Finitary Construction of Free Algebras for Equational Systems

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Abstract

The purpose of this paper is threefold: to present a general abstract, yet practical, notion of equational system; to investigate and develop the finitary construction of free algebras for equational systems; and to illustrate the use of equational systems as needed in modern applications.

Key words: Equational system; algebra; free construction; monad.

1 Introduction

The import of equational theories in theoretical computer science is by now well established. Traditional applications include the initial algebra approach to the semantics of computational languages and the specification of abstract data types pioneered by the ADJ group [14], and the abstract description of powerdomain constructions as free algebras of non-determinism advocated by Plotkin [15,18] (see also [1]). While these developments essentially belong to the realm of universal algebra, more recent applications have had to be based on the more general categorical algebra. Examples include theories of abstract syntax with variable binding [10,12], the algebraic treatment of computational effects [19,20], and models of name-passing process calculi [9,23].

In the above and most other applications of equational theories, the existence and construction of initial and/or free algebras, and consequently of monads, plays a central role; as so does the study of categories of algebras. These topics are investigated here in the context of *equational systems*, a very broad notion of equational theory. Examples of equational systems include enriched

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algebraic theories [16,22], algebras for a monad, monoids in a monoidal category, *etc.* (see Section 3).

The original motivation for the development of the theory of equational systems arose from the need of a mathematical theory readily applicable to two further examples of equational systems: (*i*) Σ-monoids (see [8, Section 6.1]), which are needed for the initial algebra approach to the semantics of languages with variable binding and capture-avoiding simultaneous substitution [10]; and (*ii*) π-algebras (see [8, Section 6.2]), which provide algebraic models of the finitary π-calculus [23]. Indeed, these two examples respectively highlight two inadequacies of enriched algebraic theories in applications: (*i*) the explicit presentation of an enriched algebraic theory may be hard to give, as it is the case with Σ-monoids; and (*ii*) models may require a theory based on more than one enrichment, as it is the case with π-algebras.

Further benefits of equational systems over enriched algebraic theories are that the theory can be developed for cocomplete, not necessarily locally presentable, categories (examples of which are the category of topological spaces, the category of directed-complete posets, and the category of complete semilattices), and that the concept of equational system is straightforwardly dualizable: a coequational system on a category is simply an equational system on the opposite category (thus, for instance, comonoids in a monoidal category are coalgebras for a coequational system). On the other hand, the price paid for all this generality is that the important connection between enriched algebraic theories and enriched Lawvere theories [21] is lost for equational systems.

An equational system $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ is defined as a parallel pair $L, R : \Sigma\text{-}\mathbf{Alg} \to \Gamma\text{-}\mathbf{Alg}$ of functors between categories of algebras over a base category $\mathscr{C}$. In this context, the endofunctor Σ on $\mathscr{C}$, which generalizes the notion of algebraic signature, is called a functorial signature; the functors L and R over $\mathscr{C}$ generalize the notion of equation and are called functorial terms; the endofunctor Γ on $\mathscr{C}$, referred to as a functorial context, corresponds to the context of the terms. The category of $\mathbb{S}$-algebras is the equalizer $\mathbb{S}\text{-}\mathbf{Alg} \hookrightarrow \Sigma\text{-}\mathbf{Alg}$ of L, R. Thus, an $\mathbb{S}$-algebra is a Σ-algebra $(X, s : \Sigma X \to X)$ such that $L(X, s) = R(X, s)$ as Γ-algebras on X.

Free constructions for equational systems are investigated in Section 4. For an equational system $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$, the existence of free $\mathbb{S}$-algebras on objects in $\mathscr{C}$ is considered in two stages: (*i*) the construction of free Σ-algebras on objects in $\mathscr{C}$, and (*ii*) the construction of free $\mathbb{S}$-algebras over Σ-algebras. The former captures the construction of freely generated terms with operations from the functorial signature Σ; the latter that of quotienting Σ-algebras by the equation $L = R$ and congruence rules. We give a finitary sufficient condition for the existence of free $\mathbb{S}$-algebras on

Σ-algebras. This condition can be used to deduce the existence of free algebras for enriched algebraic theories, but it applies more generally. The proofs of this result provides a finitary construction of free algebras that may lead to explicit descriptions. As a concrete example of this situation, we consider in Section 6 the recently introduced Nominal Equational Logic by Clouston and Pitts [7].

Monads and categories of algebras for equational systems are discussed in Section 5. In the vein of the above result, we provide a finitary condition under which the monadicity and cocompleteness of categories of algebras follow. As an application, we prove in Section 6 that the category of algebras for a Nominal Equational Logic theory is monadic and cocomplete.

We have learnt during the course of this work that variations on the concept of equational system have already been considered in the literature. For instance, Fokkinga [11] introduces the more general concept of law between transformers, but only studies initial algebras for the laws that are equational systems; Cîrstea [6] introduces the concept of coequation between abstract cosignatures, which is equivalent to our notion of coequational system, and studies final coalgebras for them; Ghani, Lüth, De Marchi, and Power [13] introduce the concept of functorial coequational presentations, which is equivalent to our notion of coequational system on a locally presentable base category with an accessible functorial signature and an accessible functorial context, and study cofree constructions for them.

Our theory of equational systems (and its dual), which we present in Sections 4 and 5, is more general and comprehensive than that of [11] and [6]; and we have briefly discussed how it relates to that of [13] in [8, Section 7].

2 Algebraic equational theories

To set our work in context, we briefly review the classical concept of algebraic equational theory and some aspects of the surrounding theory (see *e.g.* [24]).

An algebraic equational theory consists of a signature defining its operations and a set of equations describing the axioms that it should obey.

A signature $\Sigma = (O, [-])$ is given by a set of operators O together with a function $[-] : O \to \mathbb{N}$ giving an arity to each operator. The set of terms $T_\Sigma(V)$ on a set of variables V is built up from the variables and the operators of the signature Σ by the following grammar

$$t \in T_\Sigma(V) \ ::= \ v \ | \ o(t_1, \ldots, t_k)$$

where $v \in V$, o is an operator of arity k, and $t_i \in T_\Sigma(V)$ for all $i = 1, \ldots, k$.

An equation of arity V for a signature Σ, written $\Sigma \rhd V \vdash l = r$, is given by a pair of terms $l, r \in T_\Sigma(V)$.

An algebraic equational theory $\mathbb{T} = (\Sigma, E)$ is given by a signature Σ together with a set of equations E.

An algebra for a signature Σ is a pair $(X, [\![-]\!]_X)$ consisting of a carrier set X together with interpretation functions $[\![o]\!]_X : X^{[o]} \to X$ for each operator o in Σ. By structural induction, such an algebra induces interpretations $[\![t]\!]_X : X^V \to X$ of terms $t \in T_\Sigma(V)$ as follows:

$$[\![t]\!]_X = \begin{cases} X^V \xrightarrow{\pi_v} X & \text{, for } \ t = v \in V \\ X^V \xrightarrow{\langle [\![t_1]\!]_X, \ldots, [\![t_k]\!]_X \rangle} X^k \xrightarrow{[\![o]\!]_X} X & \text{, for } \ t = o(t_1, \ldots, t_k) \end{cases}$$

An algebra for the theory $\mathbb{T} = (\Sigma, E)$ is an algebra for the signature Σ that satisfies the constraints given by the equations in E, where a Σ-algebra $(X, [\![-]\!]_X)$ is said to satisfy the equation $\Sigma \rhd V \vdash l = r$ whenever $[\![l]\!]_X \vec{x} = [\![r]\!]_X \vec{x}$ for all $\vec{x} \in X^V$.

A homomorphism of $\mathbb{T}$-algebras from $(X, [\![-]\!]_X)$ to $(Y, [\![-]\!]_Y)$ is a function $h : X \to Y$ between their carrier sets that commutes with the interpretation of each operator; that is, such that $h([\![o]\!]_X(x_1, \ldots, x_k)) = [\![o]\!]_Y(h(x_1), \ldots, h(x_k))$ for all $x_i \in X$. Algebras and homomorphisms form the category $\mathbb{T}$-**Alg**.

The existence of free algebras for algebraic theories is one of the most significant properties that they enjoy. For an algebraic theory $\mathbb{T} = (\Sigma, E)$, the free algebra over a set X has as carrier the set $T_\Sigma(X)/_{\approx_E}$ of equivalence classes of terms on X under the equivalence relation $\approx_E$ defined by setting $t \approx_E t'$ iff t is provably equal to t' by the equations given in E and the congruence rules. The interpretation of each operator on $T_\Sigma(X)/_{\approx_E}$ is given syntactically: $[\![o]\!]([t_1]_{\approx_E}, \ldots, [t_k]_{\approx_E}) = [o(t_1, \ldots, t_k)]_{\approx_E}$. This construction gives rise to a left adjoint to the forgetful functor $U_\mathbb{T} : \mathbb{T}\text{-}\mathbf{Alg} \to \mathbf{Set}$. Moreover, the adjunction is monadic: $\mathbb{T}$-**Alg** is equivalent to the category of algebras for the associated monad on **Set**.

3 Equational systems

We develop abstract notions of signature and equation, leading to the concept of equational system. Free constructions for equational systems are considered in the following section.

3.1 Functorial signatures

We recall the notion of algebra for an endofunctor and how it generalizes that of algebra for a signature.

An algebra for an endofunctor Σ on a category $\mathscr{C}$ is a pair (X, s) of a carrier object X in $\mathscr{C}$ together with an algebra structure map $s : \Sigma X \to X$. A homomorphism of Σ-algebras $(X, s) \to (Y, t)$ is a map $h : X \to Y$ in $\mathscr{C}$ such that $h \cdot s = t \cdot \Sigma h$. Σ-algebras and homomorphisms form the category Σ-**Alg**, and the forgetful functor $U_\Sigma : \Sigma\text{-}\mathbf{Alg} \to \mathscr{C}$ maps an Σ-algebra (X, s) to its carrier object X.

As it is well-known, every algebraic signature can be turned into an endofunctor on **Set** preserving its algebras. Indeed, for a signature Σ, one defines the corresponding endofunctor as $\overline{\Sigma}(X) = \coprod_{o \in \Sigma} X^{[o]}$, so that Σ-**Alg** and $\overline{\Sigma}$-**Alg** are isomorphic. In this view, we will henceforth take endofunctors as a general abstract notion of signature.

Definition 3.1 (Functorial signature) *A functorial signature on a category is an endofunctor on it.*

3.2 Functorial terms

We motivate and present an abstract notion of term for functorial signatures.

Let $t \in T_\Sigma(V)$ be a term on a set of variables V for a signature Σ. Recall from the previous section that for every Σ-algebra $(X, [\![-]\!]_X)$, the term t gives an interpretation function $[\![t]\!]_X : X^V \to X$. Thus, writing Γ_V for the endofunctor $(-)^V$ on **Set**, the term t determines a function $\bar{t}$ assigning to a Σ-algebra $(X, [\![-]\!]_X)$ the Γ_V-algebra $(X, [\![t]\!]_X)$. Note that the function $\bar{t}$ does not only preserve carrier objects but, furthermore, by the uniformity of the interpretation of terms, that a Σ-homomorphism $(X, [\![-]\!]_X) \to (Y, [\![-]\!]_Y)$ is also a Γ_V-homomorphism $(X, [\![t]\!]_X) \to (Y, [\![t]\!]_Y)$. In other words, the function $\bar{t}$ extends to a functor Σ-**Alg** $\to \Gamma_V$-**Alg** over **Set**, *i.e.* a functor preserving carrier objects and homomorphisms. These considerations lead us to define an abstract notion of term in context as follows.

Definition 3.2 (Functorial term) *Let Σ be a functorial signature on a category $\mathscr{C}$. A functorial term $\mathscr{C} : \Sigma \rhd \Gamma \vdash T$ consists of an endofunctor Γ on $\mathscr{C}$, referred to as a functorial context, and a functor $T : \Sigma\text{-}\mathbf{Alg} \to \Gamma\text{-}\mathbf{Alg}$ over $\mathscr{C}$; that is, a functor such that $U_\Gamma \cdot T = U_\Sigma$.*

Typically, when a syntactic signature Σ is turned into a functorial signature $\overline{\Sigma}$

its algebras provide the models of the signature, giving interpretations to the operators. Moreover, when a term in context $\Gamma \vdash t$ is turned into a functorial term $\overline{t} : \overline{\Sigma}\text{-}\mathbf{Alg} \to \overline{\Gamma}\text{-}\mathbf{Alg}$, the object $\overline{\Gamma}X$ intuitively consists of all valuations of the context Γ in X, and the functor $\overline{t}$ encodes the process of evaluating a term to a value, parametrically on models and valuations.

We give a general example of functorial term that frequently arises in applications. To this end, let T_Σ be the free monad on a functorial signature Σ on a category $\mathscr{C}$. For an object $V \in \mathscr{C}$, to be thought of as an object of variables, the object $T_\Sigma V$ intuitively represents the syntactic terms built up from the variables by means of the signature. Under this view, thus, we obtain an abstract notion of syntactic term as a global element of $T_\Sigma V$. Assume now that $\mathscr{C}$ is monoidal closed (with structure I, $\otimes$, $[-,=]$) and that Σ is strong, with strength $\mathsf{st}_{X,V} : X \otimes \Sigma V \to \Sigma(X \otimes V)$. It follows that T_Σ is strong, say with strength $\overline{\mathsf{st}}_{X,V} : X \otimes T_\Sigma V \to T_\Sigma(X \otimes V)$ that provides a means to distribute parameters within terms as specified by st. In this situation, then, every abstract syntactic term $t : I \to T_\Sigma V$ induces a functorial term $\overline{t} : \Sigma\text{-}\mathbf{Alg} \to \Gamma_V\text{-}\mathbf{Alg}$, for the functorial context $\Gamma_V = [V, -]$, as follows:

$$\begin{aligned}&\overline{t}(X, s : \Sigma X \to X)\\ &= \big(\Gamma_V(X) \cong [V,X] \otimes I \xrightarrow{\mathrm{id}\otimes t} [V,X] \otimes T_\Sigma V \xrightarrow{\overline{\mathsf{st}}} T_\Sigma([V,X] \otimes V) \xrightarrow{T_\Sigma(ev)} T_\Sigma X \xrightarrow{\overline{s}} X\big)\end{aligned}$$

where $(X, \overline{s} : T_\Sigma X \to X)$ is the T_Σ-algebra corresponding to the Σ-algebra (X, s).

3.3 Equational systems

We define equational systems, our abstract notion of equational theory.

Definition 3.3 (Equational system) *An equational system*

$$\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$$

is given by a functorial signature Σ on a category $\mathscr{C}$, and a pair of functorial terms $\mathscr{C} : \Sigma \rhd \Gamma \vdash L$ and $\mathscr{C} : \Sigma \rhd \Gamma \vdash R$ referred to as a functorial equation.

We have restricted attention to equational systems subject to a single equation. The consideration of multi-equational systems $(\mathscr{C}, \Sigma \rhd \{\, \Gamma_i \vdash L_i = R_i \,\}_{i \in I})$ subject to a set of equations in what follows is left to the interested reader. We remark however that our development is typically without loss of generality; as, whenever $\mathscr{C}$ has I-indexed coproducts, a multi-equational system as above can be expressed as the equational system $(\mathscr{C} : \Sigma \rhd \coprod_{i \in I} \Gamma_i \vdash [L_i]_{i \in I} = [R_i]_{i \in I})$ with a single equation.

Recall that an equation $\Sigma \rhd V \vdash l = r$ in an algebraic theory is interpreted as the constraint that the interpretation functions associated with the terms l and r coincide. Hence, for an equational system $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$, it is natural to say that a Σ-algebra (X, s) satisfies the functorial equation $\Gamma \vdash L = R$ whenever $L(X, s) = R(X, s) : \Gamma X \to X$, and consequently define the category of algebras for the equational system as the full subcategory of Σ-**Alg** consisting of the Σ-algebras that satisfy the functorial equation $\Gamma \vdash L = R$. Equivalently, we introduce the following definition.

Definition 3.4 *For an equational system $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$, the category $\mathbb{S}$-**Alg** of $\mathbb{S}$-algebras is the equalizer of $L, R : \Sigma\text{-}\mathbf{Alg} \to \Gamma\text{-}\mathbf{Alg}$ (in the large category of locally small categories over $\mathscr{C}$).*

3.4 Examples

Examples of equational systems together with their induced categories of algebras follow.

(1) The equational system $\mathbb{S}_{\mathbb{T}}$ associated to the algebraic theory $\mathbb{T} = (\Sigma, E)$ is given by $(\mathbf{Set} : \Sigma_{\mathbb{T}} \rhd \Gamma_{\mathbb{T}} \vdash L_{\mathbb{T}} = R_{\mathbb{T}})$, with $\Sigma_{\mathbb{T}} X = \coprod_{o \in \Sigma} X^{[o]}$, $\Gamma_{\mathbb{T}} X = \coprod_{(V \vdash l = r) \in E} X^V$, and

$$L_{\mathbb{T}}(X, [\![-]\!]_X) = \Big(X, \big[\, [\![l]\!]_X \,\big]_{(V \vdash l = r) \in E}\Big) \ ,$$

$$R_{\mathbb{T}}(X, [\![-]\!]_X) = \Big(X, \big[\, [\![r]\!]_X \,\big]_{(V \vdash l = r) \in E}\Big) \ .$$

It follows that $\mathbb{T}$-**Alg** is isomorphic to $\mathbb{S}_{\mathbb{T}}$-**Alg**.

(2) More generally, consider an enriched algebraic theory $\mathbb{T} = (\mathscr{C}, B, E, \sigma, \tau)$ on a locally finitely presentable category $\mathscr{C}$ enriched over a suitable category $\mathscr{V}$, see [16]. Recall that this is given by functors $B, E : |\mathscr{C}_{\mathrm{fp}}| \to \mathscr{C}_0$ and a pair of morphisms $\sigma, \tau : FE \to FB$ between the free finitary monads FB and FE on $\mathscr{C}$ respectively arising from B and E. The equational system $\mathbb{S}_{\mathbb{T}}$ associated to such an enriched algebraic theory $\mathbb{T}$ is given by $(\mathscr{C}_0 : (GB)_0 \rhd (GE)_0 \vdash \overline{\sigma}_0 = \overline{\tau}_0)$, where GB and GE are the free finitary endofunctors on $\mathscr{C}$ respectively arising from B and E, and where $\overline{\sigma}$ and $\overline{\tau}$ are respectively the functors corresponding to σ and τ by the bijection between morphisms $FE \to FB$ and functors $GB\text{-}\mathbf{Alg} \cong \mathscr{C}^{FB} \to \mathscr{C}^{FE} \cong GE\text{-}\mathbf{Alg}$ over $\mathscr{C}$. It follows that $(\mathbb{T}\text{-}\mathbf{Alg})_0$ is isomorphic to $\mathbb{S}_{\mathbb{T}}$-**Alg**.

(3) The definition of Eilenberg-Moore algebras for a monad $\mathbf{T} = (T, \eta, \mu)$ on a category $\mathscr{C}$ with binary coproducts can be directly encoded as the equational system $\mathbb{S}_{\mathbf{T}} = (\mathscr{C} : T \rhd \Gamma \vdash L = R)$ with $\Gamma(X) = X + T^2 X$ and

$$L(X,s) = (\, X, [\ s \cdot \eta_X\ ,\ s \cdot \mu_X\ ]\,)\ ,$$

$$R(X,s) = (\, X, [\ \ \mathrm{id}_X\ \ ,\ s \cdot Ts\ ]\,)\ .$$

It follows that $\mathbb{S}_{\mathbf{T}}$-**Alg** is isomorphic to the category $\mathscr{C}^{\mathbf{T}}$ of Eilenberg-Moore algebras for $\mathbf{T}$.

(4) The definition of monoid in a monoidal category $(\mathscr{C}, \otimes, I, \alpha, \lambda, \rho)$ with binary coproducts yields the equational system

$$\mathbb{S}_{\mathrm{Mon}(\mathscr{C})} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$$

with $\Sigma(X) = (X \otimes X) + I$, $\Gamma(X) = \big((X \otimes X) \otimes X\big) + (I \otimes X) + (X \otimes I)$, and

$$L(X,[m,e]) = (\, X, [\ \ m \cdot (m \otimes \mathrm{id}_X)\ \ ,\ \ \lambda_X\ \ ,\ \ \rho_X\ \ ]\,)\,,$$

$$R(X,[m,e]) = (\, X, [\ m \cdot (\mathrm{id}_X \otimes m) \cdot \alpha_{X,X,X}\ ,\ m \cdot (e \otimes \mathrm{id}_X)\ ,\ m \cdot (\mathrm{id}_X \otimes e)\ ]\,)\,.$$

It follows that $\mathbb{S}_{\mathrm{Mon}(\mathscr{C})}$-**Alg** is isomorphic to the category of monoids and monoid homomorphisms in $\mathscr{C}$.

4 Free constructions for equational systems

We give a sufficient condition for the existence of free algebras for equational systems; that is, for the existence of a left adjoint to the forgetful functor $U_{\mathbb{S}} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$, for $\mathbb{S}$ an equational system. Since, by definition, the forgetful functor $U_{\mathbb{S}}$ decomposes as $\mathbb{S}\text{-}\mathbf{Alg} \overset{J_{\mathbb{S}}}{\hookrightarrow} \Sigma\text{-}\mathbf{Alg} \xrightarrow{U_\Sigma} \mathscr{C}$, its left adjoint can be described in two stages as the composition of a left adjoint to U_Σ followed by a left adjoint to $J_{\mathbb{S}}$. Conditions for the existence of the former have already been studied in the literature (see *e.g.* [3,4]). Thus, we concentrate here on obtaining a reflection to the embedding of $\mathbb{S}$-**Alg** into Σ-**Alg**.

4.1 Free Σ-algebras

The following result describes a well-known condition for the existence of free Σ-algebras (see *e.g.* [2]).

Theorem 4.1 *Let Σ be an endofunctor on a category $\mathscr{C}$ with binary coproducts. If $\mathscr{C}$ has colimits of α-chains for some infinite limit ordinal α and Σ preserves them, then the forgetful functor $U_\Sigma : \Sigma$-**Alg** $\to \mathscr{C}$ has a left adjoint.*

In the case where the ordinal α is ω, the free Σ-algebra $(TX, \tau_X : \Sigma(TX) \to TX)$ on an object $X \in \mathscr{C}$ and the unit map $\eta_X : X \to TX$ are constructed as follows. The object TX is given as a colimit of the ω-chain $\{\, f_n : X_n \to X_{n+1}\,\}_{n\geq 0}$, inductively defined by $X_0 = 0$, $f_0 = \,!$ and $X_{n+1} = X + \Sigma X_n$, $f_{n+1} = X + \Sigma f_n$ for $n \geq 0$, where 0 is an initial object and ! is the unique map. Since the functor $X + \Sigma(-)$ preserves colimits of ω-chains, the object $X + \Sigma(TX)$ is a colimit of the ω-chain $\{\, X + \Sigma f_n : X + \Sigma X_n \to X + \Sigma X_{n+1} \,\}_{n \geq 0}$. The map $[\eta_X, \tau_X]$ is the unique mediating map as follows:

$$\begin{tikzcd} & X + \Sigma 0 \arrow[r, "X+\Sigma!"] \arrow[d, "="] & X + \Sigma(X + \Sigma 0) \arrow[r] \arrow[d, "="] & \cdots\cdots & X + \Sigma(TX) \arrow[d, dashed, "\cong \,\exists!\, {[\eta_X, \tau_X]}"] \\ 0 \arrow[r, "!"] & X + \Sigma 0 \arrow[r, "X+\Sigma!"] & X + \Sigma(X + \Sigma 0) \arrow[r] & \cdots\cdots & TX \;\; \mathsf{colim} \end{tikzcd} \tag{1}$$

The intuition behind this construction of TX, in which Σ represents a signature and X an object of variables, is that of taking the union of the sequence of objects X_n of terms of depth at most n built from the operators in Σ and variables in X.

4.2 Free $\mathbb{S}$-algebras

We now turn our attention to conditions for the existence of a left adjoint to the embedding $\mathbb{S}$-**Alg** $\hookrightarrow \Sigma$-**Alg**.

Theorem 4.2 *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ be an equational system. For $\mathscr{C}$ cocomplete, if Σ and Γ preserve colimits of α-chains for some infinite limit ordinal α then $\mathbb{S}$-**Alg** is a full reflective subcategory of Σ-**Alg**.*

This result is proved by performing an iterative, possibly transfinite, construction that associates a free $\mathbb{S}$-algebra to every Σ-algebra. The cocompleteness of the base category allows one to perform the construction, whilst the other conditions guarantee that the process will eventually stop.

Here we will only give a proof for the special case of the theorem where the ordinal α is ω. This is the most relevant case in applications, and free algebras are attained in a countable number of steps. The general proof of Theorem 4.2 will be given in a future, technical paper.

4.3 Algebraic coequalizers

The construction of free algebras, explained in the following section, depends on the key concept of *algebraic coequalizer*, whose existence and explicit construction in the present setting is dealt with in this section.

Definition 4.3 *Let Σ be an endofunctor on a category $\mathscr{C}$. By a Σ-algebraic coequalizer of a parallel pair l, r in $\mathscr{C}$ into the carrier object Z of a Σ-algebra*

(Z, t) *we mean a universal* Σ*-algebra homomorphism* z *from* (Z, t) *coequalizing the parallel pair.*

$$\begin{tikzcd}
& FZ \arrow[r, "Fz"] \arrow[d, "t"] \arrow[rr, bend left, "Fh"] & FZ' \arrow[r, dashed, "Fh'"] \arrow[d, "t'"] & FW \arrow[d, "u"] \\
Y \arrow[r, shift left, "l"] \arrow[r, shift right, "r"'] & Z \arrow[r, "z"] \arrow[rr, bend right, "h"'] & Z' \arrow[r, dashed, "\exists!\, h'"] & W
\end{tikzcd}$$

The following lemma, shows how algebraic coequalizers may be seen to arise from coequalizers by *reflecting algebra cospans* to algebras.

Definition 4.4 *For* Σ *an endofunctor on a category* $\mathscr{C}$*, we let* Σ-**AlgCoSpan** *be the category with* Σ*-algebra cospans* $(\Sigma Z \to Z_1 \leftarrow Z)$ *as objects and homomorphisms* (h, h_1) *between them as follows:*

$$\begin{tikzcd}
& \Sigma Z \arrow[d] \arrow[rr, "\Sigma h"] & & \Sigma Z' \arrow[d] \\
Z \arrow[r] \arrow[rr, "h"'] & Z_1 \arrow[rr, "h_1"] & Z' \arrow[r] & Z'_1
\end{tikzcd}$$

We will henceforth regard Σ-**Alg** as a full subcategory of Σ-**AlgCoSpan** via the embedding that maps $(\Sigma Z \to Z)$ to $(\Sigma Z \to Z \stackrel{\mathrm{id}}{\leftarrow} Z)$.

Lemma 4.5 *Let* Σ *be an endofunctor on a category* $\mathscr{C}$*. If the embedding* Σ-**Alg** $\hookrightarrow$ Σ-**AlgCoSpan** *has a left adjoint, then the existence of coequalizers in* $\mathscr{C}$ *implies that of* Σ*-algebraic coequalizers.*

PROOF. Let l, r be a parallel pair into Z in $\mathscr{C}$ and let $t : \Sigma Z \to Z$ be an algebra structure. Consider the coequalizer $c : Z \twoheadrightarrow Z_1$ of l, r in $\mathscr{C}$ and let $(z, z_1) : (\Sigma Z \stackrel{c \cdot t}{\to} Z_1 \stackrel{c}{\leftarrow} Z) \longrightarrow (\Sigma Z' \stackrel{t'}{\to} Z' \stackrel{\mathrm{id}}{\leftarrow} Z')$ be a universal reflection. Then, the homomorphism $z = z_1 \cdot c : (Z, t) \to (Z', t')$ is an algebraic coequalizer of l, r. $\square$

The missing ingredient for constructing algebraic coequalizers, thus, is the construction of a reflection from Σ-**AlgCoSpan** to Σ-**Alg**. This may be achieved by generalizing the construction of free Σ-algebras recalled in Section 4.1. Indeed, the initial Σ-algebra according to this construction is trivially obtained from the construction (2) below as the reflection of the initial Σ-algebra cospan $(\Sigma 0 \overset{\mathrm{id}}{\to} \Sigma 0 \leftarrow 0)$.

Theorem 4.6 *Let Σ be an endofunctor on a category $\mathscr{C}$. For $\mathscr{C}$ cocomplete, if Σ preserves colimits of ω-chains then Σ-**Alg** is a full reflective subcategory of Σ-**AlgCoSpan**.*

PROOF. Given a Σ-algebra cospan $(t_0 : \Sigma Z_0 \to Z_1 \leftarrow Z_0 : c_0)$ we construct a Σ-algebra $t_\infty : \Sigma Z_\infty \to Z_\infty$ as follows:

$$\begin{tikzcd}
\Sigma Z_0 \arrow[r, "\Sigma c_0"] \arrow[dr, "t_0"] & \Sigma Z_1 \arrow[r, "\Sigma c_1"] \arrow[dr, "t_1"] \arrow[d, phantom, "\mathsf{po}"] & \Sigma Z_2 \arrow[r, "\Sigma c_2"] \arrow[dr, "t_2"] \arrow[d, phantom, "\mathsf{po}"] & \Sigma Z_3 & \cdots\cdots & \Sigma Z_\infty \arrow[d, dashed, "\exists!\, t_\infty"] \\
Z_0 \arrow[r, "c_0"'] & Z_1 \arrow[r, "c_1"'] & Z_2 \arrow[r, "c_2"'] & Z_3 & \cdots\cdots & Z_\infty \ \mathsf{colim}
\end{tikzcd} \tag{2}$$

where

- $Z_{n+1} \overset{c_{n+1}}{\longrightarrow} Z_{n+2} \overset{t_{n+1}}{\longleftarrow} \Sigma Z_{n+1}$ is a pushout of $Z_{n+1} \overset{t_n}{\longleftarrow} \Sigma Z_n \overset{\Sigma c_n}{\longrightarrow} \Sigma Z_{n+1}$, for all $n \geq 0$;
- Z_∞ with $\{\, \overline{c}_n : Z_n \to Z_\infty \,\}_{n\geq 0}$ is a colimit of the ω-chain $\{\, c_n \,\}_{n\geq 0}$; and
- t_∞ is the mediating map from the colimiting cone $\{\, \Sigma\overline{c}_n : \Sigma Z_n \to \Sigma Z_\infty \,\}_{n\geq 0}$ to the cone $\{\, \overline{c}_{n+1} \cdot t_n \,\}_{n\geq 0}$ of the ω-chain $\{\, \Sigma c_n \,\}_{n\geq 0}$.

We now show that the map $(\overline{c}_0, \overline{c}_1) : (\Sigma Z_0 \to Z_1 \leftarrow Z_0) \longrightarrow (\Sigma Z_\infty \to Z_\infty \overset{\mathrm{id}}{\leftarrow} Z_\infty)$ in Σ-**AlgCoSpan** is universal. To this end, consider another map $(h_0, h_1) : (\Sigma Z_0 \to Z_1 \leftarrow Z_0) \longrightarrow (\Sigma W \to W \overset{\mathrm{id}}{\leftarrow} W)$ and perform the following construction

$$\begin{tikzcd}
\Sigma Z_0 \arrow[r, "\Sigma c_0"] \arrow[dr, "t_0"] \arrow[drrrrr, "\Sigma h_0"] & \Sigma Z_1 \arrow[r, "\Sigma c_1"] \arrow[dr, "t_1"] \arrow[drrrr, "\Sigma h_1"] & \Sigma Z_2 \arrow[drrr, "\Sigma h_2"] & \cdots\cdots & \Sigma Z_\infty \arrow[d, "t_\infty"] \arrow[dr, "\Sigma h_\infty"] & \\
Z_0 \arrow[r, "c_0"] \arrow[drrrrr, "h_0"] & Z_1 \arrow[r, "c_1"] \arrow[drrrr, "h_1"] & Z_2 \arrow[drrr, "h_2"] & \cdots\cdots & Z_\infty \arrow[dr, "h_\infty"] & \Sigma W \arrow[d, "u"] \\
 & & & & & W
\end{tikzcd}$$

where

- for $n \geq 0$, h_{n+2} is the mediating map from the pushout Z_{n+2} to W with respect to the cone $(h_{n+1} : Z_{n+1} \to W \leftarrow \Sigma Z_{n+1} : u \cdot \Sigma h_{n+1})$; and
- h_∞ is the mediating map from the colimit Z_∞ with respect to the cone $\{\, h_n \,\}_{n\geq 0}$ of the ω-chain $\{\, c_n \,\}_{n\geq 0}$.

As, for all $n \geq 0$, $u \cdot \Sigma h_\infty \cdot \Sigma\overline{c}_n = h_\infty \cdot t_\infty \cdot \Sigma\overline{c}_n$, it follows that h_∞ is a Σ-algebra homomorphism. Hence, (h_0, h_1) factors as $(h_\infty, h_\infty) \cdot (\overline{c}_0, \overline{c}_1)$.

We finally establish the uniqueness of such factorizations. Indeed, for any homomorphism $h : (Z_\infty, t_\infty) \to (W, u)$ such that $h \cdot \overline{c}_1 = h_1$, it follows by induction that $h \cdot \overline{c}_n = h_n$ for all $n \geq 0$, and hence that $h = h_\infty$. □

Corollary 4.7 *Let Σ be an endofunctor on a category $\mathscr{C}$. For $\mathscr{C}$ cocomplete, if Σ preserves colimits of ω-chains then Σ-algebraic coequalizers exist. If, in addition, Σ preserves epimorphisms then Σ-algebraic coequalizers are epimorphic in $\mathscr{C}$.*

PROOF. According to Lemma 4.5 and Theorem 4.6, the algebraic coequalizer of $l, r : Y \to Z_0$ with respect to the algebra structure $t : \Sigma Z_0 \to Z_0$ is given by $\overline{c}_0 : (Z_0, t) \to (Z_\infty, t_\infty)$ in the construction (2) where c_0 is taken to be the coequalizer of l, r in $\mathscr{C}$ and t_0 is defined as $c_0 \cdot t$.

If Σ preserves epimorphisms, then the ω-chain $\{ c_n : Z_n \to Z_{n+1} \}_{n \geq 0}$ in (2) consists of epimorphisms, and hence this is also the case for its colimiting cone $\{ \overline{c}_n : Z_n \to Z_\infty \}_{n \geq 0}$. □

4.4 Finitary free algebras

The construction of free $\mathbb{S}$-algebras on Σ-algebras follows.

Theorem 4.8 *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ be an equational system. For $\mathscr{C}$ cocomplete, if Σ and Γ preserve colimits of ω-chains then $\mathbb{S}$-**Alg** is a full reflective subcategory of Σ-**Alg**.*

PROOF. Given a Σ-algebra (X_0, s_0), we construct a free $\mathbb{S}$-algebra (X_∞, s_∞) on it as follows:

$$\begin{tikzcd}
\Sigma X_0 \arrow[r, "\Sigma e_0"] \arrow[d, "s_0"] & \Sigma X_1 \arrow[r, "\Sigma e_1"] \arrow[d, "s_1"] & \Sigma X_2 \arrow[d, "s_2"] & \cdots\cdots & \Sigma X_\infty \arrow[d, dashed, "\exists!\, s_\infty"] \\
X_0 \arrow[r, "e_0", "\text{alg coeq}"'] & X_1 \arrow[r, "e_1", "\text{alg coeq}"'] & X_2 & \cdots\cdots & X_\infty \;\; \text{colim} \\
\Gamma X_0 \arrow[r, "\Gamma e_0"] \arrow[u, shift left, "L(X_0,s_0)"] \arrow[u, shift right, "R(X_0,s_0)"'] & \Gamma X_1 \arrow[r, "\Gamma e_1"] \arrow[u, shift left, "L(X_1,s_1)"] \arrow[u, shift right, "R(X_1,s_1)"'] & \Gamma X_2 \arrow[u, shift left, "L(X_2,s_2)"] \arrow[u, shift right, "R(X_2,s_2)"'] & \cdots\cdots & \Gamma X_\infty \arrow[u, dashed, "L(X_\infty,s_\infty)=R(X_\infty,s_\infty)"]
\end{tikzcd} \qquad (3)$$

where

- for $n \geq 0$, $e_n : (X_n, s_n) \to (X_{n+1}, s_{n+1})$ is an algebraic coequalizer of the parallel pair $L(X_n, s_n), R(X_n, s_n) : \Gamma X_n \to X_n$;
- X_∞ with $\{ \overline{e}_n : X_n \to X_\infty \}_{n \geq 0}$ is a colimit of the ω-chain $\{ e_n \}_{n \geq 0}$; and

- s_∞ is the mediating map from the colimiting cone $\{\, \Sigma\overline{e}_n \,\}_{n\geq 0}$ to the cone $\{\, \overline{e}_n \cdot s_n \,\}_{n\geq 0}$ of the ω-chain $\{\, \Sigma e_n \,\}_{n\geq 0}$.

As, for all $n \geq 0$, $L(X_\infty, s_\infty) \cdot \Gamma\overline{e}_n = R(X_\infty, s_\infty) \cdot \Gamma\overline{e}_n$ it follows that (X_∞, s_∞) is an $\mathbb{S}$-algebra.

We now show that the unit $\eta = \overline{e}_0 : (X_0, s_0) \to (X_\infty, s_\infty)$ satisfies the universal property that every homomorphism $(X_0, s_0) \to (W, u)$ into an $\mathbb{S}$-algebra (W, u) uniquely factors through it.

Indeed, we construct a factor $h_\infty : (X_\infty, s_\infty) \to (W, u)$ of $h_0 : (X_0, s_0) \to (W, u)$ through η as follows:

$$
\begin{tikzcd}
\Sigma X_0 \arrow[r, "\Sigma e_0"] \arrow[d, "s_0"] \arrow[drrrr, "\Sigma h_0"] & \Sigma X_1 \arrow[d, "s_1"] \arrow[drrr, "\Sigma h_1"] & \cdots\cdots & \Sigma X_\infty \arrow[d, "s_\infty"] \arrow[dr, "\Sigma h_\infty"] & \\
X_0 \arrow[r, "e_0"] \arrow[drrrr, "h_0"] & X_1 \arrow[drrr, "h_1"] & \cdots\cdots & X_\infty \arrow[dr, "h_\infty"] & \Sigma W \arrow[d, "u"] \\
\Gamma X_0 \arrow[r, "\Gamma e_0"] \arrow[u, shift left, "L(X_0,s_0)"] \arrow[u, shift right, "R(X_0,s_0)"'] \arrow[drrrr, "\Gamma h_0"] & \Gamma X_1 \arrow[u, shift left, "L(X_1,s_1)"] \arrow[u, shift right, "R(X_1,s_1)"'] \arrow[drrr, "\Gamma h_1"] & \cdots\cdots & \Gamma X_\infty \arrow[u] \arrow[dr, "\Gamma h_\infty"] & W \\
 & & & & \Gamma W \arrow[u, "L(W,u)=R(W,u)"]
\end{tikzcd}
$$

where

- for $n \geq 0$, $h_{n+1} : (X_{n+1}, s_{n+1}) \to (W, u)$ is a factor of h_n through the algebraic coequalizer e_n; and
- h_∞ is the mediating map from the colimit X_∞ to W with respect to the cone $\{\, h_n \,\}_{n\geq 0}$.

Then, $h_\infty \cdot \eta = h_0$ and, as $u \cdot \Sigma h_\infty \cdot \Sigma\overline{e}_n = h_\infty \cdot s_\infty \cdot \Sigma\overline{e}_n$ for all $n \geq 0$, it follows that h_∞ is an homomorphism $(X_\infty, s_\infty) \to (W, u)$.

We finally establish the uniqueness of such factorizations. Indeed, for any homomorphism $h : (X_\infty, s_\infty) \to (W, u)$ such that $h \cdot \eta = h_0$, it follows by induction that $h \cdot \overline{e}_n = h_n$ for all $n \geq 0$, and hence that $h = h_\infty$. $\square$

4.5 Inductive free algebras

As we have seen above, free $\mathbb{S}$-algebras on Σ-algebras may be constructed by a colimit of an ω-chain of algebraic coequalizers (Theorem 4.8), each of which is in turn constructed by a coequalizer and a colimit of an ω-chain (Corollary 4.7). Here we introduce an extra condition on the functorial signature and functorial context of an equational system to accomplish the construction of free algebras in just ω steps.

Theorem 4.9 *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ be an equational system. For $\mathscr{C}$ cocomplete, if Σ preserves colimits of ω-chains, and both Σ and Γ preserve epimorphisms, then $\mathbb{S}$-**Alg** is a full reflective subcategory of Σ-**Alg**. Furthermore, free $\mathbb{S}$-algebras on Σ-algebras are constructed in ω steps.*

PROOF. Consider the first step in the construction (3) exhibiting a free $\mathbb{S}$-algebra on the Σ-algebra (X_0, s_0). According to Corollary 4.7, the algebraic coequalizer $e_0 : (X_0, s_0) \to (X_1, s_1)$ in there is an epimorphism in $\mathscr{C}$. Thus, so is Γe_0 and, as $L(X_1, s_1) \cdot \Gamma e_0 = R(X_1, s_1) \cdot \Gamma e_0$, it follows that (X_1, s_1) is an $\mathbb{S}$-algebra; the free one on (X_0, s_0). $\square$

Consequently, under the hypothesis of Theorem 4.9, the construction of the free $\mathbb{S}$-algebra (X_∞, s_∞) on a Σ-algebra (X, s) with unit $\eta : (X, s) \to (X_\infty, s_\infty)$ is simplified as in the following diagram:

$$
\begin{tikzcd}
\Sigma X \arrow[r, two heads, "\Sigma c_0"] \arrow[d, "s"'] \arrow[dr, "s_0"] & \Sigma X_1 \arrow[r, two heads, "\Sigma c_1"] \arrow[dr, "s_1"] \arrow[d, phantom, "\mathsf{po}"] & \Sigma X_2 \arrow[r, two heads, "\Sigma c_2"] \arrow[dr, "s_2"] \arrow[d, phantom, "\mathsf{po}"] & \Sigma X_3 & \cdots\cdots & \Sigma X_\infty \arrow[d, dashed, "\exists!\, s_\infty"] \\
X \arrow[r, two heads, "c_0", "\mathsf{coeq}"'] & X_1 \arrow[r, two heads, "c_1"] & X_2 \arrow[r, two heads, "c_2"] & X_3 & \cdots\cdots & X_\infty \ \mathsf{colim} \\
\Gamma X \arrow[u, shift left, "{R(X,s)}"'] \arrow[u, shift right, "{L(X,s)}"] \arrow[rrrrr, two heads, "\Gamma\eta"] & & & & & \Gamma X_\infty \arrow[u, "{L(X_\infty,s_\infty)=R(X_\infty,s_\infty)}"']
\end{tikzcd}
\tag{4}
$$

The intuition behind the construction of X_1 from X as the coequalizer of $L(X, s)$ and $R(X, s)$ is that of quotienting the carrier object X by the equation $L = R$. The construction of X_{n+1} from X_n for $n \geq 1$ as a pushout is intuitively quotienting the object X_n by congruence rules. Therefore, the intuition behind the construction of the free algebra X_∞ is that of quotienting the object X by the equation $L = R$ and congruence rules.

5 Categories of algebras and monads for equational systems

We consider properties of categories of algebras and monads for equational systems. The results of this section jointly establish the following theorem.

Theorem 5.1 *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ be an equational system. For $\mathscr{C}$ cocomplete, if Σ and Γ preserve colimits of α-chains for some infinite limit* *ordinal α then the forgetful functor $U_{\mathbb{S}} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ is monadic, the induced monad preserves colimits of α-chains, and $\mathbb{S}\text{-}\mathbf{Alg}$ is cocomplete.*

5.1 Monadicity and cocompleteness

For an endofunctor Σ on a category $\mathscr{C}$, it is well-known that if the forgetful functor $\Sigma\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint then it is monadic. This result extends to categories of algebras for equational systems.

Proposition 5.2 *Let $\mathbb{S}$ be an equational system. If the forgetful functor $U_{\mathbb{S}} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ has a left adjoint, then it is monadic.*

PROOF. To show the monadicity of $U_{\mathbb{S}}$ by Beck's theorem [17], it is enough to show that $U_{\mathbb{S}}$ creates coequalizers of parallel pairs $f, g : (X, r) \to (Y, s)$ in $\mathbb{S}\text{-}\mathbf{Alg}$ for which $f, g : X \to Y$ has an absolute coequalizer, say $e : Y \twoheadrightarrow Z$, in $\mathscr{C}$. In this case then, Σe is a coequalizer of $\Sigma f, \Sigma g$ and Γe is a coequalizer of $\Gamma f, \Gamma g$, so that we have the following situation

$$
\begin{tikzcd}
\Sigma X \arrow[r, shift left, "\Sigma f"] \arrow[r, shift right, "\Sigma g"'] \arrow[d, "r"'] & \Sigma Y \arrow[r, "\Sigma e"] \arrow[d, "s"'] & \Sigma Z \arrow[d, dashed, "\exists!\, t"] \\
X \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & Y \arrow[r, "e", "\mathsf{coeq}"'] & Z \\
\Gamma X \arrow[r, shift left, "\Gamma f"] \arrow[r, shift right, "\Gamma g"'] \arrow[u, "L(X{,}r)=R(X{,}r)"] & \Gamma Y \arrow[r, "\Gamma e"] \arrow[u, "L(Y{,}s)=R(Y{,}s)"] & \Gamma Z \arrow[u, dashed, shift left, "L(Z{,}t)"] \arrow[u, dashed, shift right, "R(Z{,}t)"']
\end{tikzcd}
$$

for a unique Σ-algebra structure t on Z for which $L(Z, t) = R(Z, t)$.

It follows from the universal properties of e and Σe that $e : (Y, s) \to (Z, t)$ is a coequalizer of $f, g : (X, r) \to (Y, s)$ in $\Sigma\text{-}\mathbf{Alg}$, and hence also in $\mathbb{S}\text{-}\mathbf{Alg}$. $\square$

A condition for the cocompleteness of categories of algebras for equational systems follows as a corollary.

Theorem 5.3 *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ be an equational system. For $\mathscr{C}$ cocomplete, if Σ and Γ preserve colimits of α-chains for some infinite limit ordinal α then the forgetful functor $U_{\mathbb{S}} : \mathbb{S}\text{-}\mathbf{Alg} \to \mathscr{C}$ is monadic and $\mathbb{S}\text{-}\mathbf{Alg}$ is cocomplete.*

We only prove the special case of the theorem where the ordinal α is ω.

PROOF. By Theorems 4.1 and 4.8, $U_{\mathbb{S}}$ has a left adjoint and thus, by Proposition 5.2, $\mathbb{S}$-**Alg** is monadic. Furthermore, it has coequalizers since, by Theorem 4.8, it is a full reflective subcategory of Σ-**Alg** which, by Corollary 4.7, has coequalizers. Being monadic over a cocomplete category and having coequalizers, $\mathbb{S}$-**Alg** is cocomplete (see [5, Proposition 4.3.4]). $\square$

5.2 Cocontinuity

We show that the colimit-preservation properties of the functorial signature and functorial context of an equational system are inherited by the free-algebra monad.

Proposition 5.4 *Let $\mathbb{S} = (\mathscr{C} : \Sigma \rhd \Gamma \vdash L = R)$ be an equational system. For $\mathscr{C}$ cocomplete, if Σ and Γ preserve $\mathbb{I}$-indexed colimits for a small category $\mathbb{I}$, and $U_{\mathbb{S}} : \mathbb{S}$-**Alg** $\to \mathscr{C}$ has a left adjoint, then the induced monad on $\mathscr{C}$ also preserves $\mathbb{I}$-indexed colimits.*

PROOF. Write (T, η, μ) for the monad on $\mathscr{C}$ induced by the left adjoint to $U_{\mathbb{S}}$.

For a diagram $I : \mathbb{I} \to \mathscr{C}$, let $\{\, \lambda_i : Ii \to \operatorname{colim} I \,\}_{i \in \mathbb{I}}$ and $\{\, \delta_i : TIi \to \operatorname{colim} TI \,\}_{i \in \mathbb{I}}$ be colimiting cones. We show that the cones $\{\, T\lambda_i \,\}_{i \in \mathbb{I}}$ and $\{\, \delta_i \,\}_{i \in \mathbb{I}}$ are isomorphic. To this end, we construct an inverse $q : T(\operatorname{colim} I) \to \operatorname{colim} TI$ to the mediating map $p : \operatorname{colim} TI \to T(\operatorname{colim} I)$ from $\{\, \delta_i \,\}_{i \in \mathbb{I}}$ to $\{\, T\lambda_i \,\}_{i \in \mathbb{I}}$ as follows.

Let $(TX, \tau_X : \Sigma TX \to TX)$ be the free $\mathbb{S}$-algebra on $X \in \mathscr{C}$ induced by the left adjoint to $U_{\mathbb{S}}$. The family $\tau = \{\, \tau_X : \Sigma TX \to TX \,\}_{X \in \mathscr{C}}$ is natural. Hence, the family $\{\, \delta_i \cdot \tau_{Ii} : \Sigma TIi \to \operatorname{colim} TI \,\}_{i \in \mathbb{I}}$ is a cone and, as $\{\, \Sigma\delta_i \,\}_{i \in \mathbb{I}}$ is colimiting, we have a unique Σ-algebra structure t on $\operatorname{colim} TI$ such that the diagram on the top below

$$\begin{tikzcd}
\Sigma TIi \arrow[r, "\Sigma\delta_i"] \arrow[d, "\tau_{Ii}"'] & \Sigma(\operatorname{colim} TI) \arrow[d, dashed, "\exists!\, t"] \\
TIi \arrow[r, "\delta_i"] & \operatorname{colim} TI \\
\Gamma TIi \arrow[r, "\Gamma\delta_i"] \arrow[u, "L(TIi,\tau_{Ii}) = R(TIi,\tau_{Ii})"] & \Gamma(\operatorname{colim} TI) \arrow[u, dashed, shift left, "L(\operatorname{colim} TI,t)"'] \arrow[u, dashed, shift right, "R(\operatorname{colim} TI,t)"']
\end{tikzcd}$$

commutes for all $i \in \mathbb{I}$. Furthermore, the Σ-algebra $(\operatorname{colim} TI, t)$ is an $\mathbb{S}$-algebra; since $\{\, \Gamma\delta_i \,\}_{i \in \mathbb{I}}$ is colimiting and $L(\operatorname{colim} TI, t) \cdot \Gamma\delta_i = R(\operatorname{colim} TI, t) \cdot \Gamma\delta_i$ for all $i \in \mathbb{I}$.

By the universal property of free algebras, we define $q : T(\operatorname{colim} I) \to \operatorname{colim} TI$ as the unique map making the following diagram commutative:

$$
\begin{tikzcd}
\Sigma T(\operatorname{colim} I) \arrow[r, dashed, "\Sigma q"] \arrow[d, "\tau_{\operatorname{colim} I}"'] & \Sigma(\operatorname{colim} TI) \arrow[d, "t"] \\
T(\operatorname{colim} I) \arrow[r, dashed, "\exists!\, q"] & \operatorname{colim} TI \\
\operatorname{colim} I \arrow[u, "\eta_{\operatorname{colim} I}"] \arrow[ur, "\operatorname{colim} \eta_I"'] &
\end{tikzcd}
$$

This map is a morphism between the cones $\{ T\lambda_i \}_{i \in \mathbb{I}}$ and $\{ \delta_i \}_{i \in \mathbb{I}}$; as follows from the commutative diagrams below

$$
\begin{tikzcd}
\Sigma TIi \arrow[r, "\Sigma T\lambda_i"] \arrow[d, "\tau_{Ii}"'] & \Sigma T(\operatorname{colim} I) \arrow[r, "\Sigma q"] \arrow[d, "\tau_{\operatorname{colim} I}"'] & \Sigma(\operatorname{colim} TI) \arrow[d, "t"] \\
TIi \arrow[r, "T\lambda_i"] & T(\operatorname{colim} I) \arrow[r, "q"] & \operatorname{colim} TI \\
Ii \arrow[u, "\eta_{Ii}"] \arrow[r, "\lambda_i"] & \operatorname{colim} I \arrow[u, "\eta_{\operatorname{colim} I}"'] \arrow[ur, "\operatorname{colim} \eta_I"'] &
\end{tikzcd}
\qquad
\begin{tikzcd}
\Sigma TIi \arrow[rr, "\Sigma \delta_i"] \arrow[d, "\tau_{Ii}"'] & & \Sigma(\operatorname{colim} TI) \arrow[d, "t"] \\
TIi \arrow[rr, "\delta_i"] & & \operatorname{colim} TI \\
Ii \arrow[u, "\eta_{Ii}"] \arrow[r, "\lambda_i"] & \operatorname{colim} I \arrow[ur, "\operatorname{colim} \eta_I"'] &
\end{tikzcd}
$$

by the universal property of free algebras. It follows that the endomap $q \cdot p$ on $(\operatorname{colim} TI)$ is the identity, as it is an endomap on a colimiting cone.

Finally, that the endomap $p \cdot q$ on $T(\operatorname{colim} I)$ is the identity map follows from the commutativity of the diagram below

$$
\begin{tikzcd}
\Sigma T(\operatorname{colim} I) \arrow[r, "\Sigma q"] \arrow[d, "\tau_{\operatorname{colim} I}"'] & \Sigma(\operatorname{colim} TI) \arrow[r, "\Sigma p"] \arrow[d, "t"] \arrow[dr, phantom, "\text{(B)}"] & \Sigma T(\operatorname{colim} I) \arrow[d, "\tau_{\operatorname{colim} I}"] \\
T(\operatorname{colim} I) \arrow[r, "q"] \arrow[dr, phantom, "\text{(A)}"] & \operatorname{colim} TI \arrow[r, "p"] & T(\operatorname{colim} I) \\
\operatorname{colim} I \arrow[u, "\eta_{\operatorname{colim} I}"] \arrow[urr, "\eta_{\operatorname{colim} I}"'] & &
\end{tikzcd}
$$

by the universal property of free algebras.

The commutativity of the diagram (A) above follows from the commutativity of the following diagram for each $i \in \mathbb{I}$

$$
\begin{tikzcd}
Ii \arrow[r, "\lambda_i"] \arrow[dr, "\eta_{Ii}"'] \arrow[dd, "\lambda_i"'] & \operatorname{colim} I \arrow[r, "\eta_{\operatorname{colim} I}"] \arrow[dr, "\operatorname{colim} \eta_I"'] & T(\operatorname{colim} I) \arrow[d, "q"] \\
& TIi \arrow[r, "\delta_i"] \arrow[dr, "T\lambda_i"'] & \operatorname{colim} TI \arrow[d, "p"] \\
\operatorname{colim} I \arrow[rr, "\eta_{\operatorname{colim} I}"'] & & T(\operatorname{colim} I)
\end{tikzcd}
$$

because $\{ \lambda_i \}_{i \in \mathbb{I}}$ is a colimiting cone.

The commutativity of diagram (B) above follows from the commutativity of the following diagram for each $i \in \mathbb{I}$

$$\begin{tikzcd}
\Sigma T I i \arrow[rr, "\Sigma\delta_i"] \arrow[rrd, "\Sigma T\lambda_i"] \arrow[rdd, "\tau_{Ii}"] \arrow[ddd, "\Sigma\delta_i"'] & & \Sigma(\operatorname{colim} TI) \arrow[d, "\Sigma p"] \\
& & \Sigma T(\operatorname{colim} I) \arrow[dd, "\tau_{\operatorname{colim} I}"] \\
& TIi \arrow[rd, "T\lambda_i"] \arrow[d, "\delta_i"'] & \\
\Sigma(\operatorname{colim} TI) \arrow[r, "t"'] & \operatorname{colim} TI \arrow[r, "p"'] & T(\operatorname{colim} I)
\end{tikzcd}$$

because $\{\Sigma\delta_i\}_{i\in\mathbb{I}}$ is a colimiting cone. $\square$

5.3 Examples

We revisit the examples of equational systems given in Section 3 in the light of the above results.

(1) For the equational system $\mathbb{S}_{\mathbb{T}} = (\mathbf{Set} : \Sigma_{\mathbb{T}} \rhd \Gamma_{\mathbb{T}} \vdash L_{\mathbb{T}} = R_{\mathbb{T}})$ representing an algebraic theory $\mathbb{T}$, the category $\mathbb{S}_{\mathbb{T}}$-$\mathbf{Alg}$ is monadic over $\mathbf{Set}$ and cocomplete, and the free-algebra monad is finitary, by Theorem 5.1; as $\mathbf{Set}$ is cocomplete and $\Sigma_{\mathbb{T}}$ and $\Gamma_{\mathbb{T}}$ are finitary. Furthermore, $\Sigma_{\mathbb{T}}$ and $\Gamma_{\mathbb{T}}$ preserve epimorphisms and Theorem 4.9 applies.

(2) For the equational system $\mathbb{S}_{\mathbb{T}} = (\mathscr{C}_0 : (GB)_0 \rhd (GE)_0 \vdash \overline{\sigma}_0 = \overline{\tau}_0)$ representing an enriched algebraic theory $\mathbb{T} = (\mathscr{C}, B, E, \sigma, \tau)$, the category $\mathbb{S}_{\mathbb{T}}$-$\mathbf{Alg}$ is monadic over $\mathscr{C}_0$ and cocomplete, and the free-algebra monad is finitary by Theorem 5.1; as $\mathscr{C}_0$ is locally finitely presentable, and $(GB)_0$ and $(GE)_0$ are finitary.

(3) One may apply Theorem 5.1 to the equational system $\mathbb{S}_{\mathbf{T}}$ representing a monad $\mathbf{T} = (T, \eta, \mu)$ on a category $\mathscr{C}$ with binary coproducts as follows. If $\mathscr{C}$ is cocomplete and T preserves colimits of ω-chains, then $\mathbb{S}_{\mathbf{T}}$-$\mathbf{Alg} \cong \mathscr{C}^{\mathbf{T}}$ is cocomplete.

(4) To the equational system $\mathbb{S}_{\mathrm{Mon}(\mathscr{C})}$ of monoids in a monoidal category $\mathscr{C}$ with binary coproducts, we can apply Theorem 5.1 as follows. If $\mathscr{C}$ is cocomplete and the tensor product $\otimes : \mathscr{C} \times \mathscr{C} \to \mathscr{C}$ is finitary (as it happens, for instance, when it is biclosed), then $\mathbb{S}_{\mathrm{Mon}(\mathscr{C})}$-$\mathbf{Alg}$ is monadic over $\mathscr{C}$ and cocomplete, and the free-monoid monad is finitary.

6 Application: Nominal Equational Logic

Clouston and Pitts [7] have recently introduced Nominal Equational Logic (NEL) as an extension of equational logic with names and assertions on their freshness. We show in this section that every NEL-theory can be represented as an equational system in the sense that the respective categories of algebras coincide. By further showing that the equational system representation satisfies the hypothesis of Theorem 4.9, the monadicity and cocompleteness of categories of algebras for NEL-theories follows. We conclude by giving explicit descriptions of free algebras as derived from their inductive construction. For brevity, we only consider the single-sorted case; the multi-sorted one being treated analogously.

6.1 Nominal sets

For a fixed countably infinite set A of atoms, the group $\mathfrak{S}_0(\mathsf{A})$ of finite permutations of atoms consists of the bijections on A that fix all but finitely many elements of A. A $\mathfrak{S}_0(\mathsf{A})$-*action* $X = (|X|, \cdot)$ consists of a set $|X|$ equipped with a function $- \cdot = : \mathfrak{S}_0(\mathsf{A}) \times |X| \to |X|$ such that $\mathrm{id}_{\mathsf{A}} \cdot x = x$ and $\pi' \cdot (\pi \cdot x) = (\pi'\pi) \cdot x$ for all $x \in |X|$ and $\pi, \pi' \in \mathfrak{S}_0(\mathsf{A})$. $\mathfrak{S}_0(\mathsf{A})$-actions form a category with morphisms $X \to Y$ given by *equivariant* functions; that is, functions $f : |X| \to |Y|$ such that $f(\pi \cdot x) = \pi \cdot (fx)$ for all $\pi \in \mathfrak{S}_0(\mathsf{A})$ and $x \in |X|$.

By an element x of a $\mathfrak{S}_0(\mathsf{A})$-action X, denoted $x \in X$, we mean that x is a member of $|X|$. For a $\mathfrak{S}_0(\mathsf{A})$-action X, a finite subset S of A is said to *support* $x \in X$ if for all atoms $a, a' \notin S$, $(a\,a') \cdot x = x$, where the *transposition* $(a\,a')$ is the bijection that swaps a and a'. A *nominal set* is a $\mathfrak{S}_0(\mathsf{A})$-action in which every element has finite support. As an example, note that the set of atoms A becomes the *nominal set of atoms* $\mathbb{A}$ when equipped with the evaluation action $\pi \cdot a = \pi(a)$. A further example is given by $\mathfrak{S}_0(\mathsf{A})$ equipped with the conjugation action $\pi \cdot \sigma = \pi\sigma\pi^{-1}$, which we denote as $\mathfrak{S}_0(\mathbb{A})$.

The supports of an element of a nominal set are closed under intersection, and we write $\mathsf{supp}_X(x)$, or simply $\mathsf{supp}(x)$ when X is clear from the context, for the intersection of the supports of x in the nominal set X. For instance, $\mathsf{supp}_{\mathbb{A}}(a) = \{ a \}$ and $\mathsf{supp}_{\mathfrak{S}_0(\mathbb{A})}(\sigma) = \{ a \in \mathsf{A} \mid \sigma(a) \neq a \}$. For elements x and y of two, possibly distinct, nominal sets X and Y, we write $x \# y$ whenever $\mathsf{supp}_X(x)$ and $\mathsf{supp}_Y(y)$ are disjoint. Thus, for $a \in \mathbb{A}$ and $x \in X$, $a \# x$ stands for $a \notin \mathsf{supp}_X(x)$; that is, *$a$ is fresh for x*.

For an element x of a nominal set X, and $\pi, \pi' \in \mathfrak{S}_0(\mathsf{A})$ such that $\pi(a) = \pi'(a)$ for all $a \in \mathsf{supp}(x)$, we have that $\pi \cdot x = \pi' \cdot x$. Thus, for a finite set of atoms $S \supseteq \mathsf{supp}(x)$ and an injective function $\alpha : S \rightarrowtail \mathsf{A}$ it makes sense to define $\alpha \cdot x$ as $\widetilde{\alpha} \cdot x$ for $\widetilde{\alpha} \in \mathfrak{S}_0(\mathsf{A})$ any permutation extending α.

We let **Nom** be the full subcategory of $\mathfrak{S}_0(\mathsf{A})$-actions consisting of nominal sets, and briefly consider its structure relevant to us here.

The *coproduct* $\coprod_{k\in K} X_k$ of a family of nominal sets $\{ X_k \}_{k\in K}$ is the nominal set with $|\coprod_{k\in K} X_k| = \coprod_{k\in K} |X_k|$ and action $\pi \cdot (k, x) = (k, \pi \cdot x)$. As usual we write $X_1 + \cdots + X_n$ for $\coprod_{k\in\{1,\ldots,n\}} X_k$.

The *product* $\prod_{i\in I} X_i$ of a *finite* family of nominal sets $\{ X_i \}_{i\in I}$ is the nominal set with $|\prod_{i\in I} X_i| = \prod_{i\in I} |X_i|$ and action $\pi \cdot (x_i)_{i\in I} = (\pi \cdot x_i)_{i\in I}$. As usual we write $X_1 \times \cdots \times X_n$ for $\prod_{i\in\{1,\ldots,n\}} X_i$, and X^I for $\prod_{i\in I} X$. More generally, the *exponential* X^Y of nominal sets X and Y consists of all finitely supported functions $|Y| \to |X|$ with respect to the action that inversely acts on the input and directly acts on the output; *i.e.*, $\pi \cdot f = \lambda x.\, \pi \cdot f(\pi^{-1} \cdot x)$.

The *separating tensor* $\#_{i\in I} X_i$ of a *finite* family of nominal sets $\{ X_i \}_{i\in I}$ is the sub nominal set of $\prod_{i\in I} X_i$ with underlying set given by $\{ (x_i)_{i\in I} \mid x_i \# x_j \text{ for all } i \neq j \}$. We write $X_1 \# \cdots \# X_n$ for $\#_{i\in\{1,\ldots,n\}} X_i$, and $X^{\# I}$ for $\#_{i\in I} X$. Thus, for instance, the nominal set $\mathbb{A}^{\# I}$ for a finite set I is the subset of A^I consisting of the injections $I \rightarrowtail \mathsf{A}$ with action given by post-composition; *i.e.*, $\pi \cdot \imath = \pi\imath$. The separating tensor carries a symmetric monoidal closed structure.

For every set S, we define two nominal sets $\underline{S}$ and $\overline{S}$: the nominal set $\underline{S}$ has underlying set S and projection action $\pi \cdot s = s$; the nominal set $\overline{S}$ is the product $\mathfrak{S}_0(\mathbb{A}) \times \underline{S}$.

For a nominal set X, the nominal set $\mathcal{P}_0(X)$ has underlying set $\mathcal{P}_0|X|$, the set of finite subsets of $|X|$, and pointwise action $\pi \cdot S = \{ \pi \cdot x \mid x \in S \}$. In particular, $\mathcal{P}_0(\underline{S}) = \underline{\mathcal{P}_0(S)}$ for every set S. Note also that, for $A \in \mathcal{P}_0(\mathbb{A})$ and $x \in X$, $A \# x$ stands for $a \# x$ for all $a \in A$.

6.2 NEL-theories

A NEL-theory consists of a signature defining its operators together with the set of axioms that these should obey.

A *NEL-signature* $\Sigma = (\mathsf{O}, [-])$ is specified by a nominal set O of operators and an equivariant function $[-] : \mathsf{O} \to \underline{\mathbb{N}}$ associating a natural number arity to each operator. The equivariance of the arity function just stipulates that $[\pi \cdot \mathsf{o}] = [\mathsf{o}]$ for all $\mathsf{o} \in \mathsf{O}$ and $\pi \in \mathfrak{S}_0(\mathsf{A})$. Thus, $\mathsf{O} \cong \coprod_{n\in\mathbb{N}} \mathsf{O}(n)$, for $\mathsf{O}(n)$ the sub nominal set of O given by inverse image as $\{ \mathsf{o} \in \mathsf{O} \mid [\mathsf{o}] = n \}$.

The *nominal set of terms* $T_\Sigma(V)$ on a nominal set V is inductively defined by the following rules:

$$\frac{v \in V}{v \in T_\Sigma(V)} \qquad\qquad \frac{t_i \in T_\Sigma(V) \quad (i = 1, \dots, n)}{\mathsf{o}\ t_1 \dots t_n \in T_\Sigma(V)} \quad (\mathsf{o} \in \mathsf{O}(n))$$

and equipped with the action inductively defined by:

$$\begin{aligned} \pi \cdot_{T_\Sigma} v &= \pi \cdot_V v\ , \\ \pi \cdot_{T_\Sigma} (\mathsf{o}\ t_1 \dots t_n) &= (\pi \cdot_{\mathsf{O}} \mathsf{o})\ (\pi \cdot_{T_\Sigma} t_1) \dots (\pi \cdot_{T_\Sigma} t_n)\ . \end{aligned}$$

We now fix a countably infinite set V of variables. The nominal set of *freshness contexts* is defined as

$$\coprod_{S \in \mathcal{P}_0(\mathsf{V})} \Big(\mathcal{P}_0\mathbb{A} \blacktriangleright (\mathcal{P}_0\mathbb{A})^S \Big)\ ,$$

where $X \blacktriangleright Y$ denotes the sub nominal set of $X \times Y$ with underlying set $\{ (x, y) \in |X| \times |Y| \mid \mathsf{supp}(x) \supseteq \mathsf{supp}(y) \}$. Thus, it has elements $\nabla = (|\nabla|, \nabla^{\mathcal{A}}, \nabla^{\#})$ given by a finite set of variables $|\nabla| \subset \mathsf{V}$, a finite set of atoms $\nabla^{\mathcal{A}} \subset \mathsf{A}$, and a function $\nabla^{\#} : |\nabla| \to \mathcal{P}_0(\nabla^{\mathcal{A}})$ with the following action

$$\pi \cdot (|\nabla|, \nabla^{\mathcal{A}}, \nabla^{\#}) = \Big(|\nabla|, \pi(\nabla^{\mathcal{A}}), \lambda x \in |\nabla|.\, \pi(\nabla^{\#} x) \Big)\ .$$

Note that $\mathsf{supp}(\nabla) = \nabla^{\mathcal{A}}$.

If $|\nabla| = \{ x_1, \dots, x_n \}$, $\nabla^{\mathcal{A}} = \{ a_1, \dots, a_m \}$, and $\nabla^{\#}(x_i) = A_i$ for $i = 1, \dots, n$, we write ∇ as

$$a_1, \dots, a_m \blacktriangleright A_1 \# x_1, \dots, A_n \# x_n$$

where we also abbreviate $\emptyset \# x$ as x and $\{a\} \# x$ as $a \# x$.

By a term t in a freshness context ∇, written $\nabla \vdash t$, we mean $t \in T_\Sigma(\,\overline{|\nabla|}\,)$ such that $\mathsf{supp}(t) \subseteq \nabla^{\mathcal{A}}$. That is, the grammar for terms in freshness contexts is as follows:

$$\begin{aligned} t ::= {}& \sigma\, x && (\sigma \in \mathfrak{S}_0(\mathsf{A}) \text{ with } \mathsf{supp}(\sigma) \subseteq \nabla^{\mathcal{A}},\ x \in |\nabla|) \\ \mid {}& \mathsf{o}\ t_1 \dots t_n && (\mathsf{o} \in \mathsf{O}(n) \text{ with } \mathsf{supp}(\mathsf{o}) \subseteq \nabla^{\mathcal{A}}) \end{aligned}$$

where we use the notational convention of abbreviating (σ, x) as $\sigma\, x$ and further abbreviating this as x when σ is the identity. Note that $\nabla \vdash t$ implies $\pi \cdot \nabla \vdash \pi \cdot t$ for all $\pi \in \mathfrak{S}_0(\mathsf{A})$.

A *NEL-theory* is given by a NEL-signature Σ together with a set of *axioms* consisting of judgements of the form

$$\nabla \vdash t \approx t'$$

where t and t' are terms in the freshness context ∇.

We give the canonical example of NEL-theory. The NEL-signature for the untyped λ-calculus is given by the nominal set of operators

$$\{ V_a \mid a \in \mathsf{A} \} \cup \{ L_a \mid a \in \mathsf{A} \} \cup \{ A \}$$

with action

$$\pi \cdot V_a = V_{\pi(a)} \ , \quad \pi \cdot L_a = L_{\pi(a)} \ , \quad \pi \cdot A = A$$

and arities $[V_a] = 0, [L_a] = 1, [A] = 2$. The NEL-theory for $\alpha\beta\eta$-equivalence of untyped λ-terms consists of the following axioms.

$$a, a' \blacktriangleright a' \# x \vdash L_a\, x \approx L_{a'}\, ((a\, a')\, x) \tag{α}$$

$$a \blacktriangleright a \# x,\ x' \vdash A\,(L_a\, x)\, x' \approx x \tag{β-1}$$

$$a \blacktriangleright x' \vdash A\,(L_a\, V_a)\, x' \approx x' \tag{β-2}$$

$$a, a' \blacktriangleright x,\ a' \# x' \vdash A\,(L_a\,(L_{a'}\, x))\, x' \approx L_{a'}\,(A\,(L_a\, x)\, x') \tag{β-3}$$

$$a \blacktriangleright x_1,\ x_2,\ x' \vdash A\,(L_a\,(A\, x_1\, x_2))\, x' \approx A\,(A\,(L_a\, x_1)\, x')\,(A\,(L_a\, x_2)\, x') \tag{β-4}$$

$$a, a' \blacktriangleright a' \# x \vdash A\,(L_a\, x)\, V_{a'} \approx (a\, a')\, x \tag{β-5}$$

$$a \blacktriangleright a \# x \vdash x \approx L_a\,(A\, x\, V_a) \tag{η}$$

Remark *The work reported in [7] is based on judgements of the form*

$$\nabla \vdash A \# t \approx t' \ ,$$

where A is a finite set of atoms, that impose name freshness conditions on the terms of equations. However, Clouston has shown that this extension, though convenient, does not add expressive power in that every such axiom can be equivalently encoded as one without freshness conditions. For instance, the α-equivalence axiom above is the encoding of the following one

$$a \blacktriangleright x \vdash \{a\} \# L_a\, x \approx L_a\, x \ .$$

A Σ-*structure* (M, e) for a NEL-signature $\Sigma = (\mathsf{O}, [-])$ is given by a nominal set M and an $\mathbb{N}$-indexed family e of equivariant functions $\mathsf{e}_n : \mathsf{O}(n) \times M^n \to M$, referred to as *evaluation functions*. The evaluation functions extend from operators to terms to give, for each nominal set V, the equivariant function $\bar{\mathsf{e}}_V : T_\Sigma(V) \times M^V \to M$ inductively defined by:

$$\begin{aligned} \bar{\mathsf{e}}_V(v, m) &= m(v) \ , \\ \bar{\mathsf{e}}_V(\mathsf{o}\ t_1 \ldots t_n,\, m) &= \mathsf{e}_n(\mathsf{o},\, \bar{\mathsf{e}}_V(t_1, m),\, \ldots,\, \bar{\mathsf{e}}_V(t_n, m)) \ . \end{aligned}$$

By a valuation m of a freshness context ∇ in a nominal set M, we mean $m \in M^{|\nabla|}$ such that $\nabla^{\#}(x) \# m_x$ for all $x \in |\nabla|$. It follows that $\pi \cdot m$ is a valuation of $\pi \cdot \nabla$ in M for all $\pi \in \mathfrak{S}_0(\mathsf{A})$. For every valuation m of ∇, the function $\overline{m} : \overline{|\nabla|} \to M$ defined by setting $\overline{m}(\pi, x) = \pi \cdot m_x$ is finitely supported by $\mathsf{supp}(m) = \cup_{x \in |\nabla|} \mathsf{supp}(m_x)$ and hence provides an extension $\overline{m} \in M^{\overline{|\nabla|}}$ of $m \in M^{|\nabla|}$.

A Σ-structure (M, e) is said to *satisfy* the judgement $\nabla \vdash t \approx t'$ if

$$\bar{\mathsf{e}}_{\overline{|\nabla|}}(t, \overline{m}) \;=\; \bar{\mathsf{e}}_{\overline{|\nabla|}}(t', \overline{m})$$

for all valuations m of ∇ in M.

A $\mathbb{T}$-*algebra* for a NEL-theory $\mathbb{T} = (\Sigma, E)$ is a Σ-structure that satisfies every axiom in E. A *homomorphism* from a $\mathbb{T}$-algebra (M, e) to another one (M', e') is an equivariant function $h : M \to M'$ such that $h(\mathsf{e}_n(\mathsf{o}, m_1, \ldots, m_n)) = \mathsf{e}'_n(\mathsf{o}, h(m_1), \ldots, h(m_n))$ for all $n \in \mathbb{N}$, $\mathsf{o} \in \mathsf{O}(n)$, and $m_1, \ldots, m_n \in M$. $\mathbb{T}$-algebras and homomorphisms form the category $\mathbb{T}$-**Alg**.

6.3 NEL-theories as equational systems

We will now present every NEL-theory $\mathbb{T} = (\Sigma, E)$ as an equational system $\mathbb{T}' = (\mathbf{Nom} : \Sigma' \rhd \Gamma' \vdash L' = R')$ in such a way that the respective categories of algebras are isomorphic.

The *functorial signature* Σ' is simply defined as

$$\Sigma'(M) = \coprod_{n \in \mathbb{N}} \mathsf{O}_\Sigma(n) \times M^n \;,$$

so that Σ'-algebras and Σ-structures are in bijective correspondence.

Turning the set of axioms into a functorial equation is more involved. We consider first the definition of the functorial context associated to a freshness context. To this end, note that if a Σ-structure satisfies the axiom $\nabla \vdash t \approx t'$ then, by equivariance of the evaluation functions, it also satisfies the judgement $(\pi \cdot \nabla) \vdash (\pi \cdot t) \approx (\pi \cdot t')$ for all $\pi \in \mathfrak{S}_0(\mathsf{A})$ (see [7]). Hence the atoms in $\nabla^{\mathcal{A}}$ for the freshness context ∇ of a judgement can be conceptually understood as atom place-holders or meta-atoms. It follows that the functorial contexts of freshness contexts should be given by a consistent interpretation of both atoms and term variables. This is formalized by defining the *functorial context* Γ_∇ on **Nom** of a freshness context ∇ as

$$\Gamma_\nabla(M) = \{\, (\alpha, m) \in \mathbb{A}^{\#\nabla^{\mathcal{A}}} \times M^{|\nabla|} \mid m \text{ is a valuation of } \alpha \cdot \nabla \text{ in } M \,\} \;.$$

(Recall that $\alpha \cdot \nabla$ stands for $\tilde{\alpha} \cdot \nabla$ where $\tilde{\alpha} \in \mathfrak{S}_0(\mathsf{A})$ is any permutation extending $\alpha : \nabla^{\mathcal{A}} \rightarrowtail \mathsf{A}$, and that this is well-defined because $\nabla^{\mathcal{A}}$ is the support of ∇.)

For a term in a freshness context $\nabla \vdash t$, the *functorial term*

$$F_{\nabla\vdash t} : \Sigma'\text{-}\mathbf{Alg} \to \Gamma_\nabla\text{-}\mathbf{Alg}$$

then maps (M, e) to

$$F_{\nabla\vdash t}(M,\mathsf{e}) : \ \Gamma_\nabla(M) \to M : \ (\alpha, m) \mapsto \overline{\mathsf{e}}_{|\nabla|}(\alpha \cdot t, \overline{m}) \ .$$

(Recall that $\alpha \cdot t$ stands for $\widetilde{\alpha} \cdot t$ where $\widetilde{\alpha} \in \mathfrak{S}_0(\mathsf{A})$ is any permutation extending $\alpha : \nabla^{\mathcal{A}} \rightarrowtail \mathsf{A}$, and that this is well-defined because $\nabla^{\mathcal{A}}$ includes the support of t.)

The equivariance of $F_{\nabla\vdash t}(M, \mathsf{e})$ is established as follows:

$$\begin{aligned}
F_{\nabla\vdash t}(M,\mathsf{e})\Big(\pi \cdot (\alpha, m)\Big) &= F_{\nabla\vdash t}(M,\mathsf{e})(\pi\alpha, \pi \cdot m) \\
&= \overline{\mathsf{e}}((\pi\alpha) \cdot t, \overline{\pi \cdot m}) \\
&= \overline{\mathsf{e}}(\pi \cdot (\alpha \cdot t), \pi \cdot \overline{m}) \\
&= \pi \cdot \overline{\mathsf{e}}(\alpha \cdot t, \overline{m}) \\
&= \pi \cdot F_{\nabla\vdash t}(M,\mathsf{e})(\alpha, m)
\end{aligned}$$

where the third identity follows because any extension $\widetilde{\alpha}$ of α makes $\pi\widetilde{\alpha}$ into an extension of $\pi\alpha$, and because $\overline{\pi \cdot m} = \pi \cdot \overline{m}$.

The equational system $\mathbb{T}' = (\mathbf{Nom} : \Sigma' \rhd \Gamma' \vdash L' = R')$ associated to the NEL-theory $\mathbb{T} = (\Sigma, E)$ is thus defined as

$$\begin{aligned}
\Sigma' &= \textstyle\coprod_{n\in\mathbb{N}} \mathsf{O}_\Sigma(n) \times (-)^n \ , & \Gamma' &= \textstyle\coprod_{(\nabla\vdash t\approx t')\in E} \Gamma_\nabla \\
L' &= \Big[F_{\nabla\vdash t}\Big]_{(\nabla\vdash t\approx t')\in E} \ , & R' &= \Big[F_{\nabla\vdash t'}\Big]_{(\nabla\vdash t\approx t')\in E}
\end{aligned}$$

Theorem 6.1 *The categories* $\mathbb{T}$-**Alg** *and* $\mathbb{T}'$-**Alg** *are isomorphic.*

PROOF. We prove that a Σ-structure (M, e) satisfies the judgement $\nabla \vdash t \approx t'$ if and only if $F_{\nabla\vdash t}(M, [\mathsf{e}_n]_{n\in\mathbb{N}}) = F_{\nabla\vdash t'}(M, [\mathsf{e}_n]_{n\in\mathbb{N}})$.

The if part is easily shown by considering the inclusion function $\iota \in \mathsf{A}^{\#\nabla^{\mathcal{A}}}$. Indeed, for all valuations m of ∇ in M, we have that

$$\overline{\mathsf{e}}(t, \overline{m}) \ = \ F_{\nabla\vdash t}(M, [\mathsf{e}_n]_{n\in\mathbb{N}})(\iota, m) \ = \ F_{\nabla\vdash t'}(M, [\mathsf{e}_n]_{n\in\mathbb{N}})(\iota, m) \ = \ \overline{\mathsf{e}}(t', \overline{m})$$

where the first and last identities hold because the identity permutation extends ι.

To prove the only-if part, assume that (M, e) satisfies the judgement $\nabla \vdash t \approx t'$. Then, for $(\alpha, m) \in \Gamma_\nabla(M)$, as m is a valuation of $\alpha \cdot \nabla$ in M and (M, e) also satisfies $(\alpha \cdot \nabla) \vdash (\alpha \cdot t) \approx (\alpha \cdot t')$, we conclude that

$$\begin{aligned} F_{\nabla \vdash t}(M, [\mathsf{e}_n]_{n \in \mathbb{N}})(\alpha, m) &= \bar{\mathsf{e}}(\alpha \cdot t, \overline{m}) \\ &= \bar{\mathsf{e}}(\alpha \cdot t', \overline{m}) \\ &= F_{\nabla \vdash t'}(M, [\mathsf{e}_n]_{n \in \mathbb{N}})(\alpha, m) \qquad \square \end{aligned}$$

Aiming at applying Theorems 4.9 and 5.1 we establish the following result.

Theorem 6.2 *The functorial signature and functorial context of the equational system associated to a NEL-theory preserve filtered colimits and epimorphisms.*

PROOF. Let Σ' and Γ' respectively be the functorial signature and functorial context associated to a NEL-theory (Σ, E).

As the product is closed, the functor $\mathsf{O}_\Sigma(n) \times (-)^n$ preserves filtered colimits and epimorphisms for all $n \in \mathbb{N}$. Thus, the functorial signature Σ', being the (pointwise) coproduct of these functors, also preserves filtered colimits and epimorphisms.

Since the functorial context Γ' is the (pointwise) coproduct of functorial contexts of the form Γ_∇, it is enough to show that such functors preserve (i) filtered colimits and (ii) epimorphisms.

To show (i), we make the key observation that for all freshness contexts ∇, the following diagram is a pullback

$$\begin{array}{ccc} \Gamma_\nabla(M) & \xrightarrow{\jmath'_M} & \prod_{x \in |\nabla|} \left(\mathbb{A}^{\#(\nabla^\#(x))} \# M \right) \\ {\scriptstyle \imath'_M} \big\downarrow & & \big\downarrow {\scriptstyle \imath_M} \\ \mathbb{A}^{\#\nabla_{\mathcal{A}}} \times M^{|\nabla|} & \xrightarrow[\jmath_M]{} & \prod_{x \in |\nabla|} \left(\mathbb{A}^{\#(\nabla^\#(x))} \times M \right) \end{array}$$

where $\imath_M$ is induced by the embedding of the separating tensor into the product, $\imath'_M$ is the embedding determined by the definition of $\Gamma_\nabla(M)$, and $\jmath'_M$ is the restriction of the equivariant function $\jmath_M : (\alpha, m) \mapsto \big((\alpha \restriction \nabla^\#(x), m_x) \big)_{x \in |\nabla|}$ obtained by restriction and evaluation.

Thus, since the category of nominal sets is locally finitely presentable, and hence in it finite limits commute with filtered colimits, and since both the product and separating tensor are closed, and hence preserve filtered colimits, it follows that Γ_∇ preserves filtered colimits. Indeed, for D a filtered diagram of nominal sets, we have that

$$
\begin{aligned}
&\mathrm{colim}(\Gamma_\nabla D)\\
&\cong \mathrm{colim}\Big(\lim\big(\begin{tikzcd}[column sep=tiny] \mathbb{A}^{\#\nabla_{\mathcal{A}}}\times D^{|\nabla|} \arrow[dr] & & \prod_{x\in|\nabla|}\mathbb{A}^{\#(\nabla^{\#}(x))}\#D \arrow[dl, hook'] \\ & \prod_{x\in|\nabla|}\mathbb{A}^{\#(\nabla^{\#}(x))}\times D & \end{tikzcd}\big)\Big)\\
&\cong \lim\Big(\begin{tikzcd}[column sep=tiny] \mathrm{colim}(\mathbb{A}^{\#\nabla_{\mathcal{A}}}\times D^{|\nabla|}) \arrow[dr] & & \mathrm{colim}(\prod_{x\in|\nabla|}\mathbb{A}^{\#(\nabla^{\#}(x))}\#D) \arrow[dl, hook'] \\ & \mathrm{colim}(\prod_{x\in|\nabla|}\mathbb{A}^{\#(\nabla^{\#}(x))}\times D) & \end{tikzcd}\Big)\\
&\cong \lim\Big(\begin{tikzcd}[column sep=tiny] \mathbb{A}^{\#\nabla_{\mathcal{A}}}\times (\mathrm{colim}\ D)^{|\nabla|} \arrow[dr] & & \prod_{x\in|\nabla|}\mathbb{A}^{\#(\nabla^{\#}(x))}\#(\mathrm{colim}\ D) \arrow[dl, hook'] \\ & \prod_{x\in|\nabla|}\mathbb{A}^{\#(\nabla^{\#}(x))}\times (\mathrm{colim}\ D) & \end{tikzcd}\Big)\\
&\cong \Gamma_\nabla(\mathrm{colim}\ D)\ .
\end{aligned}
$$

To show (ii), we just need to show that Γ_∇ preserves surjectivity. To this end, let $f : P \twoheadrightarrow Q$ be a surjective equivariant function and let $(\alpha, q) \in \Gamma_\nabla(Q)$. Then, for every $x \in |\nabla|$, there exists $p_x \in P$ such that $f(p_x) = q_x$. Moreover, since $\mathsf{supp}(p_x) \supseteq \mathsf{supp}(q_x)$ and $\mathsf{supp}(q_x) \# (\alpha \cdot \nabla)^{\#}(x)$, there exists $\pi_x \in \mathfrak{S}_0(\mathsf{A})$ such that

$$\pi_x(a) = a \text{ for all } a \in \mathsf{supp}(q_x) \quad\text{and}\quad \pi_x(\mathsf{supp}(p_x)) \# (\alpha \cdot \nabla)^{\#}(x)\ .$$

It follows that $f(\pi_x \cdot p_x) = \pi_x \cdot f(p_x) = \pi_x \cdot q_x = q_x$ and $(\alpha \cdot \nabla)^{\#}(x) \# \pi_x \cdot p_x$. Thus, setting $p'_x = \pi_x \cdot p_x$ for all $x \in |\nabla|$, we have $(\alpha, p') \in \Gamma_\nabla(P)$ with $\Gamma_\nabla(f)(\alpha, p') = (\alpha, fp') = (\alpha, q)$ as required. □

Corollary 6.3 *The category of algebras for a NEL-theory is cocomplete and monadic over nominal sets, with the induced free-algebra monad being finitary. Moreover, free algebras on nominal sets are constructed in $\omega + \omega$ steps by the construction (1) followed by the construction (4) in Section 4.*

6.4 Presentations of free algebras

We proceed to give presentations of free algebras for NEL-theories. To this end, let $\mathbb{T} = (\Sigma, E)$ be a NEL-theory, and let (M, e) be a Σ-structure. By the construction (4) in Section 4.5, the carrier object of the free $\mathbb{T}$-algebra on (M, e) is obtained as the colimit of the ω-chain of quotients

$$M \twoheadrightarrow M_{/\approx_0} \twoheadrightarrow \cdots \twoheadrightarrow M_{/\approx_n} \twoheadrightarrow \cdots$$

$$\frac{(\alpha, m) \in \Gamma_\nabla(M)}{\overline{\mathsf{e}}(\alpha \cdot t, \overline{m}) \sim_0 \overline{\mathsf{e}}(\alpha \cdot t', \overline{m})} \ \big((\nabla \vdash t \approx t') \in E\big)$$

$$\frac{m \approx_\ell m'}{m \sim_{\ell+1} m'} \qquad \frac{m_i \approx_\ell m'_i \ \ (1 \leq i \leq n)}{\mathsf{o}\ m_1 \ldots m_n \sim_{\ell+1} \mathsf{o}\ m'_1 \ldots m'_n} \ (\mathsf{o} \in \mathsf{O}(n))$$

Fig. 1. Rules for the family of relations $\{ \sim_\ell \}_{\ell \in \mathbb{N}}$.

$$\frac{}{m \approx_E m} \qquad \frac{m \approx_E m'}{m' \approx_E m} \qquad \frac{m \approx_E m' \quad m' \approx_E m''}{m \approx_E m''}$$

$$\frac{(\alpha, m) \in \Gamma_\nabla(M)}{\overline{\mathsf{e}}(\alpha \cdot t, \overline{m}) \approx_E \overline{\mathsf{e}}(\alpha \cdot t', \overline{m})} \ \big((\nabla \vdash t \approx t') \in E\big)$$

$$\frac{m_i \approx_E m'_i \ \ (1 \leq i \leq n)}{\mathsf{o}\ m_1 \ldots m_n \approx_E \mathsf{o}\ m'_1 \ldots m'_n} \ \big(\mathsf{o} \in \mathsf{O}(n)\big)$$

Fig. 2. Rules for the relation $\approx_E$.

where $\approx_\ell$ denotes the equivalence relation generated by $\sim_\ell$, and these are given by the rules in Figure 1.

Thus, the free algebra has carrier $M_{/\approx_E}$ with $\approx_E$ given by the rules in Figure 2. Furthermore, as the quotient map $M \twoheadrightarrow M_{/\approx_E}$ is a homomorphism from (M, e) to the free algebra $(M_{/\approx_E}, [\mathsf{e}])$, the Σ-structure $[\mathsf{e}]$ on $M_{/\approx_E}$ is given by

$$[\mathsf{e}]_n(\mathsf{o}, [m_1]_{\approx_E}, \ldots, [m_n]_{\approx_E}) \ = \ [\mathsf{e}_n(\mathsf{o}, m_1, \ldots, m_n)]_{\approx_E}$$

for all $n \in \mathbb{N}$, $\mathsf{o} \in \mathsf{O}(n)$, and $m_1, \ldots, m_n \in M$.

It follows that the free $\mathbb{T}$-algebra on the empty nominal set $\emptyset$ consists of the nominal set $T_\Sigma(\emptyset)_{/\approx_E}$, of ground terms in $T_\Sigma(\emptyset)$ quotiented by $\approx_E$, equipped with the syntactic Σ-structure. As the rules for $\approx_E$ on ground terms are derivable from the NEL-rules for $\mathbb{T}$, the ground completeness result for NEL-theories of [7, Theorem 9.4] follows as a corollary.

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