A Very Short Note on the Principle of Mathematical Induction

Marcelo Fiore

Let P be a property of natural numbers. If $P(0)$ holds and if, for all natural numbers n, if $P(n)$ holds then so does $P(n+1)$, then the property P holds for all natural numbers.

Let P be a property of natural numbers. To prove that $P(n)$ is true for every natural number n by the *principle of mathematical induction* we proceed as follows.

Base case: Prove that $P(0)$ is true.

Inductive step: For all natural numbers n, prove that if $P(n)$ is true then $P(n+1)$ is also true.

Example. Prove that

$$\sum_{i=0}^{n} 2i+1 \quad = \quad (n+1)^2$$

for all natural numbers n.

Proof: We prove that the property

$$P(n): \quad \sum_{i=0}^{n} 2i+1 = (n+1)^2$$

holds for all natural numbers n by the principle of mathematical induction.

Base case: $P(0)$ is true because $\sum_{i=0}^{0} 2i+1 \ = \ 1 \ = \ (0+1)^2$.

Inductive step: For n an arbitrary natural number, we need to show that if $P(n)$ is true then so is $P(n+1)$.

So let n be an arbitrary natural number, and suppose that $P(n)$ holds; that is, we have that

$$\sum_{i=0}^{n} 2i+1 = (n+1)^2 \quad . \qquad \textbf{(IH)}$$

Then, $P(n+1)$ is also true, because

$$\begin{aligned} \sum_{i=0}^{n+1} 2i+1 &= \left(\sum_{i=0}^{n} 2i+1\right) + (2(n+1)+1) \\ &= (n+1)^2 + 2(n+1) + 1 \qquad \text{, by the Induction Hypothesis } \textbf{(IH)} \\ &= n^2 + 4n + 4 \\ &= ((n+1)+1)^2 \end{aligned}$$

The following piece of text does NOT constitute a proof:

$$\begin{aligned} \sum_{i=0}^{n+1} 2i+1 &= \left(\sum_{i=0}^{n} 2i+1\right) + (2(n+1)+1) \\ &= (n+1)^2 + 2(n+1) + 1 \\ &= ((n+1)+1)^2 \end{aligned}$$

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Principle of mathematical induction (PMI). For every property P of the natural numbers, if $P(0)$ holds and if, for all natural numbers n, if $P(n)$ holds then so does $P(n+1)$, then the property P holds for all natural numbers.

Generalised principle of mathematical induction (GPMI). For every property P of the natural numbers, if $P(0)$ holds and if, for all natural numbers n, we have that assuming that $P(0), P(1), \ldots, P(n)$ hold then also $P(n+1)$ holds, then the property P holds for all natural numbers.

Well-ordering principle (WOP). Every non-empty subset of the natural numbers has a least element.

NB. The above principles are equivalent.

1. **(PMI)** $\Rightarrow$ **(GPMI)**:

   We want to show that **(PMI)** implies **(GPMI)**. So assuming **(PMI)**, we have to show that, for P a property of the natural numbers, if

   $$P(0) \quad \text{and} \quad \forall\, n \in \mathbb{N}.\ \big(\forall\, 0 \le i \le n.\ P(i)\big) \Rightarrow P(n+1) \tag{$\dagger$}$$

   hold, then so does

   $$\forall\, n \in \mathbb{N}.\ P(n) \quad . \tag{$\ddagger$}$$

   So, letting P be a property of natural numbers we assume that ($\dagger$) holds and proceed to establish ($\ddagger$).

   Define Q to be the property of natural numbers given as follows:

   $$Q(n) : \quad \forall\, 0 \le \ell \le n.\ P(\ell) \quad .$$

   Observe that

   $$Q(0) \quad \text{and} \quad \forall\, n \in \mathbb{N}.\ Q(n) \Rightarrow Q(n+1)$$

   hold (make sure that you know why!). So, by **(PMI)** (which we have assumed —recall from above), applied to the property Q we have that

   $$\forall\, n \in \mathbb{N}.\ Q(n)$$

   holds. Finally, notice that this implies that ($\ddagger$) holds (why?). Hence the proof is finished.

2. **(GPMI)** $\Rightarrow$ **(PMI)**: Exercise.

3. **(PMI)** $\Leftrightarrow$ **(WOP)**: See lecture notes.