpha_1}$, where $[\![a_1 \cdots a_i \cdots a_n]\!]$ is the poset: $\varepsilon \leq a_1 \leq \ldots \leq (a_1 \cdots a_i) \leq \ldots \leq (a_1 \cdots a_i \cdots a_n)$.

Open-map bisimilarity. The functors $\mathbf{P}(A) \longrightarrow \mathbf{TC}_A$ and $\mathbf{M}(A)_{\lessapprox} \hookrightarrow \mathbf{TC}_A$ provide canonical choices w.r.t. which to consider the notion of open map [JNW96] and hence study process behaviour. Indeed, the notion of open map w.r.t. the inclusion $\mathbf{P}(A) \longrightarrow \mathbf{TC}_A$ corresponds to functional bisimulation (*cf.* [NC95]), whilst the notion of open map w.r.t. the embedding $\mathbf{M}(A)_{\lessapprox} \hookrightarrow \mathbf{TC}_A$ amounts to functional *back-and-forth* bisimulation on states (*cf.* [DNMV90]) —see the general theory developed in Section 2 '§ Bisimulation'. Moreover, the fibred setting allows a natural treatment of *weak bisimilarity*; this is shown in Subsection 3.1 '§ Saturation'.

1.2 Continuous Systems

We provide a treatment of continuous systems emphasising the similarities with the above view of discrete systems.

Durations. The notion of duration was introduced by Lawvere in connection to investigations of processes in continuum physics, see [Law86].

Let $\mathbf{R}$ be the additive monoid of non-negative real numbers. A *duration* is a functor $\begin{array}{c}\mathbb{X}\\ d\downarrow\\ \mathbf{R}\end{array}$ such that, for every $e \in \mathbb{X}$ and $t_0, t_1 \in \mathbf{R}$, if $d(e) = t_0 + t_1$ in $\mathbf{R}$ then there exists a unique factorisation $e = e_0 \cdot e_1$ in $\mathbb{X}$ for which $d(e_0) = t_0$ and $d(e_1) = t_1$. The category **Dur** has durations as objects and morphisms $\begin{array}{c}\mathbb{X}\\ d\downarrow\\ \mathbf{R}\end{array} \xrightarrow{h} \begin{array}{c}\mathbb{X}'\\ d'\downarrow\\ \mathbf{R}\end{array}$ given by functors $\mathbb{X} \xrightarrow{h} \mathbb{X}'$ such that $d = d'h$.

As remarked in [Law86], an important example of duration is given by solutions of differential equations. For the simplest such example, consider the differential equation

$$\frac{dx}{dt} = ax \tag{1}$$

whose initial value problem with initial condition

$$x(0) = s$$

has unique solution

$$\widetilde{s}(t) = s\, e^{a\,t} \quad .$$

It can be regarded as the duration $\begin{array}{c}\mathbf{S}\\ \downarrow\\ \mathbf{R}\end{array}$ $\begin{array}{c}s_0 \xrightarrow{t} s_1\\ \downarrow\\ t\end{array}$, where $|\ \mathbf{S}\ | \stackrel{\text{def}}{=} \mathbb{R}$ and, for $t \in \mathbb{R}_{\geq 0}$, we have $s_0 \xrightarrow{t} s_1$ in $\mathbf{S}$ if and only if there exists a solution of (1) $\sigma : I \longrightarrow \mathbb{R}$ and $t_0 \leq t_1$ in I such that $\sigma(t_0) = s_0$, $\sigma(t_1) = s_1$, and $t_1 - t_0 = t$ (*i.e.*, if and only if $\widetilde{s_0}(t) = s_1$). (Compare this construction with the *flow* of the differential equation [HS74].)

Continuous path categories. The category $\mathbf{R}_{\gtrdot}$ has $\mathbb{R}_{\geq 0}$ (the set of non-negative real numbers) as the set of objects and morphisms $t \longrightarrow t'$ given by pairs (x, y) in $\mathbb{R}_{\geq 0}$ such that $x + t + y = t'$. A morphism $t \longrightarrow t'$ can be thought of as a way of placing an interval of length t within an interval of length t'. The category $\mathbf{R}_{>}$ is the subcategory of $\mathbf{R}_{\gtrdot}$ consisting of initial placements; *viz.*, morphisms of the form $(0, y)$. Note that $\mathbf{R}_{>}$ is the poset $\mathbf{T} = (\mathbb{R}_{\geq 0}, \leq)$. We have faithful functors

$$\mathbf{T} \longrightarrow \mathbf{R}_{\gtrdot} \hookrightarrow \mathbf{Dur}$$

where the embedding $\mathbf{R}_{\gtrdot} \hookrightarrow \mathbf{Dur}$ maps a non-negative real number t to the *difference* duration $\begin{array}{c}[\![t]\!]\\ \downarrow\\ \mathbf{R}\end{array}$ $\begin{array}{c}x \leq y\\ \downarrow\\ y - x\end{array}$, where $[\![t]\!]$ is the *interval* poset $([0, t], \leq)$.

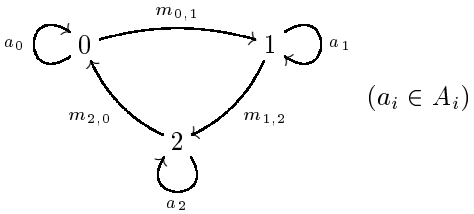
The concept of open map of durations, w.r.t. the path categories $\mathbf{T}$ and $\mathbf{R}_{\gtrdot}$, yields natural notions of bisimilarity; consider, for instance, the general results of Section 2 '§ Bisimulation' in the context of durations.

2 Fibred Models of Processes: General Theory

Processes. A *bundle* over a small category $\mathbb{C}$ is a functor $\overset{\mathbb{X}}{\downarrow}_{\mathbb{C}}$. The category $\mathbf{Cat}_{/\mathbb{C}}$ has bundles over $\mathbb{C}$ as objects and morphisms $f\overset{\mathbb{X}}{\downarrow}_{\mathbb{C}} \xrightarrow{h} f'\overset{\mathbb{X}'}{\downarrow}_{\mathbb{C}}$ given by functors $\mathbb{X} \xrightarrow{h} \mathbb{X}'$ such that $f = f'h$. The notion of process to be considered is given by certain bundles, known as ufl (*unique factorisation lifting*) functors (see , *e.g.*, [Law86, SS88, Str96]).

A *ufl functor* $f\overset{\mathbb{X}}{\downarrow}_{\mathbb{C}}$ (over a small category $\mathbb{C}$) is a functor with the property that, for every $e \in \mathbb{X}$ and $\alpha_0, \alpha_1 \in \mathbb{C}$, if $f(e) = \alpha_0 \cdot \alpha_1$ in $\mathbb{C}$ then there exists a unique factorisation $e = e_0 \cdot e_1$ in $\mathbb{X}$ for which $f(e_0) = \alpha_0$ and $f(e_1) = \alpha_1$. The category $\mathbf{Ufl}_{/\mathbb{C}}$ is the full subcategory of $\mathbf{Cat}_{/\mathbb{C}}$ with objects given by ufl functors.

The discrete and continuous systems of Section 1 provide examples of processes over monoids (*viz.*, $\mathbf{TC}_A = \mathbf{Ufl}_{/\mathbf{M}(A)}$ and $\mathbf{Dur} = \mathbf{Ufl}_{/\mathbf{R}}$) and as such force a *global* view of the state of a process. The extra generality of allowing processes over arbitrary categories may be used, for example, to provide a *distributed* view of the state. For instance, discrete processes operating in three modes (say $0, 1, 2$) and such that in mode i can either keep on computing in that mode or proceed to compute in mode $(i+1) \bmod 3$, can be naturally modelled as ufl functors over the free category on the following graph of modes:



where, for $0 \leq i \leq 2$, the sets A_i are the observable actions in mode i and the transitions $m_{i,(i+1) \bmod 3}$ represent the allowable mode changes.

Path categories. The categories of processes $\mathbf{Ufl}_{/\mathbb{C}}$ come equipped with canonical categories of *paths* generalising the constructions of Section 1.

Mac Lane's *twisted arrow category* [ML71, page 223] $\mathbb{C}_{\leftrightarrows}$ has objects given by morphisms in $\mathbb{C}$ and morphisms $\alpha \longrightarrow \beta$ given by pairs of maps (φ, γ) in $\mathbb{C}$ such that $\varphi \cdot \alpha \cdot \gamma = \beta$; such a morphism can be regarded as a context, $\varphi \cdot (_) \cdot \gamma$, within which α can be placed to yield β. The category $\mathbb{C}_{>}$ is the subcategory of $\mathbb{C}_{\leftrightarrows}$ consisting of the morphisms of the form (id, γ). We have faithful functors

$$\mathbb{C}_{>} \xrightarrow{\mathrm{j}_{\mathbb{C}}} \mathbb{C}_{\leftrightarrows} \xhookrightarrow{\mathbf{u}_{\mathbb{C}}} \mathbf{Ufl}_{/\mathbb{C}} \tag{2}$$

where the embedding $\mathbb{C}_{\divideontimes} \hookrightarrow \mathbf{Ufl}_{/\mathbb{C}}$ maps:

- an object $(a_0 \xrightarrow{\alpha} a_1) \in |\ \mathbb{C}_{\divideontimes}\ |$ to the ufl functor $\mathbf{u}_\alpha \downarrow_{\mathbb{C}}^{[\![\alpha]\!]} \downarrow_{a}^{(\alpha_0, a, \alpha_1)}$, where the *interval category* $[\![\alpha]\!]$ has objects given by factorisations $(a_0 \xrightarrow{\alpha_0} a \xrightarrow{\alpha_1} a_1)$ of α in $\mathbb{C}$, and morphisms $\epsilon : (\alpha_0, a, \alpha_1) \longrightarrow (\alpha_0', a', \alpha_1')$ given by maps $\epsilon : a \longrightarrow a'$ in $\mathbb{C}$ such that $\alpha_0 \cdot \epsilon = \alpha_0'$ and $\alpha_1 = \epsilon \cdot \alpha_1'$; and
- a morphism $(\varphi, \gamma) : \alpha \longrightarrow \beta$ in $\mathbb{C}_{\divideontimes}$ to the functor $[\![\varphi, \gamma]\!] : [\![\alpha]\!] \longrightarrow [\![\beta]\!] : (\alpha_0, \alpha_1) \longmapsto (\varphi \cdot \alpha_0, \alpha_1 \cdot \gamma)$.

Note the (natural) bijective correspondence

$$\begin{aligned} \mathbf{Ufl}_{/\mathbb{C}}(\mathbf{u}_\alpha, f) \ &\cong\ f^{-1}(\alpha) \\ h &\longmapsto h\Big((\mathrm{id}, \alpha) \xrightarrow{\alpha} (\alpha, \mathrm{id})\Big) \end{aligned} \tag{3}$$

intuitively stating that computation paths (or *runs*) of shape α of the process f (*viz.*, morphisms $\mathbf{u}_\alpha \downarrow_{\mathbb{C}}^{[\![\alpha]\!]} \longrightarrow f \downarrow_{\mathbb{C}}^{\mathbb{X}}$) correspond to evolutions in the state space taking place over α (*viz.*, morphisms $e \in \mathbb{X}$ such that $f(e) = \alpha$).

In $\mathbb{C}_{\divideontimes}$, the commutative diagrams

$$\begin{array}{ccccc} & & \mathrm{id}_b & & \\ & {\scriptstyle(\alpha, \mathrm{id}_b)} \swarrow & & \searrow {\scriptstyle(\mathrm{id}_b, \beta)} & \\ \alpha & & & & \beta \\ & {\scriptstyle(\mathrm{id}_a, \beta)} \searrow & & \swarrow {\scriptstyle(\alpha, \mathrm{id}_c)} & \\ & & \alpha \cdot \beta & & \end{array} \qquad (a \xrightarrow{\alpha} b \xrightarrow{\beta} c \text{ in } \mathbb{C}) \tag{4}$$

are pushouts, and the embedding $\mathbf{u}_\mathbb{C}$ exhibits $\mathbf{Ufl}_{/\mathbb{C}}$ as the free cocompletion of $\mathbb{C}_{\divideontimes}$ respecting these pushouts. More precisely, we have the following universal characterisation.

Theorem 2.1. *(cf. [BF99, § 2]) The category $\mathbf{Ufl}_{/\mathbb{C}}$ is cocomplete and the embedding $\mathbf{u}_\mathbb{C} : \mathbb{C}_{\divideontimes} \hookrightarrow \mathbf{Ufl}_{/\mathbb{C}}$ preserves the pushouts (4). Moreover, for every cocomplete category $\mathcal{C}$ and functor $F : \mathbb{C}_{\divideontimes} \longrightarrow \mathcal{C}$ preserving the pushouts (4), we have that*

1. *there exists a cocontinuous functor $F' : \mathbf{Ufl}_{/\mathbb{C}} \longrightarrow \mathcal{C}$ such that $F'\mathbf{u}_\mathbb{C} \mathrel{\dot\cong} F$, and*
2. *for every natural transformation $\varphi : F'\mathbf{u}_\mathbb{C} \stackrel{\cdot}{\longrightarrow} G'\mathbf{u}_\mathbb{C}$, where F' and G' are cocontinuous functors $\mathbf{Ufl}_{/\mathbb{C}} \longrightarrow \mathcal{C}$, there exists a unique natural transformation $\varphi' : F' \stackrel{\cdot}{\longrightarrow} G'$ such that $\varphi'\mathbf{u}_\mathbb{C} = \varphi$.*

Condition (1) asserts that every functor $\mathbb{C}_{\divideontimes} \longrightarrow \mathcal{C}$ preserving the pushouts (4) has (up to isomorphism) a cocontinuous extension $\mathbf{Ufl}_{/\mathbb{C}} \longrightarrow \mathcal{C}$; whilst condition (2) guarantees that such extensions are unique up to canonical isomorphism.

Theorem 2.1 is important for it shows that the processes in $\mathbf{Ufl}_{/\mathbb{C}}$ consist of *properly glued* copies of the basic processes, $\mathbf{u}_\alpha \downarrow_{\mathbb{C}}^{[\![\alpha]\!]}$, arising from computation paths. Formally, we have that, for every $f \downarrow_{\mathbb{C}}^{\mathbb{X}} \in |\, \mathbf{Ufl}_{/\mathbb{C}}\,|$,

$$f \cong \mathrm{colim}\left(\mathbb{X}_{\not\asymp} \xrightarrow{f_{\not\asymp}} \mathbb{C}_{\not\asymp} \overset{\mathbf{u}_{\mathbb{C}}}{\hookrightarrow} \mathbf{Ufl}_{/\mathbb{C}}\right)$$

where $f_{\not\asymp} : e \longmapsto f(e)$.

Pointed processes. For $u \downarrow_{\mathbb{C}}^{\mathbb{U}} \in |\, \mathbf{Ufl}_{/\mathbb{C}}\,|$, the category $u/\mathbf{Ufl}_{/\mathbb{C}}$ has: objects given by triples $(x, \mathbb{X}, f)$, where $x : \mathbb{U} \longrightarrow \mathbb{X}$ is a functor and $f \downarrow_{\mathbb{C}}^{\mathbb{X}}$ is a ufl functor, such that $fx = u$; and morphisms $(x, \mathbb{X}, f) \xrightarrow{h} (x', \mathbb{X}', f')$ given by functors $\mathbb{X} \xrightarrow{h} \mathbb{X}'$ such that $f = f'h$ and $hx = x'$.

For instance, $\mathbf{0} \downarrow_{\mathbb{C}} /\mathbf{Ufl}_{/\mathbb{C}} \cong \mathbf{Ufl}_{/\mathbb{C}}$. Further, note that for $\alpha \in \mathbb{C}$, by the correspondence (3), $\mathbf{u}_\alpha/\mathbf{Ufl}_{/\mathbb{C}}$ is (isomorphic to) the category with objects ($e \in \mathbb{X}$, $f \downarrow_{\mathbb{C}}^{\mathbb{X}} \in |\, \mathbf{Ufl}_{/\mathbb{C}}\,|$) such that $f(e) = \alpha$, and with morphisms $(e, f) \xrightarrow{h} (e', f')$ given by maps $f \xrightarrow{h} f'$ in $\mathbf{Ufl}_{/\mathbb{C}}$ such that $h(e) = e'$. In particular, $\mathbf{u}_\varepsilon/\mathbf{Ufl}_{/\mathbf{M}(A)}$ is the category of transition categories with a designated object (typically modelling an initial state).

We remark that the situation (2) induces the following one:

$$\alpha/\mathbb{C}_{>} \longrightarrow \alpha/\mathbb{C}_{\not\asymp} \hookrightarrow \mathbf{u}_\alpha/\mathbf{Ufl}_{/\mathbb{C}} \qquad (\alpha \in \mathbb{C})$$

between pointed path categories and pointed processes.

Linearly-controlled processes. We turn attention to *interleaving models*. These consist of processes varying over a category whose basic processes (or paths) are linear. Formally, we say that a category $\mathbb{C}$ is *path-linearisable* if, for every morphism $\alpha \in \mathbb{C}$, the interval category $[\![\alpha]\!]$ is a linear preorder, and we refer to ufl functors over path-linearisable categories as *linearly-controlled* process (see [BF99, § 4]).

Theorem 2.2. *([BF99, § 4]) A category $\mathbb{C}$ is path-linearisable if and only if the functor $[\![_]\!] : \mathbb{C}_{\not\asymp} \longrightarrow \mathbf{Cat} : \alpha \longmapsto [\![\alpha]\!]$ preserves the pushouts (4).*

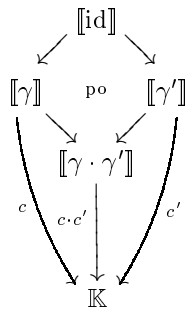
(In the setting of [BF99], this theorem corresponds to the equivalence between the properties (CFI) and (IG).)

Examples of path-linearisable categories are the monoids $\mathbf{M}(A)$ (for any set A) and $\mathbf{R}$, the posets $\mathbf{P}(A)$ (for any set A) and $\mathbf{T}$, and free categories on graphs. Moreover, path-linearisable categories are close under the sum and tensor (see '§ Tensor' in Section 3) of categories, but not under product.

There is a universal way in which to transform a bundle over a path-linearisable category into a linearly-controlled process. The construction is given by the proposition below.

Proposition 2.1. *([BN98]) For a path-linearisable category $\mathbb{C}$, the embedding $\mathbf{Ufl}_{/\mathbb{C}} \hookrightarrow \mathbf{Cat}_{/\mathbb{C}}$ has a right adjoint $\mathrm{U} : \mathbf{Cat}_{/\mathbb{C}} \longrightarrow \mathbf{Ufl}_{/\mathbb{C}}$.*

One can explicitly define $\mathrm{U}\left(\begin{smallmatrix}\mathbb{K}\\ \downarrow\\ \mathbb{C}\end{smallmatrix}\right)$ as the first-projection functor $\begin{smallmatrix}\overline{\mathbb{K}}\\ \downarrow\\ \mathbb{C}\end{smallmatrix}$ where $\overline{\mathbb{K}}$ has: objects given by pairs (C,k) with $C \in |\ \mathbb{C}\ |$ and $k \in \mathbf{Cat}_{/\mathbb{C}}\left(\begin{smallmatrix}[\![\mathrm{id}_C]\!] & \mathbb{K}\\ \mathbf{u}_{\mathrm{id}_C}\downarrow & ,\ \downarrow\\ \mathbb{C} & \mathbb{C}\end{smallmatrix}\right)$; morphisms $(C,k) \longrightarrow (C',k')$ given by pairs (γ, c) with $\gamma : C \longrightarrow C'$ in $\mathbb{C}$ and $c \in \mathbf{Cat}_{/\mathbb{C}}\left(\begin{smallmatrix}[\![\gamma]\!] & \mathbb{K}\\ \mathbf{u}_{\gamma}\downarrow & ,\ \downarrow\\ \mathbb{C} & \mathbb{C}\end{smallmatrix}\right)$ such that $k = c \circ [\![\mathrm{id}_C, \gamma]\!]$ and $c \circ [\![\gamma, \mathrm{id}_{C'}]\!] = k'$; identities $\mathrm{id}_{(C,k)} = (\mathrm{id}_C, k)$; and composition $(\gamma, c) \cdot (\gamma', c') = (\gamma \cdot \gamma', c \cdot c')$ given by the following construction



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It follows from the proposition that, for $\mathbb{C}$ path-linearisable, colimits in $\mathbf{Ufl}_{/\mathbb{C}}$ are computed as in $\mathbf{Cat}_{/\mathbb{C}}$. In particular, we have the following corollary.

Corollary 2.1. *For a linearly-controlled process* $\begin{smallmatrix}\mathbb{X}\\ f\downarrow\\ \mathbb{C}\end{smallmatrix}$,

$$\mathbb{X} \cong \mathrm{colim}\left(\mathbb{X}_{\lightning} \xrightarrow{f_{\lightning}} \mathbb{C}_{\lightning} \xrightarrow{[\![_]\!]} \mathbf{Cat}\right) \quad .$$

Bisimulation. For bundles $f\downarrow_{\mathbb{C}}^{\mathbb{X}}$ and $f'\downarrow_{\mathbb{C}}^{\mathbb{X}'}$, a relation $R \subseteq |\,\mathbb{X}\,| \times |\,\mathbb{X}'\,|$ is said to be a *simulation* (*cf.* [Mil89]) if $x_0\ R\ x_0'$ implies that, for every $x_0 \xrightarrow{e} x$ in $\mathbb{X}$, there exists $x_0' \xrightarrow{e'} x'$ in $\mathbb{X}'$ such that $f(e) = f'(e')$ and $x\ R\ x'$.

Proposition 2.2. *For a path-linearisable category $\mathbb{C}$, consider $f\downarrow_{\mathbb{C}}^{\mathbb{X}} \xrightarrow{h} f'\downarrow_{\mathbb{C}}^{\mathbb{X}'}$ in $\mathbf{Ufl}_{/\mathbb{C}}$. The functor h is open w.r.t. the inclusion $\mathbb{C}_> \longrightarrow \mathbf{Ufl}_{/\mathbb{C}}$ if and only if, for every $x_0 \in |\,\mathbb{X}\,|$ and every $h(x_0) \xrightarrow{e'} x'$ in $\mathbb{X}'$, there exists $x_0 \xrightarrow{e} x$ in $\mathbb{X}$ such that $h(e) = e'$.*

For processes $\downarrow_{\mathbb{C}}^{\mathbb{X}}$ and $\downarrow_{\mathbb{C}}^{\mathbb{X}'}$, we say that $x \in |\,\mathbb{X}\,|$ and $x' \in |\,\mathbb{X}'\,|$ such that $f(x) = f'(x')$ are *open-map bisimilar* if there exists a span of functors

$$\downarrow_{\mathbb{C}}^{\mathbb{X}} \xleftarrow{h} \downarrow_{\mathbb{C}}^{\mathbb{W}} \xrightarrow{h'} \downarrow_{\mathbb{C}}^{\mathbb{X}'}$$

which are open w.r.t. the inclusion $\mathbb{C}_> \longrightarrow \mathbf{Ufl}_{/\mathbb{C}}$ such that $h(w) = x$ and $h'(w) = x'$, for some $w \in |\,\mathbb{W}\,|$.

Corollary 2.2. *For linearly-controlled processes, the notions of bisimilarity and of open-map bisimilarity coincide.*

Relation to presheaf models. The category of processes $\mathbf{Ufl}_{/\mathbb{C}}$ is intimately related to the presheaf categories over the path categories $\mathbb{C}_{\not\asymp}$ (see [BF99]) and $\mathbb{C}_>$.

We have faithful functors

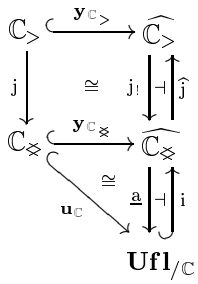
$$\mathbf{Ufl}_{/\mathbb{C}} \hookrightarrow^{\mathrm{i}} \widehat{\mathbb{C}_{\not\asymp}} \xrightarrow{\hat{\mathrm{j}}} \widehat{\mathbb{C}_>} \quad , \tag{5}$$

where $\mathrm{i}(f) \stackrel{\text{def}}{=} \mathbf{Ufl}_{/\mathbb{C}}(\mathbf{u}_{(_)}, f)$ (*cf.* (3)) and $\hat{\mathrm{j}}(P) \stackrel{\text{def}}{=} P\,\mathrm{j}^{\mathrm{op}}$, that correspond to *unfolding* a process into the presheaf of its computations (or runs). This is clearly seen when $\mathbb{C} = \mathbf{M}\,(A)$, in which case $\widehat{\mathbb{C}_>} \cong \widehat{\mathbf{P}\,(A)}$ is (equivalent to) the category $\mathbf{SF}_A$ of A-labelled synchronisation forests (see [JNW96]) and the composite (5) amounts to the functor

$$\mathbf{TC}_A \longrightarrow \mathbf{SF}_A$$

that unfolds a transition graph into the forest of synchronisation trees rooted at each state (see [WN95, JNW96]).

In more detail, we have the following relationship between the fibred viewpoint and the presheaf models



where the functors $\mathrm{j}_!$ and $\underline{\mathrm{a}}$ are obtained as free cocontinous extensions. This diagram allows us to relate the various notions of open map (and hence of bisimilarity) as follows.

Proposition 2.3. *The functors* $\mathrm{j}_!$, $\underline{\mathrm{a}}$, i, *and* $\widehat{\mathrm{j}}$ *preserve open maps (w.r.t. the embeddings* $\mathbf{y}_{\mathbb{C}_>}$, $\mathbf{y}_{\mathbb{C}_{\not<}}$, *and* $\mathbf{u}_{\mathbb{C}}$*).*

Corollary 2.3. *A functor* h *in* $\mathbf{Ufl}_{/\mathbb{C}}$ *is* $\mathbf{u}_{\mathbb{C}}$*-open if and only if the map* $\mathrm{i}(h)$ *in* $\widehat{\mathbb{C}_{\not<}}$ *is* $\mathbf{y}_{\mathbb{C}_{\not<}}$*-open.*

3 Constructions on Processes

We present a tool-kit of categorical constructions on processes that can be regarded as the basis of an abstract process description language, *cf.* [WN95, § 2] and [CW97].

Composition. We consider two versions of operations obtained by post-composition.

1. For a ufl functor $g : \mathbb{A} \longrightarrow \mathbb{B}$, we define

$$\begin{array}{rcl} \Sigma_g : \mathbf{Ufl}_{/\mathbb{A}} & \longrightarrow & \mathbf{Ufl}_{/\mathbb{B}} \\ \overset{\mathbb{X}}{f\downarrow}_{\mathbb{A}} & \longmapsto & \overset{\mathbb{X}}{gf\downarrow}_{\mathbb{B}} \end{array}$$

2. For a functor $g : \mathbb{A} \longrightarrow \mathbb{B}$ where $\mathbb{B}$ is path-linearisable, we define

$$\begin{array}{rcl} \mathrm{U}_g : \mathbf{Ufl}_{/\mathbb{A}} & \longrightarrow & \mathbf{Ufl}_{/\mathbb{B}} \\ \overset{\mathbb{X}}{f\downarrow}_{\mathbb{A}} & \longmapsto & \mathrm{U}\left(\overset{\mathbb{X}}{gf\downarrow}_{\mathbb{B}} \right) \end{array}$$

We will also need the following operation given by pre-composition: for $h : u \longrightarrow v$ in $\mathbf{Ufl}_{/\mathbb{C}}$, we define

$$\begin{array}{rcl} (_)|_h : v/\mathbf{Ufl}_{/\mathbb{C}} & \longrightarrow & u/\mathbf{Ufl}_{/\mathbb{C}} \\ (y, \mathbb{Y}, g) & \longmapsto & (yh, \mathbb{Y}, g) \end{array}$$

Pullback. For a functor $h : \mathbb{A} \longrightarrow \mathbb{B}$, we define the *pullback* (along h) functor

$$\begin{array}{rcl} h^* : \mathbf{Ufl}_{/\mathbb{B}} & \longrightarrow & \mathbf{Ufl}_{/\mathbb{A}} \\ \overset{\mathbb{X}}{f\downarrow}_{\mathbb{B}} & \longmapsto & h^*f\downarrow_{\mathbb{A}}^{h^*\mathbb{X}} \end{array}$$

where h^*f is the first-projection functor from the subcategory $h^*\mathbb{X}$ of $\mathbb{A} \times \mathbb{X}$ with morphisms $(\alpha, e) : (a, x) \longrightarrow (a', x')$ such that $h(\alpha) = f(e)$.

Saturation. The operation of *saturation* (*cf.* [FCW99]) by a functor $h : \mathbb{A} \longrightarrow \mathbb{B}$, where $\mathbb{A}$ is path-linearisable, is given by the monad on $\mathbf{Ufl}_{/\mathbb{A}}$ induced by the following composite of adjoints:

$$\mathbf{Ufl}_{/\mathbb{A}} \underset{\top}{\overset{U}{\hookleftarrow}} \mathbf{Cat}_{/\mathbb{A}} \underset{\Sigma_h}{\overset{h^*}{\underset{\top}{\leftarrow}}} \mathbf{Cat}_{/\mathbb{B}} \ .$$

Product. The *product* functor is defined as follows

$$\begin{array}{rcl} _ \times _ : \mathbf{Ufl}_{/\mathbb{A}} \times \mathbf{Ufl}_{/\mathbb{B}} & \longrightarrow & \mathbf{Ufl}_{/\mathbb{A}\times\mathbb{B}} \\ \overset{\mathbb{X}}{f\downarrow}_{\mathbb{A}} \ , \ \overset{\mathbb{Y}}{g\downarrow}_{\mathbb{B}} & \longmapsto & f \times g\downarrow_{\mathbb{A}\times\mathbb{B}}^{\mathbb{X}\times\mathbb{Y}} \end{array}$$

Sum. The *sum* functor is given by the construction

$$\begin{array}{rcl} _ + _ : \mathbf{Ufl}_{/\mathbb{A}} \times \mathbf{Ufl}_{/\mathbb{B}} & \longrightarrow & \mathbf{Ufl}_{/\mathbb{A}+\mathbb{B}} \\ \overset{\mathbb{X}}{f\downarrow}_{\mathbb{A}} \ , \ \overset{\mathbb{Y}}{g\downarrow}_{\mathbb{B}} & \longmapsto & f + g\downarrow_{\mathbb{A}+\mathbb{B}}^{\mathbb{X}+\mathbb{Y}} \end{array}$$

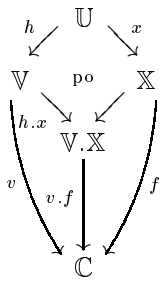
and induces the *fibred coproduct* functor $(_) \underset{\mathbb{C}}{+} (_)$ given by the composite

$$\mathbf{Ufl}_{/\mathbb{C}} \times \mathbf{Ufl}_{/\mathbb{C}} \overset{+}{\longrightarrow} \mathbf{Ufl}_{/\mathbb{C}+\mathbb{C}} \overset{\Sigma_{[\mathrm{Id}_{\mathbb{C}}, \mathrm{Id}_{\mathbb{C}}]}}{\longrightarrow} \mathbf{Ufl}_{/\mathbb{C}} \ .$$

Pushout. For $h : u \longrightarrow v$ in $\mathbf{Ufl}_{/\mathbb{C}}$ where $\mathbb{C}$ is path-linearisable, we have a *pushout* (along h) functor

$$\begin{array}{rcl} h._ : u/\mathbf{Ufl}_{/\mathbb{C}} & \longrightarrow & v/\mathbf{Ufl}_{/\mathbb{C}} \\ (x, \mathbb{X}, f) & \longmapsto & (h.x \ , \ \mathbb{V}.\mathbb{X} \ , \ v.f) \end{array}$$

defined by the diagram



.

Tensor. The *tensor* category (see, *e.g.*, [PR97, § 2] or [Str96, § 6]) $\mathbb{A} \otimes \mathbb{B}$ of the small categories $\mathbb{A}$ and $\mathbb{B}$ has objects given by pairs $a \otimes b$, with $a \in |\ \mathbb{A}\ |$ and $b \in |\ \mathbb{B}\ |$, and morphisms given by *interleaved (or shuffled) sequences* of non-identity composable maps in $\mathbb{A}$ and $\mathbb{B}$; *i.e.*, morphisms are formal sequences of non-identity morphisms of the form $a_1 \otimes b_1 \xrightarrow{\alpha_1 \oslash b_1} a_2 \otimes b_1 \xrightarrow{a_2 \oslash \beta_1} a_2 \otimes b_2 \longrightarrow \cdots$ or $a_1 \otimes b_1 \xrightarrow{a_1 \oslash \beta_1} a_1 \otimes b_2 \xrightarrow{\alpha_1 \oslash b_2} a_2 \otimes b_2 \longrightarrow \cdots$.

The tensor of categories has a universal characterisation as the following pushout square of functors

$$\begin{array}{ccc} |\ \mathbb{A}\ | \times |\ \mathbb{B}\ | & \xrightarrow{\mathrm{Id} \times \mathrm{J}_{\mathbb{B}}} & |\ \mathbb{A}\ | \times \mathbb{B} \\ \Big\downarrow{\scriptstyle \mathrm{J}_{\mathbb{A}} \times \mathrm{Id}} & \text{po} & \Big\downarrow{\scriptstyle _ \oslash _} \\ \mathbb{A} \times |\ \mathbb{B}\ | & \xrightarrow[_ \oslash _]{} & \mathbb{A} \otimes \mathbb{B} \end{array} \quad ,$$

from which one easily sees that the tensor category of two monoids is their coproduct in the category of monoids. Thus, in particular,

$$\mathbf{M}\,(A) \otimes \mathbf{M}\,(B) \cong \mathbf{M}\,(A + B) \quad .$$

Moreover, the construction induces a *tensor* functor between categories of processes:

$$\begin{array}{rcl} _ \otimes _ : \mathbf{Ufl}_{/\mathbb{A}} \times \mathbf{Ufl}_{/\mathbb{B}} & \longrightarrow & \mathbf{Ufl}_{/\mathbb{A} \otimes \mathbb{B}} \\ \mathbb{X} \quad \mathbb{Y} & & \mathbb{X} \otimes \mathbb{Y} \\ f \downarrow\ ,\ g \downarrow & \longmapsto & f \otimes g \downarrow \\ \mathbb{A} \quad \mathbb{B} & & \mathbb{A} \otimes \mathbb{B} \end{array}$$

3.1 Examples

We show that, for transition categories, the above categorical constructions yield typical operations of process calculi (*cf.* [WN95]). Hybrid systems are discussed in Section 4.

Relabelling. The operation of *relabelling* according to a function $\rho : A \longrightarrow B$ corresponds to the functor

$$_\{\rho\} \stackrel{\text{def}}{=} \Sigma_{\mathbf{M}(\rho)}(_) : \mathbf{Ufl}_{/\mathbf{M}(A)} \longrightarrow \mathbf{Ufl}_{/\mathbf{M}(B)}$$

where $\mathbf{M}(\rho)$ is the free homomorphic extension of the function mapping: $a \in A \longmapsto \rho(a)$.

Restriction. Let $A \subseteq B$, and write $\iota : A \longrightarrow B$ for the inclusion function. The *restriction* operation corresponds to the functor

$$_ \restriction \iota \stackrel{\text{def}}{=} \mathbf{M}(\iota)^*(_) : \mathbf{Ufl}_{/\mathbf{M}(B)} \longrightarrow \mathbf{Ufl}_{/\mathbf{M}(A)} \ .$$

Saturation. The main example of a saturation monad is given by the construction that associates a transition graph with the transition graph obtained from saturating transitions by the hidden insertion of silent actions; *i.e.*, the construction introduced by Milner to define weak bisimulation (see [Mil89]).

In our setting (as in [FCW99], where an account of weak bisimilarity for presheaf models required, crucially, the adoption of a fibred view of processes), the saturation by silent actions arises as the saturation monad (of Section 3 '§ Saturation') by the homomorphism $\mathbf{M}(A + \{\tau\}) \longrightarrow \mathbf{M}(A)$ that *hides* the silent τ actions (*viz.*, the free homomorphic extension of the function mapping: $a \in A \longmapsto a$, $\tau \longmapsto \varepsilon$).

Generalising from the above discussion, we have the following general definition of weak bisimulation w.r.t. a hiding functor $h : \mathbb{A} \longrightarrow \mathbb{B}$: for processes $f \downarrow^{\mathbb{X}}_{\mathbb{A}}$ and $f' \downarrow^{\mathbb{X}'}_{\mathbb{A}}$, we say that $x \in |\ \mathbb{X}\ |$ and $x' \in |\ \mathbb{X}'\ |$ are *h-bisimilar* (*cf.* [FCW99]) whenever $\eta_f(x)$ and $\eta_{f'}(x')$ are open-map bisimilar, where η denotes the unit of the h-saturation monad.

Parallel composition. We discuss synchronous and asynchronous compositions of processes.

Synchrony. As observed in [WN95, § 2.2.4], versions of parallel composition can be obtained from the product operation by restriction and relabelling. Roughly, the *synchronous parallel composition* of processes can be specified as the combined operation

$$\Sigma_\rho(\sigma^*(_ \times _)) : \mathbf{Ufl}_{/\mathbb{A}} \times \mathbf{Ufl}_{/\mathbb{B}} \longrightarrow \mathbf{Ufl}_{/\mathbb{C}}$$

where $\sigma : \mathbb{S} \longrightarrow \mathbb{A} \times \mathbb{B}$ is a *syncronisation* functor and $\rho : \mathbb{S} \longrightarrow \mathbb{C}$ a *relabelling* ufl functor.

Synchronisation à la CCS is given by the functor

$$(\sigma^*(_ \times _))\,\{\rho\} : \mathbf{Ufl}_{/\mathbf{M}(A+\{\tau\}+B)} \times \mathbf{Ufl}_{/\mathbf{M}(B+\{\tau\}+C)} \longrightarrow \mathbf{Ufl}_{/\mathbf{M}(A+\{\tau\}+C)}$$

where the synchronisation map

$$\mathbf{M}\,(A + \{(\tau,*)\} + B + \{(*,\tau)\} + C) \xrightarrow{\sigma} \mathbf{M}\,(A + \{\tau\} + B) \times \mathbf{M}\,(B + \{\tau\} + C)$$

is the free homomorphic extension of the function mapping: $a \in A \longmapsto (a,\varepsilon)$, $(\tau,*) \longmapsto (\tau,\varepsilon)$, $b \in B \longmapsto (b,b)$, $(*,\tau) \longmapsto (\varepsilon,\tau)$, $c \in C \longmapsto (\varepsilon,c)$; and where the relabelling function

$$A + \{(\tau,*)\} + B + \{(*,\tau)\} + C \xrightarrow{\rho} A + \{\tau\} + C$$

is the following mapping: $a \in A \longmapsto a$, $(\tau,*) \longmapsto \tau$, $b \in B \longmapsto \tau$, $(*,\tau) \longmapsto \tau$, $c \in C \longmapsto c$.

On the other hand, *synchronisation à la CSP*, corresponds to

$$(\sigma^*(_ \times _))\,\{\rho\} : \mathbf{Ufl}_{/\mathbf{M}(A+\{\tau\})} \times \mathbf{Ufl}_{/\mathbf{M}(A+\{\tau\})} \longrightarrow \mathbf{Ufl}_{/\mathbf{M}(A+\{\tau\})}$$

where the synchronisation map

$$\mathbf{M}\,(A + \{(\tau,*)\} + \{(*,\tau)\}) \xrightarrow{\sigma} \mathbf{M}\,(A + \{\tau\}) \times \mathbf{M}\,(A + \{\tau\})$$

is the free homomorphic extension of the function mapping: $a \in A \longmapsto (a,a)$, $(\tau,*) \longmapsto (\tau,\varepsilon)$, $(*,\tau) \longmapsto (\varepsilon,\tau)$; and where the relabelling function

$$A + \{(\tau,*)\} + \{(*,\tau)\} \xrightarrow{\rho} A + \{\tau\}$$

is the following mapping: $a \in A \longmapsto a$, $(\tau,*) \longmapsto \tau$, $(*,\tau) \longmapsto \tau$.

Asynchrony. The *asynchronous parallel composition* of processes can be described along the above lines as the functor

$$\sigma^* : \mathbf{Ufl}_{/\mathbf{M}(A)} \times \mathbf{Ufl}_{/\mathbf{M}(B)} \longrightarrow \mathbf{Ufl}_{/\mathbf{M}(A+B)}$$

where the synchronisation map

$$\mathbf{M}\,(A + B) \xrightarrow{\sigma} \mathbf{M}\,(A) \times \mathbf{M}\,(B)$$

is the free homomorphic extension of the function mapping: $a \in A \longmapsto (a,\varepsilon)$, $b \in B \longmapsto (\varepsilon,b)$.

However, there is a more direct description; namely, as the tensor functor

$$\mathbf{Ufl}_{/\mathbf{M}(A)} \times \mathbf{Ufl}_{/\mathbf{M}(B)} \xrightarrow{\otimes} \mathbf{Ufl}_{/\mathbf{M}(A)\otimes\mathbf{M}(B)}\ .$$

Non-deterministic sum. For sets A and B, let $\iota_A : A \longrightarrow A \cup B$ and $\iota_B : B \longrightarrow A \cup B$ be the respective inclusion functions.

The *non-deterministic sum* of transition categories is given by the following construction:

$$\left(_ \underset{\mathbf{M}(A\cup B)}{+} _\right) \circ (_\{\iota_A\}, _\{\iota_B\}) : \mathbf{Ufl}_{/\mathbf{M}(A)} \times \mathbf{Ufl}_{/\mathbf{M}(B)} \longrightarrow \mathbf{Ufl}_{/\mathbf{M}(A\cup B)}\ .$$

Prefix. The *prefix* operation on transition categories, for $a \in A$, arises as the composite

$$\left((_)|_{[\![\varepsilon,a]\!]}\right) \circ ([\![a,\varepsilon]\!]._) : \mathbf{u}_\varepsilon/\mathbf{Ufl}_{/\mathbf{M}(A)} \longrightarrow \mathbf{u}_\varepsilon/\mathbf{Ufl}_{/\mathbf{M}(A)}\ .$$

4 Fibred Models of Hybrid Systems

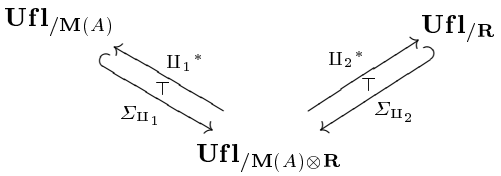
Hybrid systems are typically described as *mixed discrete-continuous* systems, and our development has been aimed at putting this slogan in mathematical form.

We have considered the categories $\mathbf{Ufl}_{/\mathbb{C}}$ as fibred models of processes, and observed, in Subsections 1.1 and 1.2, that for $\mathbb{C} = \mathbf{M}(A)$ the resulting processes correspond to discrete systems, whilst for $\mathbb{C} = \mathbf{R}$ they correspond to continuous ones. Models of hybrid systems arise from considering categories $\mathbb{C}$ where discrete actions and continuous evolutions are interleaved. The simplest example is the model

$$\mathbf{Ufl}_{/\mathbf{M}(A)\otimes\mathbf{R}}$$

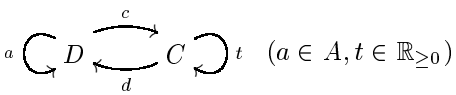
consisting of processes $\begin{array}{c}\mathbb{X}\\ \downarrow\\ \mathbf{M}(A)\otimes\mathbf{R}\end{array}$ where the evolutions in the state space $\mathbb{X}$ factor uniquely as interleavings (or shuffles) of discrete and continuous evolutions. Examples of such processes arise from *hybrid automata* (*cf.* [Hen96, Definition 1.3]).

The models of discrete, continuous, and hybrid systems introduced relate as follows



Thus the notions of open-map (and hence of bisimilarity) for hybrid systems extend that of discrete and continuous systems.

We remark that other *non-standard* models of hybrid systems can also be accomodated. For instance, hybrid systems with two modes of operation (discrete and continuous) where mode changes are explicit, can be seen as processes over the category generated by the diagram



$(a \in A, t \in \mathbb{R}_{\geq 0})$

under the condition that the composite $t \cdot t'$ be $t + t'$, for all $t, t' \in \mathbb{R}_{\geq 0}$.

4.1 Constructions on Hybrid Systems

We exemplify the use of the constructions of Section 3 on hybrid systems.

Parallel composition. The following operation corresponds to the *parallel composition* of hybrid systems as considered in [Hen96, § 1.4]:

$$\sigma^*(_ \times _) : \mathbf{Ufl}_{/\mathbf{M}(A+B)\otimes\mathbf{R}} \times \mathbf{Ufl}_{/\mathbf{M}(B+C)\otimes\mathbf{R}} \longrightarrow \mathbf{Ufl}_{/\mathbf{M}(A+B+C)\otimes\mathbf{R}}$$

where the synchronisation map

$$\mathbf{M}\,(A+B+C)\otimes\mathbf{R}\xrightarrow{\sigma}(\mathbf{M}\,(A+B)\otimes\mathbf{R})\times(\mathbf{M}\,(B+C)\otimes\mathbf{R})$$

is the unique homomorphism such that: $a\in A\longmapsto(a,\varepsilon)$, $b\in B\longmapsto(b,b)$, $c\in C\longmapsto(\varepsilon,c)$, $t\in\mathbb{R}_{>0}\longmapsto(t,t)$.

Time abstraction. We consider two versions of *time abstraction*. They both arise as the operation

$$\mathrm{U}_{\overline{\rho}}\circ\overline{h}^{*}:\mathbf{Ufl}_{/\mathbf{M}(A)\otimes\mathbf{R}}\longrightarrow\mathbf{Ufl}_{/\mathbf{M}(A+\{\tau\})}\ ,$$

where the diagram

$$\begin{array}{ccc} M & \xrightarrow{\ \overline{h}\ } & \mathbf{M}\,(A)\otimes\mathbf{R} \\ {\scriptstyle\overline{\rho}}\downarrow & & \downarrow{\scriptstyle\rho} \\ \mathbf{M}\,(A+\{\tau\}) & \xrightarrow[\ h\ ]{} & \mathbb{C} \end{array}$$

is a pullback square, for suitable choices of ρ and h.

1. The following version of time abstraction corresponds to the one considerd in [Hen96, Definition 1.3]. Take $\mathbb{C}=\mathbf{M}\,(A)\otimes\mathbf{E}$, where $\mathbf{E}$ is the monoid given by the following table

$\cdot_{\mathbf{E}}$	id	τ
id	id	τ
τ	τ	τ

   ,

   together with $\rho:\mathbf{M}\,(A)\otimes\mathbf{R}\longrightarrow\mathbf{M}\,(A)\otimes\mathbf{E}$ the unique homomorphism mapping: $a\in A\longmapsto a$, $t\in\mathbb{R}_{>0}\longmapsto\tau$ and $h:\mathbf{M}\,(A+\{\tau\})\longrightarrow\mathbf{M}\,(A)\otimes\mathbf{E}$ the free homomorphic extension of the function mapping: $a\in A\longmapsto a$, $\tau\longmapsto\tau$.
2. A slightly simpler time-abstraction operation is obtained by taking $\mathbb{C}=\mathbf{M}\,(A)$ together with $\rho:\mathbf{M}\,(A)\otimes\mathbf{R}\longrightarrow\mathbf{M}\,(A)$ the unique homomorphism mapping: $a\in A\longmapsto a$, $t\in\mathbb{R}_{\geq 0}\longmapsto\varepsilon$ and $h:\mathbf{M}\,(A+\{\tau\})\longrightarrow\mathbf{M}\,(A)$ the free homomorphic extension of the function mapping: $a\in A\longmapsto a$, $\tau\longmapsto\varepsilon$.

   In this case, for a hybrid system $\begin{array}{c}\mathbb{X}\\ {\scriptstyle f}\downarrow\\ \mathbf{M}\,(A)\otimes\mathbf{R}\end{array}$, the transition category $\mathrm{U}_{\overline{\rho}}\left(\overline{h}^{*}f\right)$ obtained by time abstraction corresponds to the transition graph with set of nodes $|\,\mathbb{X}\,|$, set of edges $\{\,(a,e)\in A\times\mathbb{X}\mid f(e)=a\vee f(e)=t\cdot a\vee f(e)=a\cdot t'\vee f(e)=t\cdot a\cdot t'$, for some $t,t'\in\mathbb{R}_{>0}\,\}\cup\{\,(\tau,e)\in\{\tau\}\times\mathbb{X}\mid f(e)=\varepsilon\vee f(e)=t$, for some $t\in\mathbb{R}_{>0}\,\}$, with domains and codomains inherited from $\mathbb{X}$ and with labelling function given by first projection.

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