The Bicategory-Theoretic Solution of Recursive Domain Equations

Dedicated to Gordon Plotkin on the occasion of his 60th birthday

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Abstract

We generalise the traditional approach of Smyth and Plotkin to the solution of recursive domain equations from order-enriched structures to bicategorical ones and thereby develop a bicategorical theory for recursively defined domains in accordance with Axiomatic Domain Theory.

Keywords: Bicategory theory, domain theory, recursive domain equations, bicategorical limit/colimit coincidence, algebraic completeness, algebraic compactness, Axiomatic Domain Theory.

Introduction

One of the many areas of theoretical computer science to which Gordon Plotkin has made fundamental contributions is domain theory, and within it the theory of recursively defined domains—the theme of this paper. As it is well-known, the subject started with Dana Scott's D_∞ construction [21,22], which he developed to provide a mathematical model of the untyped lambda calculus in the category of continuous lattices. The D_∞ model arises from an ω-chain of embedding/projection pairs

[1] The core of the results of this paper were first announced, without proof, in [7, Sections 2 and 3].

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$\langle\, D_n \underset{\hookrightarrow}{\overset{\twoheadleftarrow}{\top}} D_{n+1} \,\rangle$, with $D_{n+1} = [D_n \to D_n]$, both as the limit of the ω^{op}-chain of projections $\langle\, D_n \twoheadleftarrow D_{n+1} \,\rangle$ and as the colimit of the ω-chain of embeddings $\langle\, D_n \hookrightarrow D_{n+1} \,\rangle$. This is the all important *limit/colimit coincidence* of domain theory. Indeed, already in [22, page 129], Dana Scott mentions a suggestion of Bill Lawvere to the extent that his proof of the remarkable fact that $D_\infty \cong [D_\infty \to D_\infty]$ could be explained by a fixpoint argument hinging on the limit/colimit coincidence. He further remarks in the references that no doubt the overall idea of the construction could be put into a more general abstract context, leaving open the possibility for such a development.

A category-theoretic framework for the limit/colimit coincidence was provided by Mike Smyth and Gordon Plotkin in [23]. There, building on previous work of Mitch Wand [28], they showed that the limit/colimit coincidence depended solely on the underlying order-theoretic structure of the category of discourse, and further gave a systematic treatment of the solution of recursive domain equations applicable to the whole range of categories of domains. A fundamental contribution of their analysis was to make explicit an aspect of the theory that mediates between the limit and colimit views of recursive domains and sets the theory in motion. This is what we call the *local characterisation* of limits of ω^{op}-chains of projections and colimits of ω-chains of embeddings. By the former we mean the equivalence between the global condition that a cocone $\langle \gamma_n : D_\infty \to D_n \rangle$ for an ω^{op}-chain of projections $\langle\, D_n \twoheadleftarrow D_{n+1} \,\rangle$ is a limit and the local condition that the cocone consists of projections such that $\bigvee \varphi_n \gamma_n = 1_{D_\infty}$, where $\varphi_n : D_n \hookrightarrow D_\infty$ is the embedding associated to the projection $\gamma_n : D_\infty \twoheadrightarrow D_n$. Dually, and equivalently, the local characterisation of colimits of ω-chains of embeddings amounts to the fact that the global condition that a cone $\langle \varphi_n : D_n \to D_\infty \rangle$ for an ω-chain of embeddings $\langle\, D_n \hookrightarrow D_{n+1} \,\rangle$ is a colimit is equivalent to the local condition that the cone consists of embeddings such that $\bigvee \varphi_n \gamma_n = 1_{D_\infty}$, where $\gamma_n : D_\infty \twoheadrightarrow D_n$ is the projection associated to the embedding $\varphi_n : D_n \hookrightarrow D_\infty$.

The local-characterisation theorem of Mike Smyth and Gordon Plotkin [23, Theorem 2] is the starting point of our work. Specifically, we present in Section 1 a generalisation of it in two directions as follows. Firstly, following the folklore, we generalise from the consideration of embedding/projection pairs (referred to as coreflections in the categorical jargon) to that of adjoint pairs. Secondly, in the spirit of higher-dimensional category theory, we move up a level from the order-enriched setting to a bicategorical one—hence the title of the paper.

The paper [23] remained the state of the art in solving recursive domain equations until further impulse to the subject was injected by Peter Freyd with his analysis of inductive and recursive types, leading to the concepts and theory of *algebraic completeness* and *algebraic compactness* [11,12,13]. In studying this work, and being already familiar with [23] from the LFCS Theory Postgraduate Course on Domain Theory by Gordon Plotkin, Marcelo Fiore, then his Ph.D. student, recognised the relevance of the old ideas to the new ones and, in collaboration, a theory of recursive domains was developed; see [9], and also [8,10,20]. This theory of order-enriched algebraic completeness and compactness, which was to provide an axiomatic treatment for domain-theoretic models, is the background for the second part of the present work, where, in Section 2, a generalisation to the bicategorical setting is presented.

Thus we provide a bicategorical theory for recursively defined domains in accordance with Axiomatic Domain Theory. Our motivations for pursuing these investigations in the context of models of computation, and at this level of generality, have been expounded in [7,6]. A further natural setting for the application of our work is that of generalisations of categories of domains from order-theoretic structures to category-theoretic ones; see, *e.g.*, [17,18,1,25,2,7,6,26,27,14]. However, we also hope that our results will become relevant to the body of work on higher-dimensional category theory.

1 Local-characterisation theorem

This section presents a central result of the paper, Theorem 1.5, that generalises the local characterisation of colimits of ω-chains of embeddings in ω**Cpo**-categories [23, Theorem 2] yielding the limit/colimit coincidence [22].

We start however with some basic bicategorical definitions and facts (Sections 1.1–1.3) needed in the main development (Sections 1.4–1.6). Throughout, familiarity with (unbiased) bicategories, for which the reader is referred to [3,5] (and to [19]), is assumed.

1.1 Pseudo cells

An invertible 2-cell in a bicategory is henceforth referred to as a *pseudo cell*, and we write $\mathcal{K}_{\overset{\cong}{\Rightarrow}}$ for the sub-bicategory of a bicategory $\mathcal{K}$ with the same objects and arrows, but only the pseudo cells.

1.2 Adjunctions and mates

An adjunction $(\eta, \varepsilon : f \dashv g : B \to A)$ in a bicategory consists of objects A and B, arrows $f : A \to B$ and $g : B \to A$, and 2-cells $\eta : 1_A \Rightarrow gf$ and $\varepsilon : fg \Rightarrow 1_B$ such that the triangle laws

$$\big(\, f \cong f\,1_A \xRightarrow{f\eta} f(gf) \cong (fg)f \xRightarrow{\varepsilon f} 1_B f \cong f \,\big) \;=\; 1_f$$

$$\big(\, g \cong 1_A\, g \xRightarrow{\eta g} (gf)g \cong g(fg) \xRightarrow{g\varepsilon} g\, 1_B \cong g \,\big) \;=\; 1_g$$

hold.

For adjunctions $(\eta, \varepsilon : f \dashv g : B \to A)$ and $(\eta', \varepsilon' : f' \dashv g' : B \to A)$, the *mates* (see [15]) $\sigma^M : g' \Rightarrow g$ and $\tau^M : f' \Rightarrow f$ of 2-cells $\sigma : f \Rightarrow f'$ and $\tau : g \Rightarrow g'$ are respectively given by the composites

$$g' \cong 1_A g' \xRightarrow{\eta g'} (gf)g' \cong gfg' \xRightarrow{g\sigma g'} gf'g' \cong g(f'g') \xRightarrow{g\varepsilon'} g1_B \cong g$$

and

$$f' \cong f'1_A \overset{f'\eta}{\Longrightarrow} f'(gf) \cong f'gf \overset{f'\tau f}{\Longrightarrow} f'g'f \cong (f'g')f \overset{\varepsilon' f}{\Longrightarrow} 1_B f \cong f \quad .$$

Lemma 1.1 *For adjunctions $(\eta, \varepsilon : f \dashv g : B \to A)$ and $(\eta', \varepsilon' : f' \dashv g' : B \to A)$, and 2-cells $\sigma : f \Rightarrow f'$ and $\tau : g \Rightarrow g'$, the following are equivalent.*

(i) *The 2-cell σ is invertible and τ is the mate of its inverse.*

(ii) *The 2-cell τ is invertible and σ is the mate of its inverse.*

(iii) *The identities $\eta' = (\tau\sigma) \cdot \eta$ and $\varepsilon = \varepsilon' \cdot (\sigma\tau)$ hold.*

Definition 1.2 For a bicategory $\mathcal{K}$, we define the *bicategory of adjunctions* $\mathcal{K}^{\mathrm{adj}}$ as follows. The objects of $\mathcal{K}^{\mathrm{adj}}$ are those of $\mathcal{K}$, the arrows $A \to B$ of $\mathcal{K}^{\mathrm{adj}}$ are given by adjunctions $(\eta, \varepsilon : f \dashv g : B \to A)$ in $\mathcal{K}$, the 2-cells $(\eta, \varepsilon : f \dashv g) \Rightarrow (\eta', \varepsilon' : f' \dashv g')$ of $\mathcal{K}^{\mathrm{adj}}$ are given by pairs $\sigma : f \Rightarrow f'$ and $\tau : g \Rightarrow g'$ of 2-cells in $\mathcal{K}$ such that $\eta' = (\tau\sigma) \cdot \eta$ and $\varepsilon = \varepsilon' \cdot (\sigma\tau)$. The identities and composition of arrows, and the horizontal and vertical identities and composition of 2-cells are as expected; as are the coherence isomorphisms.

By Lemma 1.1 every 2-cell in $\mathcal{K}^{\mathrm{adj}}$ is invertible. In fact, $\mathcal{K}^{\mathrm{adj}}$ is biequivalent to the sub-bicategory of $\mathcal{K}_{\overset{\sim}{\Rightarrow}}$ determined by the left adjoints in $\mathcal{K}$.

1.3 Bicategorical colimits

We give an explicit elementary definition of bicategorical colimits (elsewhere bicolimits [24]) of ω-chains.

Definition 1.3 An *ω-chain* in a bicategory is given by an ω-indexed family of arrows $\langle f_n : A_n \to A_{n+1} \rangle$. A *pseudo cone* for such a chain is given by an object A, an ω-indexed family of arrows $\langle \varphi_n : A_n \to A \rangle$, and an ω-indexed family of pseudo cells $\langle \Phi_n : \varphi_{n+1} f_n \overset{\sim}{\Longrightarrow} \varphi_n \rangle$. Further, such a pseudo cone is a *bicategorical colimit* if it satisfies the following universal property.

(i) For every pseudo cone $\langle \Psi_n : \psi_{n+1} f_n \overset{\sim}{\Longrightarrow} \psi_n : A_n \to X \rangle$ there exists an arrow $u : A \to X$ and an ω-indexed family of pseudo cells $\langle \mu_n : u\varphi_n \overset{\sim}{\Longrightarrow} \psi_n \rangle$ such that the square

$$\begin{tikzcd}
u(\varphi_{n+1} f_n) \cong (u\varphi_{n+1}) f_n \arrow[r, Rightarrow, "\mu_{n+1} f_n"] \arrow[d, Rightarrow, "u\Phi_n"'] & \Psi_{n+1} f_n \arrow[d, Rightarrow, "\Psi_n"] \\
u\varphi_n \arrow[r, Rightarrow, "\mu_n"'] & \psi_n
\end{tikzcd}$$

commutes for all $n \in \omega$.

(ii) For every pair of arrows $u, v : A \to X$ and every ω-indexed family of 2-cells

$\langle \xi_n : u\varphi_n \Rightarrow v\varphi_n \rangle$ such that the square

$$
\begin{tikzcd}
u(\varphi_{n+1}f_n) \cong (u\varphi_{n+1})f_n \arrow[r, Rightarrow, "\xi_{n+1}f_n"] \arrow[d, Rightarrow, "u\Phi_n"'] & (v\varphi_{n+1})f_n \cong v(\varphi_{n+1}f_n) \arrow[d, Rightarrow, "v\Phi_n"] \\
u\varphi_n \arrow[r, Rightarrow, "\xi_n"'] & v\varphi_n
\end{tikzcd}
$$

commutes for all $n \in \omega$, there exists a unique 2-cell $\xi : u \Rightarrow v$ such that $\xi_n = \xi\varphi_n$, for all $n \in \omega$.

1.4 The canonical local cones of a pseudo cone of adjunctions

We need to consider the structure of pseudo cones of ω-chains in bicategories of adjunctions in detail. To this end, we start by spelling out in terms of data from $\mathcal{K}$ what pseudo cones of ω-chains in $\mathcal{K}^{\mathrm{adj}}$ amount to.

Let $\mathcal{K}$ be a bicategory. A *pseudo cone of adjunctions*

$$
(\Phi_n, \Gamma_n) : (\iota_{n+1}, \jmath_{n+1} : \varphi_{n+1} \dashv \gamma_{n+1})(\eta_n, \varepsilon_n : f_n \dashv g_n) \xRightarrow{\sim} (\iota_n, \jmath_n : \varphi_n \dashv \gamma_n)
$$

for an ω-chain $\langle \eta_n, \varepsilon_n : f_n \dashv g_n : A_{n+1} \longrightarrow A_n \rangle$ in $\mathcal{K}^{\mathrm{adj}}$ consists of an object A, an ω-indexed family $\langle \iota_n, \jmath_n : \varphi_n \dashv \gamma_n : A \to A_n \rangle$ of adjunctions in $\mathcal{K}$, and an ω-indexed family $\langle \Phi_n, \Gamma_n \rangle$ of pseudo cells $\Phi_n : \varphi_{n+1}f_n \xRightarrow{\sim} \varphi_n$ and $\Gamma_n : g_n\gamma_{n+1} \xRightarrow{\sim} \gamma_n$ such that the squares

$$
\begin{tikzcd}
1_{A_n} \arrow[r, Rightarrow, "\eta_n"] \arrow[d, Rightarrow, "\iota_n"'] & g_n f_n \arrow[r, Rightarrow, "\sim"] \arrow[d, phantom, "\mathcal{I}_n"] & g_n 1_{A_{n+1}} f_n \arrow[d, Rightarrow, "g_n \iota_{n+1} f_n"] \\
\gamma_n \varphi_n & (g_n\gamma_{n+1})(\varphi_{n+1}f_n) \arrow[l, Rightarrow, "\Gamma_n\Phi_n"] & g_n(\gamma_{n+1}\varphi_{n+1})f_n \arrow[l, Rightarrow, "\sim"']
\end{tikzcd}
\tag{1}
$$

$$
\begin{tikzcd}
\varphi_n\gamma_n \arrow[r, Rightarrow, "{\Phi_n}^{-1}{\Gamma_n}^{-1}"] \arrow[d, Rightarrow, "\jmath_n"'] & (\varphi_{n+1}f_n)(g_n\gamma_{n+1}) \arrow[r, Rightarrow, "\sim"] \arrow[d, phantom, "\mathcal{J}_n"] & \varphi_{n+1}(f_n g_n)\gamma_{n+1} \arrow[d, Rightarrow, "\varphi_{n+1}\varepsilon_n\gamma_{n+1}"] \\
1_A & \varphi_{n+1}\gamma_{n+1} \arrow[l, Rightarrow, "\jmath_{n+1}"] & \varphi_{n+1}1_{A_{n+1}}\gamma_{n+1} \arrow[l, Rightarrow, "\sim"']
\end{tikzcd}
\tag{2}
$$

commute for all $n \in \omega$.

It is important to observe that a pseudo cone of adjunctions induces local ω-chains, together with cones, in $\mathcal{K}(A_n, A_n)$ and $\mathcal{K}(A, A)$ as follows.

- For every $n \in \omega$, we have the ω-chain $\langle g_{\ell,n} f_{n,\ell} \rangle_{\ell \geq n}$ in $\mathcal{K}(A_n, A_n)$, where $g_{n,n}f_{n,n} = 1_{A_n}$ and, for $n < \ell$, $g_{\ell,n}f_{n,\ell} = g_n(g_{\ell,n+1}f_{n+1,\ell})f_n$, given as follows

$$
\begin{tikzcd}
1_{A_n} \arrow[r, Rightarrow, "\eta_n"'] \arrow[rr, Rightarrow, bend left] & g_n f_n \cong g_n 1_{A_{n+1}} f_n \arrow[r, Rightarrow, "g_n\eta_{n+1}f_n"'] \arrow[rr, Rightarrow, bend left] & g_n(g_{n+1}f_{n+1})f_n \cong g_n(g_{n+1}1_{A_{n+2}}f_{n+1})f_n \arrow[r, Rightarrow] \arrow[r, Rightarrow, bend left] & \cdots
\end{tikzcd}
$$

with cone

$$
\begin{tikzcd}
1_{A_n} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & {} \arrow[r, Rightarrow, "\sim"] \arrow[d, phantom, "\mathcal{I}_n"] & g_n 1_{A_{n+1}} f_n \arrow[r, Rightarrow] \arrow[d, Rightarrow] & {} \arrow[r, Rightarrow, "\sim"] \arrow[d, phantom, "g_n \mathcal{I}_{n+1} f_n"] & g_n(g_{n+1} 1_{A_{n+2}} f_{n+1}) f_n \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \cdots \\
\gamma_n \varphi_n & {} \arrow[l, Rightarrow] & g_n(\gamma_{n+1}\varphi_{n+1}) f_n \arrow[l, Rightarrow, "\sim"'] & {} \arrow[l, Rightarrow] & g_n(g_{n+1}(\gamma_{n+2}\varphi_{n+2}) f_{n+1}) f_n \arrow[l, Rightarrow, "\sim"] & \cdots \arrow[l, Rightarrow]
\end{tikzcd}
$$

- In $\mathcal{K}(A, A)$, we have the ω-chain $\langle \varphi_n \gamma_n \rangle$ given as follows

$$
\varphi_0\gamma_0 \xRightarrow{{\Phi_0}^{-1}{\Gamma_0}^{-1}} \xRightarrow{\sim} \xRightarrow{\varphi_1\varepsilon_0\gamma_1} \xRightarrow{\sim} \varphi_1\gamma_1 \xRightarrow{{\Phi_1}^{-1}{\Gamma_1}^{-1}} \xRightarrow{\sim} \xRightarrow{\varphi_2\varepsilon_1\gamma_2} \xRightarrow{\sim} \varphi_2\gamma_2 \Longrightarrow \cdots
$$

with cone

$$
\begin{tikzcd}
\varphi_0\gamma_0 \arrow[r, Rightarrow] \arrow[drrrr, Rightarrow] & {} \arrow[r, Rightarrow, "\sim"] & {} \arrow[r, Rightarrow] & {} \arrow[r, Rightarrow, "\sim"] \arrow[d, phantom, "\mathcal{J}_0"] & \varphi_1\gamma_1 \arrow[r, Rightarrow] \arrow[d, Rightarrow] & {} \arrow[r, Rightarrow, "\sim"] \arrow[d, phantom, "\mathcal{J}_1"] & {} \arrow[r, Rightarrow] \arrow[d, phantom, "\cdots"] & {} \arrow[r, Rightarrow, "\sim"] & \varphi_2\gamma_2 \arrow[r, Rightarrow] \arrow[dllll, Rightarrow] & \cdots \\
& & & & 1_A & & & & &
\end{tikzcd}
$$

Henceforth, we shall refer to the above local cones $\langle g_{\ell,n} f_{n,\ell} \rangle_\ell \stackrel{\cdot}{\Longrightarrow} \gamma_n\varphi_n$ and $\langle \varphi_n\gamma_n \rangle \stackrel{\cdot}{\Longrightarrow} 1_A$ as the *canonical local cones* of the pseudo cone of adjunctions.

1.5 Local-characterisation theorem

We consider ω**Cat**-bicategories, a bicategorical generalisation of ω**Cpo**-categories.

Definition 1.4 An ω**Cat**-*bicategory* is a bicategory whose hom-categories have colimits of ω-chains and whose composition functors preserve them.

The announced generalisation of [23, Theorem 2] follows.

Theorem 1.5 *Let $\mathcal{K}$ be an ω*Cat*-bicategory.*

For an ω-chain of adjunctions $\langle \eta_n, \varepsilon_n : f_n \dashv g_n : A_{n+1} \to A_n \rangle$ and a pseudo cone $\langle \Phi_n : \varphi_{n+1} f_n \stackrel{\sim}{\Longrightarrow} \varphi_n : A_n \to A \rangle$ for the ω-chain $\langle f_n : A_n \to A_{n+1} \rangle$ in $\mathcal{K}$, the following are equivalent.

(i) *The pseudo cone $\langle \Phi_n : \varphi_{n+1} f_n \stackrel{\sim}{\Longrightarrow} \varphi_n : A_n \to A \rangle$ is a bicategorical colimit of the ω-chain $\langle f_n : A_n \to A_{n+1} \rangle$ in $\mathcal{K}$.*

(ii) *The pseudo cone $\langle \Phi_n : \varphi_{n+1} f_n \stackrel{\sim}{\Longrightarrow} \varphi_n : A_n \to A \rangle$ is a bicategorical colimit of the ω-chain $\langle f_n : A_n \to A_{n+1} \rangle$ in $\mathcal{K}_{\stackrel{\sim}{\Rightarrow}}$.*

(iii) *There is a pseudo cone of adjunctions*

$$
(\Phi_n, \Gamma_n) : (\varphi_{n+1} \dashv \gamma_{n+1})(f_n \dashv g_n) \stackrel{\sim}{\Longrightarrow} (\varphi_n \dashv \gamma_n)
$$

such that its canonical local cones $\langle \varphi_n\gamma_n \rangle \stackrel{\cdot}{\Longrightarrow} 1_A$ and $\langle g_{\ell,n} f_{n,\ell} \rangle_\ell \stackrel{\cdot}{\Longrightarrow} \gamma_n\varphi_n$ are colimiting.

In the case of an ω-chain of coreflections, all the 2-cells in the canonical local cones $\langle g_{\ell,n} f_{n,\ell} \rangle_\ell \stackrel{\cdot}{\Longrightarrow} \gamma_n\varphi_n$ are invertible and we have the following simplified version of the theorem.

Corollary 1.6 *Let $\mathcal{K}$ be an ω**Cat**-bicategory.*

For an ω-chain of coreflections $\langle f_n \dashv g_n : A_{n+1} \to A_n \rangle$ and a pseudo cone $\langle \Phi_n : \varphi_{n+1} f_n \stackrel{\sim}{\Longrightarrow} \varphi_n : A_n \to A \rangle$ for the ω-chain $\langle f_n : A_n \to A_{n+1} \rangle$ in $\mathcal{K}$, the following are equivalent.

(i) *The pseudo cone $\langle \Phi_n : \varphi_{n+1} f_n \stackrel{\sim}{\Longrightarrow} \varphi_n : A_n \to A \rangle$ is a bicategorical colimit of the ω-chain $\langle f_n : A_n \to A_{n+1} \rangle$ in $\mathcal{K}$.*

(ii) *The pseudo cone $\langle \Phi_n : \varphi_{n+1} f_n \stackrel{\sim}{\Longrightarrow} \varphi_n : A_n \to A \rangle$ is a bicategorical colimit of the ω-chain $\langle f_n : A_n \to A_{n+1} \rangle$ in $\mathcal{K}_{\stackrel{\sim}{\Rightarrow}}$.*

(iii) *There is a pseudo cone of coreflections*

$$(\Phi_n, \Gamma_n) : (\varphi_{n+1} \dashv \gamma_{n+1})(f_n \dashv g_n) \stackrel{\sim}{\Longrightarrow} (\varphi_n \dashv \gamma_n)$$

such that its canonical local cone $\langle \varphi_n \gamma_n \rangle \stackrel{\cdot}{\Longrightarrow} 1_A$ is colimiting.

Clearly, the condition about the canonical local cone $\langle \varphi_n \gamma_n \rangle \stackrel{\cdot}{\Longrightarrow} 1_A$ being colimiting generalises the analogous condition of [23, Theorem 2] asserting that $\bigvee \varphi_n \gamma_n = 1_A$.

1.6 Limit/colimit coincidence

Theorem 1.5 and its dual with respect to the bicategorical limit of the ω^{op}-chain of right adjoints provide the following corollary about the coincidence of limits and colimits.

Corollary 1.7 *For a pseudo cone of adjunctions*

$$(\Phi_n, \Gamma_n) : (\varphi_{n+1} \dashv \gamma_{n+1})(f_n \dashv g_n) \stackrel{\sim}{\Longrightarrow} (\varphi_n \dashv \gamma_n)$$

*in an ω**Cat**-bicategory, where $\langle f_n \dashv g_n : A_{n+1} \to A_n \rangle$ is an ω-chain of adjunctions, the following are equivalent.*

(i) *The pseudo cone $\langle \Phi_n : \varphi_{n+1} f_n \stackrel{\sim}{\Longrightarrow} \varphi_n : A_n \to A \rangle$ is a bicategorical colimit of the ω-chain $\langle f_n : A_n \to A_{n+1} \rangle$.*

(ii) *The pseudo cocone $\langle \Gamma_n : g_n \gamma_{n+1} \stackrel{\sim}{\Longrightarrow} \gamma_n : A \to A_n \rangle$ is a bicategorical limit of the ω^{op}-chain $\langle g_n : A_{n+1} \to A_n \rangle$.*

Such coincidences of limits and colimits in bicategorical settings are part of the categorical folklore [24,29].

2 Algebraic completeness and compactness

Algebraic completeness and compactness are universal properties due to Freyd [12,13] that respectively provide canonical interpretations of inductive and recursive types. We consider them in the context of ω**Cat**-bicategories in Sections 2.2 and 2.3, where we generalise part of the definitions and results of [8, Chapter 7].

2.1 Initial algebras

We are interested in fixed points of pseudo functors up to equivalence and, more specifically, in initial such. Hence the following definition (*cf.* [4]).

Definition 2.1 A *pseudo initial algebra* for a pseudo endofunctor T on a bicategory is an algebra $a : TA \to A$ satisfying the following universal property.

(i) For every algebra $x : TX \to X$ there exists $(\mathrm{it}(x), \iota_x)$ as in the following diagram

$$\begin{tikzcd} TA \arrow[r, "a"] \arrow[d, "T(\mathrm{it}(x))"'] \arrow[dr, phantom, "\overset{\iota_x}{\cong}"] & A \arrow[d, "\mathrm{it}(x)"] \\ TX \arrow[r, "x"'] & X \end{tikzcd}$$

(ii) For every

$$\begin{tikzcd} TA \arrow[r, "a"] \arrow[d, "Tu"'] \arrow[dr, phantom, "\overset{\mu}{\cong}"] & A \arrow[d, "u"] \\ TX \arrow[r, "x"'] & X \end{tikzcd} \qquad \begin{tikzcd} TA \arrow[r, "a"] \arrow[d, "Tv"'] \arrow[dr, phantom, "\overset{\nu}{\cong}"] & A \arrow[d, "v"] \\ TX \arrow[r, "x"'] & X \end{tikzcd}$$

there exists a (necessarily invertible) unique 2-cell $\xi : u \Rightarrow v$ such that $(\xi a)\cdot\mu = \nu \cdot (x\, T\xi)$.

The following result generalises Lambek's lemma [16], *viz.* that initial algebras of endofunctors are isomorphisms, to the bicategorical setting.

Lemma 2.2 *Pseudo initial algebras of pseudo endofunctors are equivalences.*

Recall that an *equivalence* in a bicategory is an arrow with a pseudo inverse, where a *pseudo inverse* for an arrow $f : A \to B$ is an arrow $g : B \to A$ such that $gf \cong 1_A$ and $fg \cong 1_B$.

The 'basic lemma' below, generalising [23, Lemma 2], provides a tool for constructing pseudo initial algebras.

Definition 2.3 A *pseudo initial object* in a bicategory is an object 0 satisfying the following universal property.

(i) For every object X, there exists an arrow $0 \to X$.

(ii) For every pair of arrows $u, v : 0 \to X$, there exists a (necessarily invertible) unique 2-cell $u \Rightarrow v$.

Lemma 2.4 *Let $\mathcal{K}$ be a bicategory with pseudo initial object 0 and let T be a pseudo endofunctor on $\mathcal{K}$. For $\perp : 0 \to T0$ consider the ω-chain $\langle T^n\perp : T^n 0 \to T^{n+1}0\rangle$ and let $\langle \Phi_n : \varphi_{n+1} f_n \overset{\sim}{\Longrightarrow} \varphi_n : T^n 0 \to A\rangle$ be a bicategorical colimit for it.*

If

$$\left\langle \Phi'_n = T(\Phi_n)\cdot T_{f_n,\varphi_{n+1}} : T(\varphi_{n+1})T(f_n) \overset{\sim}{\Longrightarrow} T(\varphi_{n+1} f_n) \overset{\sim}{\Longrightarrow} T\varphi_n : T^{n+1}0 \to TA \right\rangle$$

is a bicategorical colimit for the ω-chain $\langle T^{n+1}\perp : T^{n+1}0 \to T^{n+2}0\rangle$ and $a : TA \to A$

mediates between the pseudo cones $\langle \Phi'_n \rangle$ and $\langle \Phi_{n+1} \rangle$, then a is a pseudo initial algebra.

2.2 Algebraic completeness

We consider algebraic completeness with respect to pseudo ω**Cat**-functors.

Definition 2.5 A *pseudo ω**Cat**-functor* $T : \mathcal{K} \to \mathcal{L}$ between ω**Cat**-bicategories is a pseudo functor such that, for every pair of objects A, B of $\mathcal{K}$, the functor $T_{A,B} : \mathcal{K}(A, B) \to \mathcal{L}(TA, TB)$ preserves colimits of ω-chains.

Definition 2.6 An ω**Cat**-bicategory is *pseudo ω**Cat**-algebraically complete* if every pseudo ω**Cat**-endofunctor on it has a pseudo initial algebra.

The concept of pseudo ω**Cat**-algebraic completeness is robust with respect to parameterisation, which is crucial for the interpretation of mutually-inductive types.

Theorem 2.7 *Let $\mathcal{K}$ and $\mathcal{L}$ be ω**Cat**-bicategories with $\mathcal{L}$ pseudo ω**Cat**-algebraically complete, and let $T : \mathcal{K} \times \mathcal{L} \to \mathcal{L}$ be a pseudo ω**Cat**-functor. For every object A in $\mathcal{K}$, let $\mu T(A)$ be the object underlying a chosen pseudo initial algebra for the pseudo functor $T(A, -) : \mathcal{L} \to \mathcal{L}$ defined by setting the first component of T to always be the object A, or the arrow 1_A, or the 2-cell 1_{1_A}. Then, the mapping $A \mapsto \mu T(A)$ canonically extends to a pseudo ω**Cat**-functor $\mu T : \mathcal{K} \to \mathcal{L}$.*

We have the following result analogous to Bekič's lemma.

Proposition 2.8 *Let $F : \mathcal{K} \times \mathcal{L} \to \mathcal{K}$ and $G : \mathcal{K} \times \mathcal{L} \to \mathcal{L}$ be pseudo ω**Cat**-functors with $\mathcal{K}$ and $\mathcal{L}$ pseudo ω**Cat**-algebraically complete.*

For every pseudo initial $F(-, \mu G(-))$-algebra $a : F(A, \mu G(A)) \to A$ and every pseudo initial $G(A, -)$-algebra $b : G(A, \mu G(A)) \to \mu G(A)$, the pair (a, b) is a pseudo initial $\langle F, G \rangle$-algebra.

Corollary 2.9 *If $\mathcal{K}$ and $\mathcal{L}$ are pseudo ω**Cat**-algebraically complete, then so is $\mathcal{K} \times \mathcal{L}$.*

The basic lemma (Lemma 2.4) motivates the following definition.

Definition 2.10 An ω**Cat**-bicategory is *pseudo ω**Cat**-algebraically ω-complete* if it has a pseudo initial object, say 0, and for every pseudo ω**Cat**-endofunctor T on it the ω-chain $\langle T^n\bot : T^n 0 \to T^{n+1} 0 \rangle$, where $\bot : 0 \to T0$, has a bicategorical colimit and T preserves it.

Clearly then, ω-completeness implies completeness. Further, as an application of the local-characterisation theorem (Theorem 1.5) and the basic lemma (Lemma 2.4) we have the following result.

Corollary 2.11 *For an ω**Cat**-bicategory $\mathcal{K}$ with a pseudo initial object 0 such that every arrow from it is a coreflection, the following are equivalent.*

(i) *$\mathcal{K}$ is pseudo ω**Cat**-algebraically complete.*

(ii) *For every pseudo* $\omega\mathbf{Cat}$*-endofunctor* T *on* $\mathcal{K}$*, the* ω*-chain* $\langle T^n\bot : T^n 0 \to T^{n+1}0\rangle$*, where* $\bot : 0 \to T0$*, has a bicategorical colimit.*

(iii) $\mathcal{K}$ *is pseudo* $\omega\mathbf{Cat}$*-algebraically* ω*-complete.*

2.3 Algebraic compactness

Algebraic compactness arises from the coincidence of initial algebras and final coalgebras.

A *mediating map* between an algebra $a : TA \to A$ and a coalgebra $b : B \to TB$ is an arrow $f : A \to B$ such that

$$\begin{tikzcd} TA \arrow[r, "a"] \arrow[d, "Tf"'] \arrow[dr, phantom, "\cong"] & A \arrow[d, "f"] \\ TB & B \arrow[l, "b"] \end{tikzcd}$$

If $a : TA \to A$ is pseudo initial and $b : B \to TB$ is pseudo final, then a mediating map between them always exists, and any two such are isomorphic.

Definition 2.12 An $\omega\mathbf{Cat}$-bicategory is *pseudo* $\omega\mathbf{Cat}$*-algebraically compact* if every pseudo $\omega\mathbf{Cat}$-endofunctor on it has a pseudo initial algebra and a pseudo final coalgebra, and the mediating maps between them are equivalences.

Proposition 2.13 *If* $\mathcal{K}$ *and* $\mathcal{L}$ *are pseudo* $\omega\mathbf{Cat}$*-algebraically compact, then so are* $\mathcal{K}^{\mathrm{op}}$ *and* $\mathcal{K} \times \mathcal{L}$*.*

It follows that, for $\mathcal{K}$ pseudo $\omega\mathbf{Cat}$-algebraically compact, every pseudo $\omega\mathbf{Cat}$-bifunctor $T : \mathcal{K}^{\mathrm{op}} \times \mathcal{K} \to \mathcal{K}$ has a *pseudo free dialgebra* $T(A, A) \simeq A$ characterised by the following universal property: for every $x' : X' \to T(X, X')$ and $x : T(X', X) \to X$, we have

$$\begin{tikzcd} X' \arrow[r, "x'"] \arrow[d, "\mathrm{coit}(x',x)"'] \arrow[dr, phantom, "\cong"] & T(X, X') \arrow[d, "T(\mathrm{it}(x',x),\mathrm{coit}(x',x))"] \\ A \arrow[r, "\simeq"] & T(A, A) \end{tikzcd} \qquad \begin{tikzcd} T(A, A) \arrow[r, "\simeq"] \arrow[d, "T(\mathrm{coit}(x',x),\mathrm{it}(x',x))"'] \arrow[dr, phantom, "\cong"] & A \arrow[d, "\mathrm{it}(x',x)"] \\ T(X', X) \arrow[r, "x"'] & X \end{tikzcd}$$

given uniquely up to canonical coherent isomorphism (as defined for pseudo initial algebras).

For the following class of $\omega\mathbf{Cat}$-bicategories, the notions of pseudo $\omega\mathbf{Cat}$-algebraic completeness and compactness coincide.

Definition 2.14 An $\omega\mathbf{Cat}_\bullet$*-bicategory* is an $\omega\mathbf{Cat}$-bicategory whose hom-categories have initial object and whose composition functors are initial-object preserving in each argument separately.

Proposition 2.15 *In an* $\omega\mathbf{Cat}_\bullet$*-bicategory, any pair of arrows* $f : 0 \rightleftarrows A : g$ *with* 0 *pseudo initial, forms a coreflection* $f \dashv g$*, and hence* 0 *is also pseudo terminal.*

Theorem 2.16 *Every pseudo $\omega\mathbf{Cat}$-algebraically complete $\omega\mathbf{Cat}_\bullet$-bicategory is pseudo $\omega\mathbf{Cat}$-algebraically compact.*

Finally, we identify a class of $\omega\mathbf{Cat}_\bullet$-bicategories for which pseudo $\omega\mathbf{Cat}$-algebraic compactness is guaranteed. These may be seen as enriched bicategorical analogues of ω-complete pointed partial orders.

Definition 2.17 A *Kbicat* is an $\omega\mathbf{Cat}_\bullet$-bicategory with pseudo initial object and bicategorical colimits of ω-chains of coreflections.

There are plenty of Kbicats. For example, the category of sets and partial functions, with hom-sets ordered by graph inclusion; the category of sets and relations, with hom-sets ordered by inclusion; the category of ω-cpos and partial ω-continuous functions, with hom-sets ordered pointwise; the bicategory of profunctors. Moreover, Kbicats are closed under duals and products.

Corollary 2.18 *Kbicats are pseudo $\omega\mathbf{Cat}$-algebraically compact.*

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A Proof of the local-characterisation theorem

This technical section proves the main result of the paper. In view of the fact that every $\omega\mathbf{Cat}$-bicategory is pseudo equivalent to an $\omega\mathbf{Cat}$-category, Theorem 1.5 and Corollary 1.6 need only be established for $\omega\mathbf{Cat}$-categories.

Theorem A.1 *Let $\mathcal{K}$ be an $\omega\mathbf{Cat}$-category.*

For an ω-chain of adjunctions $\langle \eta_n, \varepsilon_n : f_n \dashv g_n : A_{n+1} \to A_n \rangle$ and a pseudo cone $\langle \Phi_n : \varphi_{n+1} f_n \overset{\sim}{\Longrightarrow} \varphi_n : A_n \to A \rangle$ for the ω-chain $\langle f_n : A_n \to A_{n+1} \rangle$, the following are equivalent.

(i) *The pseudo cone $\langle \Phi_n : \varphi_{n+1} f_n \overset{\sim}{\Longrightarrow} \varphi_n : A_n \to A \rangle$ is a bicategorical colimit of the ω-chain $\langle f_n : A_n \to A_{n+1} \rangle$ in $\mathcal{K}$.*

(ii) *The pseudo cone $\langle \Phi_n : \varphi_{n+1} f_n \overset{\sim}{\Longrightarrow} \varphi_n : A_n \to A \rangle$ is a bicategorical colimit of the ω-chain $\langle f_n : A_n \to A_{n+1} \rangle$ in $\mathcal{K}_{\overset{\sim}{\Rightarrow}}$.*

(iii) *There is a pseudo cone of adjunctions*

$$(\Phi_n, \Gamma_n) : (\iota_{n+1}, \jmath_{n+1} : \varphi_{n+1} \dashv \gamma_{n+1})(\eta_n, \varepsilon_n : f_n \dashv g_n) \overset{\sim}{\Longrightarrow} (\iota_n, \jmath_n : \varphi_n \dashv \gamma_n)$$

such that its canonical local cones $\langle \varphi_n \gamma_n \rangle \overset{\cdot}{\Longrightarrow} 1_A$ and $\langle g_{\ell,n} f_{n,\ell} \rangle_\ell \overset{\cdot}{\Longrightarrow} \gamma_n \varphi_n$ are colimiting.

Proof. We prove the following chain of implications: $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (i)$. The first implication holds trivially because bicategorical colimits in $\mathcal{K}$ are bicategorical colimits in $\mathcal{K}_{\overset{\sim}{\Rightarrow}}$. As for the other two implications one argues as follows.

$(ii) \Rightarrow (iii)$: It is convenient to introduce the following definitions: for $m \geq n \in \omega$, let $f_{n,m} : A_n \to A_m$ be the arrow inductively defined by $f_{n,n} = 1_{A_n}$ and $f_{n,m+1} = f_m f_{n,m}$; analogously, let $g_{m,n} : A_m \to A_n$ be the arrow inductively defined by $g_{n,n} = 1_{A_n}$ and $g_{m+1,n} = g_{m,n} g_m$.

We start by presenting the necessary constructions leading to the definitions of $\gamma_n : A \to A_n$ and $\iota_n : 1_{A_n} \Rightarrow \gamma_n \varphi_n$ for each $n \in \omega$.

For $m \geq n \in \omega$, let the following cone

$$\mathcal{G}_{m,n} : \quad \begin{tikzcd} g_{m,n} \arrow[r, Rightarrow, "g_{m,n}\eta_m"] \arrow[dr, Rightarrow, "\iota_m^{m,n}"'] & g_{m+1,n}f_m \arrow[r, Rightarrow, "g_{m+1,n}\eta_{m+1}f_m"] \arrow[d, Rightarrow, "\iota_{m+1}^{m,n}"] & g_{m+2,n}f_{m,m+2} \arrow[r, Rightarrow] \arrow[dl, Rightarrow, "\iota_{m+2}^{m,n}"] & \cdots \\ & \overline{g_{m,n}} & \cdots & \end{tikzcd}$$

be colimiting in the hom-category $\mathcal{K}(A_m, A_n)$. Since by precomposing the cone $\mathcal{G}_{m+1,n}$ with f_m one obtains a colimiting cone for

$$g_{m+1,n}f_m \xRightarrow{g_{m+1,n}\eta_{m+1}f_m} g_{m+2,n}f_{m,m+2} \xRightarrow{g_{m+2,n}\eta_{m+2}f_{m,m+2}} \cdots$$

it follows that there exists a universal pseudo-cell $\varpi^f_{m,n} : \overline{g_{m,n}} \overset{\sim}{\Longrightarrow} \overline{g_{m+1,n}} f_m$ such that

$$\varpi^f_{m,n} \cdot \iota_k^{m,n} \;=\; \iota_k^{m+1,n} f_m \tag{A.1}$$

for all $k \geq m+1$. Similarly, by post composing $\mathcal{G}_{m,n+1}$ with g_n one derives the existence of a universal pseudo-cell $\varpi^g_{m,n} : \overline{g_{m,n}} \overset{\sim}{\Longrightarrow} g_n \overline{g_{m,n+1}}$, such that

$$\varpi^g_{m,n} \cdot \iota_k^{m,n} \;=\; g_n \iota_k^{m,n+1} \tag{A.2}$$

for all $k \geq m$. Consequently, let $\varpi_{m,n} : \overline{g_{m,n}} \overset{\sim}{\Longrightarrow} g_n \overline{g_{m+1,n+1}} f_m$ be the universal pseudo-cell such that

$$\varpi_{m,n} \cdot \iota_k^{m,n} \;=\; g_n \iota_k^{m+1,n+1} f_m$$

for all $k \geq m+1$. It follows that

$$\varpi_{m,n} \;=\; (\varpi^g_{m+1,n} f_m) \cdot \varpi^f_{m,n} \tag{A.3}$$

$$\;=\; (g_n \varpi^f_{m,n+1}) \cdot \varpi^g_{m,n} \,. \tag{A.4}$$

In particular, since $\iota_n^{n,n} = \iota_{n+1}^{n,n} \cdot \eta_n$, we have from (A.3), (A.1), and (A.2) that

$$\varpi_{n,n} \cdot \iota_n^{n,n} \;=\; (g_n \iota_{n+1}^{n+1,n+1} f_n) \cdot \eta_n \,. \tag{A.5}$$

Consider now the pseudo cone $\langle\, (\varpi^f_{m,n})^{-1} : \overline{g_{m+1,n}} f_m \overset{\sim}{\Longrightarrow} \overline{g_{m,n}} : A_m \to A_n \,\rangle_{m \geq n}$ for the ω-chain $\langle f_m : A_m \to A_{m+1} \rangle_{m \geq n}$. Since the pseudo cone $\langle \Phi_m : \varphi_{m+1} f_m \overset{\sim}{\Longrightarrow} \varphi_m : A_m \to A \rangle_{m \geq n}$ is a bicategorical colimit, there exists an arrow

$$\gamma_n : A \to A_n \tag{A.6}$$

and pseudo-cells $\varpi^n_m : \gamma_n\varphi_m \overset{\sim}{\Longrightarrow} \overline{g_{m,n}}$, for all $m \geq n$, such that

$$\varpi^n_m \cdot (\gamma_n \Phi_m) \;=\; (\varpi^f_{m,n})^{-1} \cdot (\varpi^n_{m+1} f_m) \quad . \tag{A.7}$$

Moreover, define

$$\iota_n \;=\; (\varpi^n_n)^{-1} \cdot \iota^{n,n}_n : 1_{A_n} \Rightarrow \gamma_n\varphi_n \quad . \tag{A.8}$$

We now proceed to define pseudo cells $\Gamma_n : g_n\gamma_{n+1} \overset{\sim}{\Longrightarrow} \gamma_n : A \to A_n$ for each $n \in \omega$. To this end, consider first the following commuting diagram with $m \geq n+1$

$$\begin{tikzcd}
g_n\gamma_{n+1}\varphi_{m+1}f_m \arrow[r, Rightarrow, "g_n\varpi^{n+1}_{m+1}f_m"] \arrow[d, Rightarrow, "g_n\gamma_{n+1}\Phi_m"'] & g_n\overline{g_{m+1,n+1}}f_m \arrow[r, Rightarrow, "(\varpi^g_{m+1,n})^{-1}f_m"] \arrow[d, Rightarrow, "g_n(\varpi^f_{m,n+1})^{-1}"'] \arrow[dr, Rightarrow, "(\varpi_{m,n})^{-1}"] & \overline{g_{m+1,n}}f_m \arrow[r, Rightarrow, "(\varpi^n_{m+1})^{-1}f_m"] \arrow[d, Rightarrow, "(\varpi^f_{m,n})^{-1}"] & \gamma_n\varphi_{m+1}f_m \arrow[d, Rightarrow, "\gamma_n\Phi_m"] \\
g_n\gamma_{n+1}\varphi_m \arrow[r, Rightarrow, "g_n\varpi^{n+1}_m"'] & g_n\overline{g_{m,n+1}} \arrow[r, Rightarrow, "(\varpi^g_{m,n})^{-1}"'] & \overline{g_{m,n}} \arrow[r, Rightarrow, "(\varpi^n_m)^{-1}"'] & \gamma_n\varphi_m
\end{tikzcd}$$

where the leftmost and rightmost squares commute by equation (A.7), whilst the central ones commute by equations (A.3) and (A.4). Furthermore, for $m \geq n+1$, let

$$\Upsilon^n_m \;=\; (\varpi^n_m)^{-1} \cdot (\varpi^g_{m,n})^{-1} \cdot (g_n\varpi^{n+1}_m) : g_n\gamma_{n+1}\varphi_m \overset{\sim}{\Longrightarrow} \gamma_n\varphi_m : A_m \to A_n$$

be the bottom pseudo cell in the above diagram, so that

$$\Upsilon^n_m \cdot (g_n\gamma_{n+1}\Phi_m) \;=\; (\gamma_n\Phi_m) \cdot (\Upsilon^n_{m+1} f_m) \tag{A.9}$$

for all $m \geq n+1$. Then, since the pseudo cone $\langle \Phi_m : \varphi_{m+1}f_m \overset{\sim}{\Longrightarrow} \varphi_m \rangle_{m \geq n+1}$ is a bicategorical colimit and the family $\langle \Upsilon^n_m : g_n\gamma_{n+1}\varphi_m \overset{\sim}{\Longrightarrow} \gamma_n\varphi_m : A_m \to A_n \rangle_{m \geq n+1}$ satisfies equation (A.9), it follows that there exists a unique pseudo cell

$$\Gamma_n : g_n\gamma_{n+1} \overset{\sim}{\Longrightarrow} \gamma_n : A \to A_n$$

such that $\Upsilon^n_m = \Gamma_n\varphi_m$ for all $m \geq n+1$.

We now establish the commutativity of diagram (1); *viz.*, that

$$\iota_n = (\Gamma_n\Phi_n) \cdot (g_n\iota_{n+1}f_n) \cdot \eta_n$$

for all $n \in \omega$. Aiming at this, observe first of all the following:

$$\begin{aligned}
\Gamma_n\Phi_n \;&=\; (\gamma_n\Phi_n) \cdot (\Gamma_n\varphi_{n+1}f_n) \\
&\qquad\qquad \text{(by the interchange law)} \\
&=\; (\gamma_n\Phi_n) \cdot (\Upsilon^n_{n+1}f_n) \\
&\qquad\qquad \text{(by the universal property of } \Gamma_n\text{)} \\
&=\; (\gamma_n\Phi_n) \cdot \big((\varpi^n_{n+1})^{-1}f_n\big) \cdot \big((\varpi^g_{n+1,n})^{-1}f_n\big) \cdot (g_n\varpi^{n+1}_{n+1}f_n) \\
&\qquad\qquad \text{(by definition of } \Upsilon^n_{n+1}\text{)} \\
&=\; (\gamma_n\Phi_n) \cdot (\gamma_n(\Phi_n)^{-1}) \cdot (\varpi^n_n)^{-1} \cdot (\varpi^f_{n,n})^{-1} \cdot \big((\varpi^g_{n+1,n})^{-1}f_n\big) \cdot (g_n\varpi^{n+1}_{n+1}f_n) \\
&\qquad\qquad \text{(by equation (A.7))} \\
&=\; (\varpi^n_n)^{-1} \cdot (\varpi_{n,n})^{-1} \cdot (g_n\varpi^{n+1}_{n+1}f_n) \qquad\qquad (\text{A.10}) \\
&\qquad\qquad \text{(by equation (A.3))}
\end{aligned}$$

Hence

$$
\begin{aligned}
\iota_n &= (\varpi_n^n)^{-1}\cdot\iota_n^{n,n} && \big(\text{by definition (A.8)}\big)\\
&= (\varpi_n^n)^{-1}\cdot(\varpi_{n,n})^{-1}\cdot(g_n\iota_{n+1}^{n+1,n+1}f_n)\cdot\eta_n && (\text{by equation (A.5)})\\
&= (\varpi_n^n)^{-1}\cdot(\varpi_{n,n})^{-1}\cdot(g_n\varpi_{n+1}^{n+1}f_n)\cdot(g_n\iota_{n+1}f_n)\cdot\eta_n && (\text{by definition of } \iota_{n+1})\\
&= (\Gamma_n\Phi_n)\cdot(g_n\iota_{n+1}f_n)\cdot\eta_n && (\text{by equation (A.10)})
\end{aligned}
$$

as required.

The fact that the canonical local cone

$$\langle g_{m,n}f_{n,m}\rangle_{m\geq n}\overset{\cdot}{\Longrightarrow}\gamma_n\varphi_n$$

is colimiting is an immediate consequence of the fact that it is obtained from the following pasting of diagrams:

$$
\begin{tikzcd}
1_{A_n} \arrow[r, Rightarrow, "\eta_n"] \arrow[d, Rightarrow, "\iota_n^{n,n}"'] & g_nf_n \arrow[r, Rightarrow, "g_n\eta_{n+1}f_n"] \arrow[d, Rightarrow, "g_n\iota_{n+1}^{n+1,n+1}f_n"] & g_{n+2,n}f_{n,n+2} \arrow[r, Rightarrow] \arrow[d, Rightarrow, "g_{n+2,n}\iota_{n+2}^{n+2.n+2}f_{n,n+2}"] & \cdots \\
\overline{g_{n,n}} \arrow[d, Rightarrow, "(\varpi_n^n)^{-1}"'] & g_n\overline{g_{n+1,n+1}}f_n \arrow[l, Rightarrow, "(\varpi_{n,n})^{-1}"'] \arrow[d, Rightarrow, "g_n(\varpi_{n+1}^{n+1})^{-1}f_n"] & g_{n+2,n}\overline{g_{n+2,n+2}}f_{n,n+2} \arrow[l, Rightarrow, "g_n(\varpi_{n+1,n+1})^{-1}f_n"'] \arrow[d, Rightarrow, "g_{n+2,n}(\varpi_{n+2,n+2})^{-1}f_{n,n+2}"] & \cdots \arrow[l, Rightarrow] \\
\gamma_n\varphi_n & g_n\gamma_{n+1}\varphi_{n+1}f_n \arrow[l, Rightarrow, "\Gamma_n\Phi_n"] & g_{n+2,n}\gamma_{n+2}\varphi_{n+2}f_{n,n+2} \arrow[l, Rightarrow, "g_n\Gamma_{n+1}\Phi_{n+1}f_n"] & \cdots \arrow[l, Rightarrow]
\end{tikzcd}
$$

where the upper one, because of equation (A.5), is the one that defines $\overline{g_{n,n}}$ as a colimit, whilst the lower one, which commutes by equation (A.10), consists of pseudo cells.

We now consider the other canonical local cone. Note that we can describe its corresponding chain in the hom-category $\mathcal{K}(A,A)$ as follows:

$$\big\langle\ \varphi_n\gamma_n \xRightarrow{(\Phi_n)^{-1}(\Gamma_n)^{-1}} \varphi_{n+1}f_ng_n\gamma_{n+1} \xRightarrow{\varphi_{n+1}\varepsilon_n\gamma_{n+1}} \varphi_{n+1}\gamma_{n+1}\ \big\rangle_n\ .$$

Let

$$\langle\alpha_n:\varphi_n\gamma_n\Rightarrow a\rangle \tag{A.11}$$

be a colimiting cone for this chain. We will now proceed to find a pseudo cell $\jmath : a \overset{\sim}{\Longrightarrow} 1_A$, so that setting

$$\langle\jmath_n=\jmath\cdot\alpha_n:\varphi_n\gamma_n\Rightarrow 1_A\rangle_n \tag{A.12}$$

we obtain a colimiting canonical local cone that will make diagram (2) commute by construction.

The hint for proving $a\cong 1_A$ comes from the following calculation with $k\in\omega$:

$$
\begin{align}
(\operatorname{colim}_{n\in\omega}\ \varphi_n\gamma_n)\varphi_k &\cong \operatorname{colim}_{n\in\omega}\ \varphi_n\gamma_n\varphi_k \notag\\
&\cong \operatorname{colim}_{n\geq k}\ \varphi_n\gamma_n\varphi_k \notag\\
&\cong \operatorname{colim}_{n\geq k}\ \varphi_n\gamma_n\varphi_n f_{k,n} \notag\\
&\cong \operatorname{colim}_{n\geq k}\ \varphi_n(\operatorname{colim}_{l\geq n}\ g_{l,n}f_{n,l})f_{k,n} \notag\\
&\cong \operatorname{colim}_{n\geq k}\ \operatorname{colim}_{l\geq n}\ \varphi_n g_{l,n}f_{n,l}f_{k,n} \tag{A.13}\\
&\cong \operatorname{colim}_{n\geq k}\ \varphi_n g_{n,n}f_{n,n}f_{k,n} \tag{A.14}\\
&\cong \operatorname{colim}_{n\geq k}\ \varphi_n f_{k,n} \notag\\
&\cong \operatorname{colim}_{n\geq k}\ \varphi_k \notag\\
&\cong \varphi_k \notag
\end{align}
$$

In fact we need to have a closer look at the matrix (A.13) and in particular at its diagonal (A.14); see Figure A.1 for a sketch of the matrix completed with the colimit arrows and their associated chain.

With the definitions of Figure A.1 one verifies that

$$
\begin{aligned}
x^{(k)}_{n+1,n}\cdot y^{(k)}_{n,n} &= (\Phi_n)^{-1}f_{k,n} && \text{for } n\geq k\\
x^{(k)}_{l+1,n}\cdot y^{(k)}_{l,n} &= y^{(k)}_{l,n+1}\cdot x^{(k)}_{l,n} && \text{for } l\geq k, n+1\\
(\varphi_{n+1}\gamma_{n+1}\Phi_{k,n+1})\cdot x^{(k)}_{\infty,n} &= x'^{(k)}_{\infty,n}\cdot(\varphi_n\gamma_n\Phi_{k,n}) && \text{for } n\geq k\\
(\varphi_{k+1}\gamma_{k+1}\Phi_k)\cdot x^{(k)}_{\infty,k} &= x'^{(k)}_{\infty,k}
\end{aligned}
$$

where the first identity follows from the triangular law $(\varepsilon_n f_n)\cdot(f_n\eta_n) = 1_{f_n}$, for $n\in\omega$. We therefore have, for every $k\in\omega$, a colimiting cone as follows

$$
\begin{tikzcd}
\varphi_k \arrow[r, Rightarrow, "(\Phi_k)^{-1}"] \arrow[d, Rightarrow, "\varphi_k\iota_k"'] & \varphi_{k+1}f_k \arrow[r, Rightarrow, "(\Phi_{k+1})^{-1}f_k"] \arrow[d, Rightarrow, "\varphi_{k+1}((\gamma_{k+1}\Phi_k)\cdot(\iota_{k+1}f_k))"] & \varphi_{k+2}f_{k,k+2} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \cdots \\
\varphi_k\gamma_k\varphi_k \arrow[r, Rightarrow, "x'^{(k)}_{\infty,k}"] \arrow[dr, Rightarrow, "\alpha_k\varphi_k"'] & \varphi_{k+1}\gamma_{k+1}\varphi_k \arrow[r, Rightarrow, "x'^{(k)}_{\infty,k+1}"] \arrow[d, Rightarrow, "\alpha_{k+1}\varphi_k"] & \varphi_{k+2}\gamma_{k+2}\varphi_k \arrow[r, Rightarrow] \arrow[dl, Rightarrow, "\alpha_{k+2}\varphi_k"] & \cdots \\
& a\varphi_k & \cdots &
\end{tikzcd}
\tag{A.15}
$$

induced by Figure A.1 and definition (A.11). Since the top chain consists of pseudo cells, it follows that the components of the cone consist of pseudo cells too. Hence, in particular, the 2-cells $(\alpha_k\varphi_k)\cdot(\varphi_k\iota_k) : \varphi_k \Rightarrow a\varphi_k$ are invertible for all $k\in\omega$. Furthermore, this family is subject to the following property

$$
\begin{aligned}
(\alpha_k\varphi_k)\cdot(\varphi_k\iota_k)\cdot\Phi_k &= (\alpha_{k+1}\varphi_k)\cdot(\varphi_{k+1}\gamma_{k+1}\Phi_k)\cdot(\varphi_{k+1}\iota_{k+1}f_k)\\
&\qquad\qquad (\text{see diagram (A.15)})\\
&= (a\Phi_k)\cdot(\alpha_{k+1}\varphi_{k+1}f_k)\cdot(\varphi_{k+1}\iota_{k+1}f_k)
\end{aligned}
$$

$$
\begin{tikzcd}
 & \varphi_{k+1}\gamma_{k+1}\varphi_k \arrow[r, Rightarrow, "x'^{(k)}_{\infty,k+1}"] & \varphi_{k+2}\gamma_{k+2}\varphi_k \arrow[r, dotted, no head] & \cdots \\
\varphi_k\gamma_k\varphi_k \arrow[ur, Rightarrow, "x'^{(k)}_{\infty,k}"] \arrow[r, Rightarrow, "x^{(k)}_{\infty,k}"'] \arrow[d, dotted, no head] & \varphi_{k+1}\gamma_{k+1}\varphi_{k+1}f_k \arrow[u, Rightarrow, "\varphi_{k+1}\gamma_{k+1}\Phi_k"'] \arrow[r, Rightarrow, "x^{(k)}_{\infty,k+1}"'] \arrow[d, dotted, no head] & \varphi_{k+2}\gamma_{k+2}\varphi_{k+2}f_{k,k+2} \arrow[u, Rightarrow, "\varphi_{k+2}\gamma_{k+2}\Phi_{k,k+2}"'] \arrow[r, dotted, no head] \arrow[d, dotted, no head] & \cdots \\
{} \arrow[r, dotted, no head] \arrow[d, dotted, no head] & {} \arrow[r, dotted, no head] \arrow[d, dotted, no head] & {} \arrow[r, dotted, no head] \arrow[d, dotted, no head] & \cdots \\
{} \arrow[r, dotted, no head] & {} \arrow[r, dotted, no head] & {} \arrow[r, dotted, no head] & \cdots \\
\varphi_k g_{k+2,k} f_{k,k+2} \arrow[u, Rightarrow, "y^{(k)}_{k+2,k}"] \arrow[r, Rightarrow, "x^{(k)}_{k+2,k}"] & \varphi_{k+1} g_{k+1} f_{k,k+2} \arrow[u, Rightarrow, "y^{(k)}_{k+2,k+1}"] \arrow[r, Rightarrow, "x^{(k)}_{k+2,k+1}"] & \varphi_{k+2} f_{k,k+2} \arrow[u, Rightarrow, "y^{(k)}_{k+2,k+2}"] \arrow[ur, dotted, no head] & \\
\varphi_k g_k f_k \arrow[u, Rightarrow, "y^{(k)}_{k+1,k}"] \arrow[r, Rightarrow, "x^{(k)}_{k+1,k}"] & \varphi_{k+1} f_k \arrow[u, Rightarrow, "y^{(k)}_{k+1,k+1}"] \arrow[ur, Rightarrow, "(\Phi_{k+1})^{-1} f_k"'] & & \\
\varphi_k \arrow[u, Rightarrow, "y^{(k)}_{k,k}"] \arrow[ur, Rightarrow, "(\Phi_k)^{-1}"'] & & &
\end{tikzcd}
$$

$$
\begin{aligned}
x^{(k)}_{l,n} &= \Big((\varphi_{n+1}\varepsilon_n) \cdot \big((\Phi_n)^{-1} g_n\big)\Big) g_{l,n+1} f_{k,l} && \text{for } l \geq k, n+1 \\
y^{(k)}_{l,n} &= \varphi_n g_{l,n} \eta_l f_{k,l} && \text{for } l \geq k, n \\
x^{(k)}_{\infty,n} &= \big((\varphi_{n+1}\varepsilon_n\gamma_{n+1}) \cdot ((\Phi_n)^{-1}(\Gamma_n)^{-1})\big)\big((\Phi_n)^{-1} f_{k,n}\big) && \text{for } n \geq k \\
x'^{(k)}_{\infty,n} &= \big((\varphi_{n+1}\varepsilon_n\gamma_{n+1}) \cdot ((\Phi_n)^{-1}(\Gamma_n)^{-1})\big)\varphi_k && \\
\Phi_{k,k} &= 1_{\varphi_k} && \\
\Phi_{k,n+1} &= \Phi_{k,n} \cdot (\Phi_n f_{k,n}) && \text{for } n \geq k
\end{aligned}
$$

Fig. A.1. Sketch of the matrix (A.13).

---

and, by the universal property of bicategorical colimits, there exists a unique pseudo cell

$$\jmath : a \overset{\sim}{\Longrightarrow} 1_A \tag{A.16}$$

such that

$$\jmath^{-1}\varphi_k \;=\; (\alpha_k\varphi_k) \cdot (\varphi_k\iota_k) \tag{A.17}$$

for all $k \in \omega$.

We finally prove that with the definitions (A.6), (A.8), (A.12), and (A.16), we obtain adjunctions

$$\iota_n, \jmath_n : \varphi_n \dashv \gamma_n \quad .$$

Expanding the definition of $\jmath_n$ and using equation (A.17), one obtains the triangular identity $(\jmath_n \varphi_n) \cdot (\varphi_n \iota_n) = 1_{\varphi_n}$. It follows from this that the composite $(\gamma_n \jmath_n) \cdot (\iota_n \gamma_n)$ is an idempotent. Thus, to deduce the other triangular identity, *viz.* $(\gamma_n \jmath_n) \cdot (\iota_n \gamma_n) = 1_{\gamma_n}$, we need only show that the composite $(\gamma_n \jmath_n) \cdot (\iota_n \gamma_n)$ is an isomorphism. To this end note that using a matrix analogous to that of Figure A.1 and that the cone (A.12) is colimiting, the following cone

$$\begin{tikzcd}
\gamma_n \arrow[rr, Rightarrow, "(\Gamma_n)^{-1}"] \arrow[d, Rightarrow, "\iota_n\gamma_n"'] & & g_n\gamma_{n+1} \arrow[rr, Rightarrow, "g_n(\Gamma_{n+1})^{-1}"] \arrow[d, Rightarrow, "((\Gamma_n\varphi_{n+1})\cdot(g_n\iota_{n+1}))\gamma_{n+1}"] & & g_{n+2,n}\gamma_{n+2} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \cdots \\
\gamma_n\varphi_n\gamma_n \arrow[rr, Rightarrow] \arrow[drr, Rightarrow, "\gamma_n\jmath_n"'] & & \gamma_n\varphi_{n+1}\gamma_{n+1} \arrow[rr, Rightarrow] \arrow[d, Rightarrow, "\gamma_n\jmath_{n+1}"] & & \gamma_n\varphi_{n+2}\gamma_{n+2} \arrow[r, Rightarrow] \arrow[dll, Rightarrow, "\gamma_n\jmath_{n+2}"] & \cdots \\
& & \gamma_n & & \ldots &
\end{tikzcd}$$

is colimiting too. Thus, as the top chain above consists of pseudo cells, the components of the cone are invertible.

(*iii*) $\Rightarrow$ (*i*): Recall that to prove that $\langle \Phi_n : \varphi_{n+1} f_n \xRightarrow{\sim} \varphi_n : A_n \to A \rangle$ is a bicategorical colimit of the ω-chain $\langle f_n : A_n \to A_{n+1} \rangle$ we need to show that the properties (i) and (ii) of Definition 1.3 hold.

Proof of property (i). To find a suitable $u : A \to X$ consider the chain

$$\langle\, \psi_n\gamma_n \xRightarrow{(\Psi_n)^{-1}(\Gamma_n)^{-1}} \psi_{n+1} f_n g_n \gamma_{n+1} \xRightarrow{\psi_{n+1}\varepsilon_n\gamma_{n+1}} \psi_{n+1}\gamma_{n+1} \,\rangle_n$$

and let $\langle \Upsilon_n : \psi_n\gamma_n \Rightarrow u \rangle_n$ be a colimiting cone for it. We need to describe a family $\langle \mu_k : u\varphi_k \Rightarrow \psi_k \rangle_k$ such that the following diagram commutes:

$$\begin{tikzcd}
u\varphi_{k+1} f_k \arrow[r, Rightarrow, "\mu_{k+1} f_k"] \arrow[d, Rightarrow, "u\Phi_k"'] & \psi_{k+1} f_k \arrow[d, Rightarrow, "\Psi_k"] \\
u\varphi_k \arrow[r, Rightarrow, "\mu_k"'] & \psi_k
\end{tikzcd} \tag{A.18}$$

For a fixed $k \in \omega$ observe, once again using a matrix analogous to that of Figure A.1 and that the cone $\langle \Upsilon_n \rangle_n$ is colimiting, that the following diagram

$$\begin{tikzcd}
\psi_k \arrow[rr, Rightarrow, "(\Psi_k)^{-1}"] \arrow[d, Rightarrow, "\psi_k\iota_k"'] & & \psi_{k+1} f_k \arrow[rr, Rightarrow, "(\Psi_{k+1})^{-1} f_k"] \arrow[d, Rightarrow, "\psi_{k+1}((\gamma_{k+1}\Phi_k)\cdot(\iota_{k+1} f_k))"] & & \psi_{k+2} f_{k,k+2} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \cdots \\
\psi_k\gamma_k\varphi_k \arrow[rr, Rightarrow] \arrow[drr, Rightarrow, "\Upsilon_k\varphi_k"'] & & \psi_{k+1}\gamma_{k+1}\varphi_k \arrow[rr, Rightarrow] \arrow[d, Rightarrow, "\Upsilon_{k+1}\varphi_k"] & & \psi_{k+2}\gamma_{k+2}\varphi_k \arrow[r, Rightarrow] \arrow[dll, Rightarrow, "\Upsilon_{k+2}\varphi_k"] & \cdots \\
& & u\varphi_k & & \ldots &
\end{tikzcd} \tag{A.19}$$

yields a colimiting cone for the chain $\langle (\Psi_n)^{-1} f_{k,n} : \psi_n f_{k,n} \overset{\sim}{\Longrightarrow} \psi_{n+1} f_{k,n+1} \rangle_{n \geq k}$. Since this chain consists of pseudo cells, then so does the cone. Thus, in particular, $(\Upsilon_k \varphi_k) \cdot (\psi_k \iota_k)$ is invertible. Letting $\mu_k : u\varphi_k \overset{\sim}{\Longrightarrow} \psi_k$ be its inverse, we have that (A.18) holds because

$$
\begin{aligned}
(\Upsilon_k \varphi_k) \cdot (\psi_k \iota_k) \cdot \Psi_k &= (\Upsilon_{k+1} \varphi_k) \cdot (\psi_{k+1} \gamma_{k+1} \Phi_k) \cdot (\psi_{k+1} \iota_{k+1} f_k) \\
&\qquad \big(\text{see diagram (A.19)}\big) \\
&= (u \Phi_k) \cdot (\Upsilon_{k+1} \varphi_{k+1} f_k) \cdot (\psi_{k+1} \iota_{k+1} f_k)
\end{aligned}
$$

Proof of property (ii). Consider arrows $u, v : A \to X$ and an ω-indexed family of 2-cells $\langle \Upsilon_n : u\varphi_n \Rightarrow v\varphi_n \rangle_n$, as in condition (ii) of Definition 1.3. We seek a 2-cell $\Upsilon : u \Rightarrow v$, such that $\Upsilon_n = \Upsilon \varphi_n$, for every $n \in \omega$.

Since $1_{A_n} = \operatorname{colim} \langle \varphi_n \gamma_n \rangle_n$, we have that $u = \operatorname{colim} \langle u \varphi_n \gamma_n \rangle_n$ and that $v = \operatorname{colim} \langle v \varphi_n \gamma_n \rangle_n$. From the properties of the family $\langle \Upsilon_n \rangle_n$, one can describe two colimiting cones together with a map between them:

$$
\begin{tikzcd}
& \vdots & \vdots & \\
& u\varphi_n\gamma_n \arrow[r, Rightarrow, "\Upsilon_n\gamma_n"] \arrow[d, Rightarrow] \arrow[dddl, Rightarrow, "u\jmath_n"'] & v\varphi_n\gamma_n \arrow[d, Rightarrow] \arrow[dddr, Rightarrow, "v\jmath_n"] & \\
& u\varphi_{n+1}\gamma_{n+1} \arrow[r, Rightarrow, "\Upsilon_{n+1}\gamma_{n+1}"] \arrow[ddl, Rightarrow, "u\jmath_{n+1}"] & v\varphi_{n+1}\gamma_{n+1} \arrow[ddr, Rightarrow, "v\jmath_{n+1}"'] & \\
& \vdots & \vdots & \\
u \arrow[rrr, Rightarrow, "\Upsilon"'] & & & v
\end{tikzcd}
$$

where $\Upsilon : u \Rightarrow v$ is the unique 2-cell such that

$$
\Upsilon \cdot (u\jmath_n) = (v\jmath_n) \cdot (\Upsilon_n \gamma_n) \tag{A.20}
$$

for all $n \in \omega$. We claim that Υ satisfies the required property. Indeed, since ι_n and $\jmath_n$ are respectively the unit and counit of the adjunction $\varphi_n \dashv \gamma_n$, we have that

$$
\begin{aligned}
\Upsilon \varphi_n &= (\Upsilon \varphi_n) \cdot u\big((\jmath_n \varphi_n) \cdot (\varphi_n \iota_n)\big) && \text{(by a triangular identity)} \\
&= \big(\Upsilon \cdot (u\jmath_n)\big)\varphi_n \cdot (u\varphi_n \iota_n) \\
&= \big((v\jmath_n) \cdot (\Upsilon_n \gamma_n)\big)\varphi_n \cdot (u\varphi_n \iota_n) && \big(\text{by equation (A.20)}\big) \\
&= (v\jmath_n \varphi_n) \cdot (v\varphi_n \iota_n) \cdot \Upsilon_n && \text{(by the interchange law)} \\
&= \Upsilon_n && \text{(by a triangular identity)}
\end{aligned}
$$

Moreover Υ is uniquely determined by the property $\Upsilon \varphi_n = \Upsilon_n$ for all $n \in \omega$. Indeed, this property implies

$$\Upsilon \cdot (u\jmath_n) \;=\; (v\jmath_n) \cdot (\Upsilon \varphi_n \gamma_n) \;=\; (v\jmath_n) \cdot (\Upsilon_n \gamma_n) \text{ for all } n \in \omega$$

and, by the universality of colimits, there exists a unique such Υ. □

For an ω-chain of coreflections $\langle f_n \dashv g_n \rangle$, all the 2-cells in the canonical cones $\langle g_{l,n} f_{n,l} \rangle_l \stackrel{\cdot}{\Longrightarrow} \gamma_n \varphi_n$ are pseudo-cells. Hence, the condition about these cones being colimiting is vacuous and we have the following simplified version of the theorem.

Corollary A.2 *In an ω**Cat**-category $\mathcal{K}$, for an ω-chain of coreflections $\langle f_n \dashv g_n : A_{n+1} \to A_n \rangle$ and a pseudo cone*

$$\langle \Phi_n : \varphi_{n+1} f_n \stackrel{\sim}{\Longrightarrow} \varphi_n : A_n \to A \rangle$$

for the ω-chain $\langle f_n : A_n \to A_{n+1} \rangle$, the following are equivalent.

(i) *$\langle \Phi_n : \varphi_{n+1} f_n \stackrel{\sim}{\Longrightarrow} \varphi_n : A_n \to A \rangle$ is a pseudo colimit for $\langle f_n : A_n \to A_{n+1} \rangle$ in $\mathcal{K}$.*

(ii) *$\langle \Phi_n : \varphi_{n+1} f_n \stackrel{\sim}{\Longrightarrow} \varphi_n : A_n \to A \rangle$ is a pseudo colimit for $\langle f_n : A_n \to A_{n+1} \rangle$ in $\mathcal{K}_{\stackrel{\sim}{\Rightarrow}}$.*

(iii) *There is a pseudo cone of coreflections*

$$(\Phi_n, \Gamma_n) : (\varphi_{n+1} \dashv \gamma_{n+1})(f_n \dashv g_n) \stackrel{\sim}{\Longrightarrow} (\varphi_n \dashv \gamma_n)$$

such that the canonical cone $\langle \varphi_n \gamma_n \rangle \stackrel{\cdot}{\Longrightarrow} 1_A$ is colimiting.

Proof. We just need to check that every ι_n as defined in (A.8) is a pseudo cell. Recall, then, that $\iota_n = (\varpi_n^n)^{-1} \cdot \iota_n^{n,n} : 1_{A_n} \Rightarrow \gamma_n \varphi_n$ where ϖ_n^n is a pseudo cell, whilst $\iota_n^{n,n}$ is the first component of the colimiting cone

$$\mathcal{G}_{n,n} : \quad \begin{tikzcd} g_{n,n} \arrow[r, Rightarrow, "g_{n,n}\eta_n"] \arrow[dr, Rightarrow, "\iota_n^{n,n}"'] & g_{n+1,n} f_n \arrow[r, Rightarrow, "g_{n+1,n}\eta_{n+1} f_n"] \arrow[d, Rightarrow, "\iota_{n+1}^{n,n}"] & g_{n+2,n} f_{n,n+2} \arrow[r, Rightarrow] \arrow[dl, Rightarrow, "\iota_{n+2}^{n,n}"] & \cdots \\ & \overline{g_{m,n}} & \cdots & \end{tikzcd}$$

But, as each η_n is a pseudo cell, the above local ω-chain consists of pseudo cells; and consequently so does its colimiting cone. □