Generalised Species of Structures: Cartesian Closed and Differential Structure (Working Draft)

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Abstract

We generalise Joyal's notion of species of structures and develop their combinatorial calculus. In particular, we provide operations for their composition, addition, multiplication, pairing and projection, abstraction and evaluation, and differentiation; developing both the cartesian closed and linear structures of species.

Contents

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1 Categorical background

1.1 Monoidal categories

Monoidal categories. A *monoidal category* is a tuple $(\mathcal{C}, \otimes, I, a, l, r)$ where $\mathcal{C}$ is a category, $\_ \otimes =$ is a bifunctor $\mathcal{C} \times \mathcal{C} \longrightarrow \mathcal{C}$, I is an object of $\mathcal{C}$, and a, l, r are natural isomorphisms with components $a_{A,B,C} : (A \otimes B) \otimes C \longrightarrow A \otimes (B \otimes C)$, $l_A : I \otimes A \longrightarrow A$, $r_A : A \otimes I \longrightarrow A$ subject to coherence axioms [Kel82]. We have a *strict monoidal category* when these isomorphisms are identities. A *monoidal functor* $F : (\mathcal{C}, \otimes, I, a, l, r) \longrightarrow (\mathcal{C}', \otimes', I', a', l', r')$ is a functor $F : \mathcal{C} \longrightarrow \mathcal{C}'$ equipped with a morphism $I \longrightarrow F(I)$ and a natural transformation with components $F(A) \otimes' F(B) \longrightarrow F(A \otimes B)$ subject to coherence axioms [EK66, Law73]. We have a *strict monoidal functor* if these morphisms are identities.

A *symmetric (strict) monoidal category* is a (strict) monoidal category equipped with a natural isomorphism c, called the symmetry, with components $c_{A,B} : A \otimes B \longrightarrow B \otimes A$ satisfying further coherence axioms [Kel82]. A *symmetric (strict) monoidal functor* between symmetric (strict) monoidal categories is a (strict) monoidal functor that satisfies a further coherence axiom associated to the symmetries [EK66, Law73].

Free symmetric strict monoidal completion. We write $\mathcal{C}\boldsymbol{at}$ for the category of small categories and functors, and $\mathcal{SMC}\boldsymbol{at}$ for the category of small symmetric strict monoidal categories and strict monoidal functors. The forgetful functor $\mathcal{SMC}\boldsymbol{at} \longrightarrow \mathcal{C}\boldsymbol{at} : (\mathcal{C}, \otimes, I) \longmapsto \mathcal{C}$ has a left adjoint $!(\_) : \mathcal{C}\boldsymbol{at} \longrightarrow \mathcal{SMC}\boldsymbol{at}$ that maps a small category into its *symmetric strict monoidal completion*.

An explicit description of $!\mathbb{C}$ is given by the category with objects consisting of finite sequences $\langle\!| c_i |\!\rangle_{i=1,n}$ $(n \in \mathbb{N})$ of objects of $\mathbb{C}$ with $!\mathbb{C}\big[\langle\!| c_i |\!\rangle_{i=1,k}, \langle\!| d_j |\!\rangle_{j=1,\ell}\big] = \emptyset$ iff $k \neq \ell$ and morphisms $\langle\!| c_i |\!\rangle_{i=1,n} \longrightarrow \langle\!| c'_i |\!\rangle_{i=1,n}$ given by pairs $\big(\sigma, \langle\!| f_i |\!\rangle_{i=1,n}\big)$ consisting of a permutation $\sigma \in \mathfrak{S}_n$ and a sequence of maps $\langle\!| f_i : c_i \longrightarrow c'_{\sigma i} |\!\rangle_{i=1,n}$ in $\mathbb{C}$. (Composition is essentially given pointwise modulo permutation

$$\big(\sigma', \langle\!| f'_i |\!\rangle_{i=1,n}\big) \circ \big(\sigma, \langle\!| f_i |\!\rangle_{i=1,n}\big) = \big(\sigma' \circ \sigma, \langle\!| f'_{\sigma i} \circ f_i |\!\rangle_{i=1,n}\big)$$

and identities are given pointwise.) The symmetric strict monoidal structure of $!\mathbb{C}$ is given by concatenation with unit the empty sequence and the obvious symmetry.

We will use the following notational conventions. For $C \in !\mathbb{C}$, we write $c \in C$ to indicate that c is an index ranging through the length of C and we let $C_{@c} \in \mathbb{C}$ be the element of C at index c. Thus, we have that $C = \bigotimes_{c \in C} \langle\!| C_{@c} |\!\rangle$. Further, for $\gamma : C \longrightarrow C'$ in $!\mathbb{C}$ and $c \in C$, we write $\gamma c \in C'$ for the index associated to c by the permutation underlying γ and let $\gamma_{@c} : C_{@c} \longrightarrow C'_{@\gamma c}$ be the corresponding map in $\mathbb{C}$.

It is important to note that the symmetric strict monoidal completion comes equipped with canonical natural coherent equivalences as follows

$$\begin{array}{ccc} \mathbf{1} & \xrightarrow{\cong} & !\mathbf{0} \\ () & \longmapsto & \langle\!| \, |\!\rangle \end{array}$$

$$\begin{array}{ccc} !\mathbb{C}_1 \times !\mathbb{C}_2 & \xrightarrow[\simeq]{\oplus} & !(\mathbb{C}_1 + \mathbb{C}_2) \\ (C_1, C_2) & \longmapsto & !\amalg_1(C_1) \otimes !\amalg_2(C_2) \end{array} \tag{1}$$

For the purpose of the development below, we fix a quasi inverse to $\oplus$ according to the following notation

$$!(\mathbb{C}_1 + \mathbb{C}_2) \longrightarrow !\mathbb{C}_1 \times !\mathbb{C}_2 \ : \ C \longmapsto (C._1, C._2)$$

For instance, one could take $(\_).{}_i : !(\mathbb{C}_1 + \mathbb{C}_2) \longrightarrow !\mathbb{C}_i$ to be the free strict monoidal extension of the functor $\mathbb{C}_1 + \mathbb{C}_2 \longrightarrow !\mathbb{C}_i$ mapping $\amalg_i(c)$ to the singleton sequence $\langle\!\langle c \rangle\!\rangle$ and $\amalg_j(c)$ with $j \neq i$ to the empty sequence $\langle\!\langle\,\rangle\!\rangle$.

1.2 Bicategories

Bicategories. A *bicategory* $\mathcal{X}$ consists of the following data.

- A set $\mathcal{X}_0$ of 0-cells.
- A family of categories $\mathcal{X}[X, Y]$ for $X, Y \in \mathcal{X}_0$, with objects and arrows respectively called 1-cells and 2-cells.
- A composition law given by a family of functors $m_{X,Y,Z} : \mathcal{X}[Y, Z] \times \mathcal{X}[X, Y] \longrightarrow \mathcal{X}[X, Z]$ for $X, Y, Z \in \mathcal{X}_0$. (The action of m on a pair of 1-cells (g, f) is written $g \circ f$.)
- Units, given by 1-cells $1_X \in \mathcal{X}[X, X]$ for $X \in \mathcal{X}_0$.
- An associativity law given by a natural isomorphism with components $a_{h,g,f} : (h \circ g) \circ f \Longrightarrow h \circ (g \circ f)$ for $f \in \mathcal{X}[X, Y]$, $g \in \mathcal{X}[Y, Z]$, and $h \in \mathcal{X}[Z, W]$.
- Left and right unit laws respectively given by natural isomorphisms with components $l_f : 1_Y \circ f \Longrightarrow f$ and $r_f : f \circ 1_X \Longrightarrow f$ for $f \in \mathcal{X}[X, Y]$.

The associativity, and right and left unit laws are subject to coherence axioms [Bén67]. A 2-*category* is a bicategory in which the associativity, left and right unit laws are identities.

We recall the notion of morphism of bicategories. A *pseudo-functor* $F : \mathcal{X} \longrightarrow \mathcal{A}$ between bicategories consists of the following data.

- A function $F : \mathcal{X}_0 \longrightarrow \mathcal{A}_0$.
- A family of functors $F_{X,Y} : \mathcal{X}[X, Y] \longrightarrow \mathcal{A}[FX, FY]$ for $X, Y \in \mathcal{X}_0$.
- For all $X, Y, Z \in \mathcal{X}_0$, natural isomorphisms with components $\phi_{g,f} : F(g) \circ F(f) \Longrightarrow F(g \circ f)$ in $\mathcal{A}[FX, FZ]$ for $f \in \mathcal{X}[X, Y]$ and $g \in \mathcal{X}[Y, Z]$.
- For all $X \in \mathcal{X}_0$, an isomorphism $\phi_X : 1_{FX} \Longrightarrow F(1_X)$ in $\mathcal{A}[FX, FX]$.

These data are subject to coherence axioms [Bén67]. A pseudo-functor between 2-categories is said to be a 2-*functor* if its natural isomorphisms are identities.

Pseudo-adjoints. A *right pseudo-adjoint* to a pseudo-functor $F : \mathcal{X} \longrightarrow \mathcal{A}$ is given by the following data.

- A function $G : \mathcal{A}_0 \longrightarrow \mathcal{X}_0$.

- A family of 1-cells $\epsilon_A \in \mathcal{A}[FGA, A]$, for $A \in \mathcal{A}_0$, such that for all $X \in \mathcal{X}_0$ and $A \in \mathcal{X}_0$ the functor

$$\begin{array}{ccc} \mathcal{X}[X, GA] & \xrightarrow{E_{X,A}} & \mathcal{A}[FX, A] \\ f & \mapsto & \epsilon_A \circ F(f) \end{array} \tag{2}$$

is an equivalence of categories.

Hence, to give a right pseudo-adjoint to F we need provide functors $D_{X,A} : \mathcal{A}[FX, A] \longrightarrow \mathcal{X}[X, GA]$ together with natural isomorphisms

$$\mathrm{Id}_{\mathcal{X}[X,GA]} \Rightarrow D_{X,A}E_{X,A} \ , \quad E_{X,A}D_{X,A} \Rightarrow \mathrm{Id}_{\mathcal{A}[FX,A]}$$

These data canonically yield a pseudo-functor $G : \mathcal{A} \longrightarrow \mathcal{X}$. A *right 2-adjoint* to a 2-functor is determined by a right pseudo-adjoint for which the functors in (2) are isomorphisms.

1.3 Profunctors

Coends. $\int$ is the coend operation, whose definition and basic properties can be found in [Mac71, Chapter X].

Presheaves. For a small category $\mathbb{C}$, we write $\widehat{\mathbb{C}}$ for the functor category $[\mathbb{C}^\circ, \mathcal{S}\boldsymbol{et}]$ of *presheaves* on $\mathbb{C}$ and natural transformations, and let $y_{\mathbb{C}}$ denote the *Yoneda embedding* $\mathbb{C} \hookrightarrow \widehat{\mathbb{C}} : c \mapsto \mathbb{C}[\_, c]$.

Profunctors. For $(\mathbb{C}, \otimes, I)$ a (symmetric) monoidal category, the presheaf category $\widehat{\mathbb{C}}$ acquires a (symmetric) monoidal structure via *Day's tensor product* construction [Day70, IK86] given, for $X_1, X_2 \in \widehat{\mathbb{C}}$, as

$$X_1 \widehat{\otimes} X_2 \ = \ \int^{c_1, c_2 \in \mathbb{C}} X_1(c_1) \times X_2(c_2) \times y_{\mathbb{C}}(c_1 \otimes c_2)$$

The unit for Day's tensor product $\widehat{\otimes}$ is $y_{\mathbb{C}}(I)$.

For small categories $\mathbb{A}$ and $\mathbb{B}$, an $(\mathbb{A}, \mathbb{B})$-*profunctor*, indicated as $\mathbb{A} \nrightarrow \mathbb{B}$, is a functor $\mathbb{A} \longrightarrow \widehat{\mathbb{B}}$. Small categories, profunctors, and natural transformations between them form a bicategory [Bén00]. The profunctor composition $V \circ U : \mathbb{A} \nrightarrow \mathbb{C}$ of $U : \mathbb{A} \nrightarrow \mathbb{B}$ and $V : \mathbb{B} \nrightarrow \mathbb{C}$ is given by

$$(V \circ U)(a)(c) \ = \ \int^{b \in \mathbb{B}} V(b)(c) \times U(a)(b) \tag{3}$$

with identities $y_{\mathbb{C}} : \mathbb{C} \nrightarrow \mathbb{C}$.

We will use the following construction on profunctors. The *dual* of an $(\mathbb{A}, \mathbb{B})$-profunctor P is the $(\mathbb{B}^\circ, \mathbb{A}^\circ)$-profunctor $P^\perp$ given by

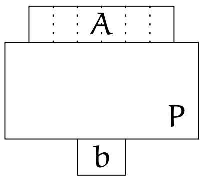
$$P^\perp(b)(a) \ = \ P(a)(b)$$

For more on the structure of the bicategory of profunctors see [Bén00, Law73, CW03].

2 The calculus of generalised species

For small categories $\mathbb{A}$ and $\mathbb{B}$, an $(\mathbb{A}, \mathbb{B})$-*species of structures* is a profunctor $!\mathbb{A} \nrightarrow \mathbb{B}$. In particular, $(\mathbb{A}, \mathbf{1})$-species are referred to as $\mathbb{A}$-species. The notation $\mathsf{P} : \mathbb{A} \;!\!\!\longrightarrow \mathbb{B}$ is used to indicate that P is an $(\mathbb{A}, \mathbb{B})$-species.

Structures in $\mathsf{P}(\mathsf{A})(\mathsf{b})$, for a species $\mathsf{P} : \mathbb{A} \;!\!\!\longrightarrow \mathbb{B}$, $\mathsf{A} \in\, !\mathbb{A}$, and $\mathsf{b} \in \mathbb{B}$, are pictorially represented as follows



Concrete examples of *combinatorial species* abound in the literature.

- Joyal's *species* [Joy81] are $\mathbf{1}$-species, and k-*sorted (or* k-*coloured) species* [Joy81, MN93, BLL98] are $\big(\sum_{i=1}^{k} \mathbf{1}\big)$-species.
- *Permutationals* [Joy81, Ber87] are **CP**-species for **CP** the groupoid of finite cyclic permutations.
- *Partitionals* [NR85] are $\mathbf{B}^*$-species for $\mathbf{B}^*$ the groupoid of non-empty finite sets.

Further examples that fit into generalised species are I-*permutationals* [MN93], and *species on graphs and digraphs* [Mén96].

Basic general examples of species follow.

- Presheaves on $\mathbb{C}$ are essentially species $\mathbf{0} \;!\!\!\longrightarrow \mathbb{C}$, whilst presheaves on $!\mathbb{C}$ also correspond to species $\mathbb{C} \;!\!\!\longrightarrow \mathbf{1}$.
- The Yoneda embedding $\mathsf{y}_{!\mathbb{C}}$ is a $\mathbb{C} \;!\!\!\longrightarrow\, !\mathbb{C}$ species.
- The species $\epsilon_{\mathbb{C}} : !\mathbb{C} \;!\!\!\longrightarrow \mathbb{C}$ is defined as $\epsilon_{\mathbb{C}}(\mathsf{C}) = \,!!\mathbb{C}\,\big[\, \langle\!|\, \langle\!|\_|\!\rangle \,|\!\rangle, \mathsf{C} \,\big]$.
- The species $\mathrm{S}_{\mathbb{C}} : \mathbb{C} \;!\!\!\longrightarrow \mathbb{C}$ is defined as

$$\mathrm{S}_{\mathbb{C}}(\mathsf{C}) \;=\; \sum_{\mathsf{c} \in \mathsf{C}} \mathsf{y}_{\mathbb{C}}(\mathsf{C}_{@\mathsf{c}}) \tag{4}$$

- The species $\mathrm{E}_{\mathbb{A},\mathbb{B}} : \mathbb{A} \;!\!\!\longrightarrow \mathbb{B}$ is defined by $\mathrm{E}_{\mathbb{A},\mathbb{B}}(\mathsf{A}) = 1$.

2.1 The bicategory of species

We introduce the bicategory $\boldsymbol{\mathcal{ES}}$ (*Espèces de Structures*) of generalised species of structures.

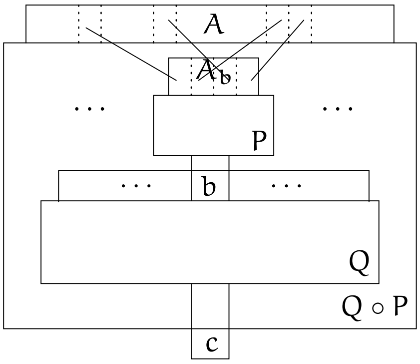
Composition. For species $P : \mathbb{A} \,!\!\!\longrightarrow \mathbb{B}$ and $Q : \mathbb{B} \,!\!\!\longrightarrow \mathbb{C}$, the *composition* $Q \circ P : \mathbb{A} \,!\!\!\longrightarrow \mathbb{C}$ is defined as

$$(Q \circ P)(A)(c) \;=\; \int^{B \in !\mathbb{B}} Q(B)(c) \times P^{\#}(A)(B) \qquad (A \in !\mathbb{A}, c \in \mathbb{C}^{\circ}) \tag{5}$$

where

$$P^{\#}(A)(B) \;=\; \int^{A_b \in !\mathbb{A} \ (b \in B)} \big(\textstyle\prod_{b \in B} P(A_b)(B_{@b})\big) \times !\mathbb{A}\left[\bigotimes_{b \in B} A_b, A\right] \qquad (A \in !\mathbb{A}, B \in !\mathbb{B}^{\circ})$$

One can visualise the structures in $(Q \circ P)(A)(c)$ as follows



Lemma 2.1 *For* $P : \mathbb{A} \,!\!\!\longrightarrow \mathbb{B}$*, we have that*

$$P^{\#}(A)\langle\!|b|\!\rangle \;\cong\; P(A)(b)$$

naturally in $A \in !\mathbb{A}$ *and* $b \in \mathbb{B}^{\circ}$*, and*

$$P^{\#}(A)(B_1 \otimes B_2) \;\cong\; \int^{A_1, A_2 \in !\mathbb{A}} P^{\#}(A_1)(B_1) \times P^{\#}(A_2)(B_2) \times !\mathbb{A}\,[A_1 \otimes A_2, A]$$

naturally in $A \in !\mathbb{A}$ *and* $B_1, B_2 \in !\mathbb{B}^{\circ}$*.*

PROOF:

$$P^{\#}(A)\langle\!|b|\!\rangle \;=\; \int^{A' \in !\mathbb{A}} P(A')(b) \times !\mathbb{A}\,[A', A] \;\cong\; P(A)(b)$$

$$\begin{aligned}
& P^{\#}(A)(B_1 \otimes B_2) \\
\cong\;& \int^{A_b \in !\mathbb{A} \ (b \in B_1 \otimes B_2)} \big(\textstyle\prod_{b \in B_1 \otimes B_2} P(A_b)((B_1 \otimes B_2)_{@b})\big) \times !\mathbb{A}\big[\bigotimes_{b \in B_1 \otimes B_2} A_b, A\big] \\
\cong\;& \int^{A_{b_1} \in !\mathbb{A} \ (b_1 \in B_1)} \int^{A_{b_2} \in !\mathbb{A} \ (b_2 \in B_2)} \big(\textstyle\prod_{b_1 \in B_1} P(A_{b_1})((B_1)_{@b_1})\big) \times \big(\textstyle\prod_{b_2 \in B_2} P(A_{b_2})((B_2)_{@b_2})\big) \\
& \qquad \times !\mathbb{A}\big[\big(\bigotimes_{b_1 \in B_1} A_{b_1}\big) \otimes \big(\bigotimes_{b_2 \in B_2} A_{b_2}\big), A\big] \\
\cong\;& \int^{A_1, A_2 \in !\mathbb{A}} \int^{A_{b_1} \in !\mathbb{A} \ (b_1 \in B_1)} \int^{A_{b_2} \in !\mathbb{A} \ (b_2 \in B_2)} \\
& \quad \big(\textstyle\prod_{b_1 \in B_1} P(A_{b_1})((B_1)_{@b_1})\big) \times \big(\textstyle\prod_{b_2 \in B_2} P(A_{b_2})((B_2)_{@b_2})\big) \\
& \quad \times !\mathbb{A}\big[A_1 \otimes A_2, A\big] \times !\mathbb{A}\big[\bigotimes_{b_1 \in B_1} A_{b_1}, A_1\big] \times !\mathbb{A}\big[\bigotimes_{b_2 \in B_2} A_{b_2}, A_2\big] \\
\cong\;& \int^{A_1, A_2 \in !\mathbb{A}} \int^{A_{b_1} \in !\mathbb{A} \ (b_1 \in B_1)} \big(\textstyle\prod_{b_1 \in B_1} P(A_{b_1})((B_1)_{@b_1})\big) \times !\mathbb{A}\big[\bigotimes_{b_1 \in B_1} A_{b_1}, A_1\big] \\
& \qquad \times \int^{A_{b_2} \in !\mathbb{A} \ (b_2 \in B_2)} \big(\textstyle\prod_{b_2 \in B_2} P(A_{b_2})((B_2)_{@b_2})\big) \times !\mathbb{A}\big[\bigotimes_{b_2 \in B_2} A_{b_2}, A_2\big] \\
& \qquad \times !\mathbb{A}\big[A_1 \otimes A_2, A\big] \\
=\;& \int^{A_1, A_2 \in !\mathbb{A}} P^{\#}(A_1)(B_1) \times P^{\#}(A_2)(B_2) \times !\mathbb{A}\,[A_1 \otimes A_2, A]
\end{aligned}$$

□

Abstractly, we have that $Q \circ P$ is the composite of profunctors

$$!\mathbb{A} \xrightarrow{P^{\#}} !\mathbb{B} \xrightarrow{Q} \mathbb{C}$$

where $(P^{\#})^{\perp} : !\mathbb{B}^{\circ} \longrightarrow \widehat{!\mathbb{A}^{\circ}}$ is the free monoidal extension of $P^{\perp} : \mathbb{B}^{\circ} \longrightarrow \widehat{!\mathbb{A}^{\circ}}$ induced by Day's tensor product on $\widehat{!\mathbb{A}^{\circ}}$.

Note also that a succint description of the coend defining composition can be given as follows

$$(Q \circ P)(A)(c) \cong \int^{F \in !(!\mathbb{A}^{\circ} \times \mathbb{B})} Q(!\pi_2 F)(c) \;\times\; \widehat{!\mathbb{A}^{\circ} \times \mathbb{B}}\left[SF, \overline{P}\right] \;\times\; !\mathbb{A}\left[\bigotimes(!\pi_1 F), A\right] \tag{6}$$

where S is as in (4), $\overline{P}(A, b) = P(A)(b)$, and $\bigotimes$ is the multiplication of !.

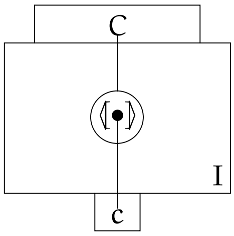
Example 2.2 *We give explicit descriptions of sample pre- and post-compositions with a species* $P : \mathbb{A} \mathrel{!\!\longrightarrow} \mathbb{B}$.

1. *For* $b \in \mathbb{B}$, *the composite species* $\mathbb{A} \xrightarrow{P}_{!} \mathbb{B} \xrightarrow{y_{!\mathbb{B}}\langle\!|b|\!\rangle}_{!} \mathbf{1}$ *is isomorphic to the species* $P_b : \mathbb{A} \mathrel{!\!\longrightarrow} \mathbf{1}$ *defined as*
$$P_b(A)() \;=\; P(A)(b) \tag{7}$$
Indeed,
$$\begin{aligned} (y\langle\!|b|\!\rangle \circ P)(A)() &= \int^{B \in !\mathbb{B}} !\mathbb{B}(B, \langle\!|b|\!\rangle) \times P^{\#}(A)(B) \\ &\cong \int^{b' \in \mathbb{B}} \mathbb{B}(b', b) \times P^{\#}(A)\langle\!|b'|\!\rangle \\ &\cong P^{\#}(A)\langle\!|b|\!\rangle \\ &= \int^{A' \in !\mathbb{A}} P(A')(b) \times !\mathbb{A}\left[A', A\right] \\ &\cong P(A)(b) \end{aligned}$$

2. *For* $X \in \widehat{\mathbb{A}}$, *the composite* $\mathbf{0} \xrightarrow{X}_{!} \mathbb{A} \xrightarrow{P}_{!} \mathbb{B}$, *is as follows*
$$(P \circ X)\langle\!|\,|\!\rangle(b) \cong \int^{A \in !\mathbb{A}} P(A)(b) \times \widehat{\mathbb{A}}\left[S_{\mathbb{A}}A, X\langle\!|\,|\!\rangle\right] \tag{8}$$
where $S_{\mathbb{A}} : !\mathbb{A} \longrightarrow \widehat{\mathbb{A}}$ *is as in (4). Indeed,*
$$\begin{aligned} (P \circ X)\langle\!|\,|\!\rangle(b) &= \int^{A \in !\mathbb{A}} P(A)(b) \times \int^{Z_a \in !\mathbf{0}} \left(\prod_{a \in A} X(Z_a)(A_{@a})\right) \times !\mathbf{0}\left[\bigotimes_{a \in A} Z_a, \langle\!|\,|\!\rangle\right] \\ &\cong \int^{A \in !\mathbb{A}} P(A)(b) \times \prod_{a \in A} X\langle\!|\,|\!\rangle(A_{@a}) \\ &\cong \int^{A \in !\mathbb{A}} P(A)(b) \times \prod_{a \in A} \widehat{\mathbb{A}}\left[y_{\mathbb{A}}(A_{@a}), X\langle\!|\,|\!\rangle\right] \\ &\cong \int^{A \in !\mathbb{A}} P(A)(b) \times \widehat{\mathbb{A}}\left[S_{\mathbb{A}}(A), X\langle\!|\,|\!\rangle\right] \end{aligned}$$

Identities. The *identity* species $I_{\mathbb{C}} : \mathbb{C} \mathrel{!\!\longrightarrow} \mathbb{C}$ is defined as

$$I_{\mathbb{C}}(C) \;=\; !\mathbb{C}\left[\langle\!|\_|\!\rangle, C\right] \tag{9}$$



Proposition 2.3 *For* $\mathsf{P} : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$, $\mathsf{Q} : \mathbb{B} \mathrel{!\!\!\longrightarrow} \mathbb{C}$, *and* $\mathsf{R} : \mathbb{C} \mathrel{!\!\!\longrightarrow} \mathbb{D}$, *we have canonical natural coherent isomorphisms as follow*

$$\boxed{\begin{array}{c} \mathrm{I}_{\mathbb{B}} \circ \mathsf{P} \cong \mathsf{P} \cong \mathsf{P} \circ \mathrm{I}_{\mathbb{A}} \\ (\mathsf{R} \circ \mathsf{Q}) \circ \mathsf{P} \cong \mathsf{R} \circ (\mathsf{Q} \circ \mathsf{P}) \end{array}} \tag{10}$$

establishing the unit laws of identities and the associativity of composition.

Proof:

$$\begin{aligned}
&(\mathrm{I}_{\mathbb{B}} \circ \mathsf{P})(A)(b) \\
&\quad = \int^{B \in !\mathbb{B}} !\mathbb{B}\left[\langle\!| b |\!\rangle, B\right] \times \int^{A_{b'} \in !\mathbb{A}\ (b' \in B)} \left(\prod\nolimits_{b' \in B} \mathsf{P}(A_{b'})(B_{@b'})\right) \times !\mathbb{A}\left[\bigotimes\nolimits_{b' \in B} A_{b'}, A\right] \\
&\quad \cong \int^{b' \in \mathbb{B}} \mathbb{B}\,[b, b'] \times \int^{A' \in !\mathbb{A}} \mathsf{P}(A')(b') \times \mathbb{A}\,[A', A] \\
&\quad \cong \int^{b' \in \mathbb{B}} \mathbb{B}\,[b, b'] \times \mathsf{P}(A)(b') \\
&\quad \cong \mathsf{P}(A)(b)
\end{aligned} \tag{11}$$

$$\begin{aligned}
&(\mathsf{P} \circ \mathrm{I}_{\mathbb{A}})(A)(b) \\
&\quad = \int^{A' \in !\mathbb{A}} \mathsf{P}(A')(b) \times \int^{X_a \in !\mathbb{A}\ (a \in A')} !\mathbb{A}\left[\langle\!| A'_{@a} |\!\rangle, X_a\right] \times !\mathbb{A}\left[\bigotimes\nolimits_{a \in A'} X_a, A\right] \\
&\quad \cong \int^{A' \in !\mathbb{A}} \mathsf{P}(A')(b) \times !\mathbb{A}\left[\bigotimes\nolimits_{a \in A'} \langle\!| A'_{@a} |\!\rangle, A\right] \\
&\quad \cong \mathsf{P}(A)(b)
\end{aligned} \tag{12}$$

$$\begin{aligned}
&((\mathsf{R} \circ \mathsf{Q}) \circ \mathsf{P})(A)(d) \\
&= \int^{B \in !\mathbb{B}} (\mathsf{R} \circ \mathsf{Q})(B)(d) \times \int^{A_b \in !\mathbb{A}\ (b \in B)} \left(\prod\nolimits_{b \in B} \mathsf{P}(A_b)(B_{@b})\right) \times !\mathbb{A}\left[\bigotimes\nolimits_{b \in B} A_b, A\right] \\
&= \int^{B \in !\mathbb{B}} \left(\int^{C \in !\mathbb{C}} \mathsf{R}(C)(d) \times \int^{B_c \in !\mathbb{B}\ (c \in C)} \left(\prod\nolimits_{c \in C} \mathsf{Q}(B_c)(C_{@c})\right) \times !\mathbb{B}\left[\bigotimes\nolimits_{c \in C} B_c, B\right]\right) \\
&\qquad \times \int^{A_b \in !\mathbb{A}\ (b \in B)} \left(\prod\nolimits_{b \in B} \mathsf{P}(A_b)(B_{@b})\right) \times !\mathbb{A}\left[\bigotimes\nolimits_{b \in B} A_b, A\right] \\
&\cong \int^{C \in !\mathbb{C}} \mathsf{R}(C)(d) \times \int^{B_c \in !\mathbb{B}\ (c \in C)} \left(\prod\nolimits_{c \in C} \mathsf{Q}(B_c)(C_{@c})\right) \\
&\quad \times \int^{A_b \in !\mathbb{A}\ (b \in \bigotimes_{c \in C} B_c)} \left(\prod\nolimits_{b \in (\bigotimes_{c \in C} B_c)} \mathsf{P}(A_b)\big((\bigotimes\nolimits_{c \in C} B_c)_{@b}\big)\right) \times !\mathbb{A}\left[\bigotimes\nolimits_{b \in (\bigotimes_{c \in C} B_c)} A_b, A\right] \\
&\cong \int^{C \in !\mathbb{C}} \mathsf{R}(C)(d) \times \int^{B_c \in !\mathbb{B}\ (c \in C)} \left(\prod\nolimits_{c \in C} \mathsf{Q}(B_c)(C_{@c})\right) \\
&\quad \times \int^{A_{b,c} \in !\mathbb{A}\ (c \in C, b \in B_c)} \left(\prod\nolimits_{c \in C, b \in B_c} \mathsf{P}(A_{b,c})\big((B_c)_{@b}\big)\right) \times !\mathbb{A}\left[\bigotimes\nolimits_{c \in C} \bigotimes\nolimits_{b \in B_c} A_{b,c}, A\right]
\end{aligned} \tag{13}$$

$$
\begin{aligned}
&(R \circ (Q \circ P))(A)(d) \\
&= \int^{C \in !\mathbb{B}} R(C)(d) \times \int^{A_c \in !\mathbb{A}\ (c \in C)} \Big(\prod_{c \in C} (Q \circ P)(A_c)(C_{@c}) \Big) \times !\mathbb{A}\left[\bigotimes_{c \in C} A_c, A\right] \\
&= \int^{C \in !\mathbb{B}} R(C)(d) \\
&\qquad \times \int^{A_c \in !\mathbb{A}\ (c \in C)} \Big(\prod_{c \in C} \int^{B \in !\mathbb{B}} Q(B)(C_{@c}) \\
&\qquad\qquad \times \int^{X_b \in !\mathbb{A}\ (b \in B)} \Big(\prod_{b \in B} P(X_b)(B_{@b}) \Big) \\
&\qquad\qquad\qquad \times !\mathbb{A}\left[\bigotimes_{b \in B} X_b, A_c\right] \Big) \\
&\qquad\qquad \times !\mathbb{A}\left[\bigotimes_{c \in C} A_c, A\right] \\
&\cong \int^{C \in !\mathbb{B}} R(C)(d) \\
&\qquad \times \int^{A_c \in !\mathbb{A}\ (c \in C)} \Big(\int^{B_c \in !\mathbb{B}\ (c \in C)} \Big(\prod_{c \in C} Q(B_c)(C_{@c}) \Big) \\
&\qquad\qquad \times \Big(\prod_{c \in C} \int^{X_b \in !\mathbb{A}\ (b \in B_c)} \Big(\prod_{b \in B_c} P(X_b)((B_c)_{@b}) \Big) \\
&\qquad\qquad\qquad \times !\mathbb{A}\left[\bigotimes_{b \in B_c} X_b, A_c\right] \Big) \Big) \\
&\qquad\qquad \times !\mathbb{A}\left[\bigotimes_{c \in C} A_c, A\right] \\
&\cong \int^{C \in !\mathbb{B}} R(C)(d) \\
&\qquad \times \int^{A_c \in !\mathbb{A}\ (c \in C)} \Big(\int^{B_c \in !\mathbb{B}\ (c \in C)} \Big(\prod_{c \in C} Q(B_c)(C_{@c}) \Big) \\
&\qquad\qquad \times \Big(\int^{X_{b,c} \in !\mathbb{A}\ (c \in C, b \in B_c)} \Big(\prod_{c \in C} \prod_{b \in B_c} P(X_{b,c})((B_c)_{@b}) \Big) \\
&\qquad\qquad\qquad \times \prod_{c \in C} !\mathbb{A}\left[\bigotimes_{b \in B_c} X_{b,c}, A_c\right] \Big) \Big) \\
&\qquad\qquad \times !\mathbb{A}\left[\bigotimes_{c \in C} A_c, A\right] \\
&\cong \int^{C \in !\mathbb{B}} R(C)(d) \\
&\qquad \times \int^{B_c \in !\mathbb{B}\ (c \in C)} \Big(\prod_{c \in C} Q(B_c)(C_{@c}) \Big) \\
&\qquad\qquad \times \int^{X_{b,c} \in !\mathbb{A}\ (c \in C, b \in B_c)} \Big(\prod_{c \in C} \prod_{b \in B_c} P(X_{b,c})((B_c)_{@b}) \Big) \\
&\qquad\qquad\qquad \times !\mathbb{A}\left[\bigotimes_{c \in C} \bigotimes_{b \in B_c} X_{b,c}, A\right]
\end{aligned} \tag{14}
$$

□

Theorem 2.4 *Small categories, species, and natural transformations form a bicategory.*

PROOF: The composition and identities, the left and right unit laws, and the associativity law are as in (5), (9), and (10) respectively. See Appendix A for details. □

2.2 Addition and multiplication

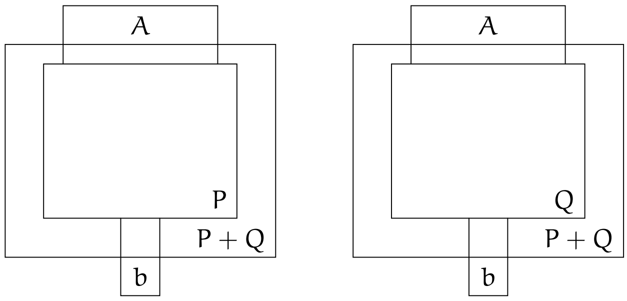
Each hom-category $\mathcal{ES}[\mathbb{A}, \mathbb{B}]$ acquires a commutative rig structure given by the addition and multiplication of species.

Addition. For $P, Q : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$, the *addition* $P + Q : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ is defined by

$$(P + Q)(A)(b) \;=\; P(A)(b) + Q(A)(b) \qquad (A \in !\mathbb{A}, b \in \mathbb{B}^\circ)$$

That is,

$$P + Q \;=\; \big(!\mathbb{A} \xrightarrow{\Delta_{!\mathbb{A}}} !\mathbb{A} \times !\mathbb{A} \xrightarrow{P \times Q} \widehat{\mathbb{B}} \times \widehat{\mathbb{B}} \xrightarrow{+} \widehat{\mathbb{B}}\big)$$



More generally, for $X_i \in \widehat{\mathbb{B}}$ and $P_i : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ $(i \in I)$, the *linear combination* $\sum_{i \in I} X_i P_i : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ is defined by

$$\Big(\sum_{i \in I} X_i P_i\Big)(A)(b) \;=\; \sum_{i \in I} X_i(b) \times P_i(A)(b)$$

Addition together with the species $\underline{0} : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ defined as

$$\underline{0}(A)(b) \;=\; \emptyset \qquad (A \in !\mathbb{A}, b \in \mathbb{B}^\circ)$$

satisfy commutative monoid laws.

Proposition 2.5 *For* $P, Q, R : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$*, we have*

$$\boxed{\begin{array}{c} (P + Q) + R \cong P + (Q + R) \\ P + \underline{0} \cong P \qquad\qquad P + Q \cong Q + P \end{array}}$$

Further, for $P, Q : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ *and* $R : \mathbb{C} \mathrel{!\!\!\longrightarrow} \mathbb{A}$*, we have*

$$\boxed{(P + Q) \circ R \cong (P \circ R) + (Q \circ R)}$$

PROOF: We only show that $(P + Q) \circ R \cong (P \circ R) + (Q \circ R)$.

$$\begin{aligned}
\big((P + Q) \circ R\big)(C)(b) &\cong \int^{A \in !\mathbb{A}} \big(P(A)(b) + Q(A)(b)\big) \times R^{\#}(C)(A) \\
&\cong \int^{A \in !\mathbb{A}} \big(P(A)(b) \times R^{\#}(C)(A)\big) + \big(Q(A)(b) \times R^{\#}(C)(A)\big) \\
&\cong \Big(\int^{A \in !\mathbb{A}} P(A)(b) \times R^{\#}(C)(A)\Big) + \Big(\int^{A \in !\mathbb{A}} Q(A)(b) \times R^{\#}(C)(A)\Big) \\
&= \big((P \circ R) + (Q \circ R)\big)(C)(b)
\end{aligned}$$

□

Multiplication. For $P, Q : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$, the *multiplication* $P \cdot Q : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ is defined by

$$(P \cdot Q)(A)(b) \;=\; \int^{A_1, A_2 \in !\mathbb{A}} P(A_1)(b) \times Q(A_2)(b) \times !\mathbb{A}(A_1 \otimes A_2, A) \tag{15}$$

That is, using (7),

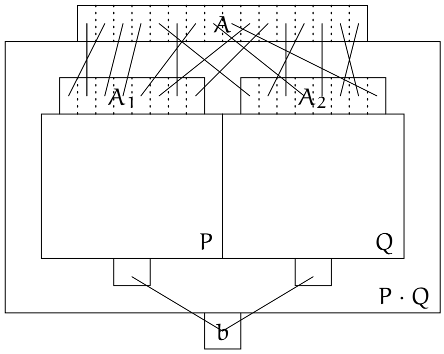
$$(P \cdot Q)_b = P_b \widehat{\otimes} Q_b$$

for all $b \in \mathbb{B}^\circ$.

Abstractly, $(P \cdot Q)^\perp$ is the composite of functors

$$\mathbb{B}^\circ \xrightarrow{\langle P^\perp, Q^\perp \rangle} \widehat{!\mathbb{A}^\circ} \times \widehat{!\mathbb{A}^\circ} \xrightarrow{\widehat{\otimes}} \widehat{!\mathbb{A}^\circ}$$

where $\widehat{\otimes}$ is Day's tensor product.



Multiplication together with the species $\underline{1} : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ defined by

$$\underline{1}(A)(b) \;=\; !\mathbb{A}\big[\langle\!|\,|\!\rangle, A\big]$$

satisfy commutative monoid and distributive laws.

Proposition 2.6 *For* $P, Q, R : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$, *we have*

$$\boxed{\begin{array}{c} (P \cdot Q) \cdot R \cong P \cdot (Q \cdot R) \\ P \cdot \underline{1} \cong P \qquad\qquad P \cdot Q \cong Q \cdot P \\ P \cdot \underline{0} \cong \underline{0} \qquad P \cdot (Q + R) \cong (P \cdot Q) + (P \cdot R) \end{array}}$$

Further, for $P, Q : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ *and* $R : \mathbb{C} \mathrel{!\!\!\longrightarrow} \mathbb{A}$, *we have*

$$\boxed{(P \cdot Q) \circ R \cong (P \circ R) \cdot (Q \circ R)}$$

PROOF:

- Note that

$$\begin{aligned}
&\big((P \cdot Q) \cdot R\big)(A)(b) \\
&= \int^{A_1, A_2 \in !\mathbb{A}} (P \cdot Q)(A_1)(b) \times R(A_2)(b) \times !\mathbb{A}\,[A_1 \otimes A_2, A] \\
&= \int^{A_1, A_2 \in !\mathbb{A}} \Big(\int^{A_3, A_4 \in !\mathbb{A}} P(A_3)(b) \times Q(A_4)(b) \times !\mathbb{A}\,[A_3 \otimes A_4, A_1] \Big) \\
&\qquad \times R(A_2)(b) \times !\mathbb{A}\,[A_1 \otimes A_2, A] \\
&\cong \int^{A_2, A_3, A_4 \in !\mathbb{A}} P(A_3)(b) \times Q(A_4)(b) \times R(A_2)(b) \times !\mathbb{A}\,[A_3 \otimes A_4 \otimes A_2, A]
\end{aligned}$$

and that

$$
\begin{aligned}
&\big(P\cdot(Q\cdot R)\big)(A)(b)\\
&\quad= \int^{A_1,A_2\in !\mathbb{A}} P(A_1)(b)\times(Q\cdot R)(A_2)(b)\times !\mathbb{A}\,[A_1\otimes A_2, A]\\
&\quad= \int^{A_1,A_2\in !\mathbb{A}} \; P(A_1)(b)\times\Big(\int^{A_3,A_4\in !\mathbb{A}} Q(A_3)(b)\times R(A_4)(b)\times !\mathbb{A}\,[A_3\otimes A_4, A_2]\Big)\\
&\qquad\qquad \times !\mathbb{A}\,[A_1\otimes A_2, A]\\
&\quad\cong \int^{A_1,A_3,A_4\in !\mathbb{A}} P(A_1)(b)\times Q(A_3)(b)\times R(A_4)(b)\times !\mathbb{A}\,[A_1\otimes A_3\otimes A_4, A]
\end{aligned}
$$

- $$
\begin{aligned}
(P\cdot\underline{1})(A)(b) &= \int^{A_1,A_2\in !\mathbb{A}} P(A_1)(b)\times !\mathbb{A}\,\big[\langle\!|\,|\!\rangle, A_2\big]\times !\mathbb{A}\,[A_1\otimes A_2, A]\\
&\cong \int^{A_1\in !\mathbb{A}} P(A_1)(b)\times !\mathbb{A}\,\big[A_1\otimes\langle\!|\,|\!\rangle, A\big]\\
&\cong P(A)(b)
\end{aligned}
$$

- $$
(P\cdot\underline{0})(A)(b) \;=\; \int^{A_1,A_2\in !\mathbb{A}} P(A_1)(b)\times\underline{0}(A_2)(b)\times !\mathbb{A}\,[A_1\otimes A_2, A] \;=\; \emptyset
$$

- $$
\begin{aligned}
\big(P\cdot(Q+R)\big)(A)(b) &= \int^{A_1,A_2\in !\mathbb{A}} P(A_1)(b)\times(Q+R)(A_2)(b)\times !\mathbb{A}\,[A_1\otimes A_2, A]\\
&\cong \int^{A_1,A_2\in !\mathbb{A}} \; \big(P(A_1)(b)\times Q(A_2)(b)\times !\mathbb{A}\,[A_1\otimes A_2, A]\,\big)\\
&\qquad\qquad + \big(P(A_1)(b)\times R(A_2)(b)\times !\mathbb{A}\,[A_1\otimes A_2, A]\,\big)\\
&\cong \big(\;\int^{A_1,A_2\in !\mathbb{A}} P(A_1)(b)\times Q(A_2)(b)\times !\mathbb{A}\,[A_1\otimes A_2, A]\,\big)\\
&\qquad + \big(\int^{A_1,A_2\in !\mathbb{A}} P(A_1)(b)\times R(A_2)(b)\times !\mathbb{A}\,[A_1\otimes A_2, A]\,\big)\\
&= \big((P\cdot Q)+(P\cdot R)\big)(A)(b)
\end{aligned}
$$

- $$
\begin{aligned}
&\big((P\cdot Q)\circ R\big)(C)(b)\\
&\quad= \int^{A\in !\mathbb{A}}(P\cdot Q)(A)(b)\times R^{\#}(C)(A)\\
&\quad= \int^{A\in !\mathbb{A}}\big(\int^{A_1,A_2\in !\mathbb{A}} P(A_1)(b)\times Q(A_2)(b)\times !\mathbb{A}\,[A_1\otimes A_2, A]\,\big)\times R^{\#}(C)(A)\\
&\quad\cong \int^{A_1,A_2\in !\mathbb{A}} P(A_1)(b)\times Q(A_2)(b)\times R^{\#}(C)(A_1\otimes A_2)\\
&\quad\cong \int^{A_1,A_2\in !\mathbb{A}} \; P(A_1)(b)\times Q(A_2)(b)\\
&\qquad\qquad \times \int^{C_1,C_2\in !\mathbb{C}} R^{\#}(C_1)(A_1)\times R^{\#}(C_2)(A_2)\times !\mathbb{C}\,[C_1\otimes C_2, C]\\
&\qquad\qquad\quad \text{, by Lemma 2.1}\\
&\quad\cong \int^{C_1,C_2\in !\mathbb{C}} \; \big(\int^{A_1\in !\mathbb{A}} P(A_1)(b)\times R^{\#}(C_1)(A_1)\big)\\
&\qquad\qquad \times\big(\int^{A_2\in !\mathbb{A}} Q(A_2)(b)\times R^{\#}(C_2)(A_2)\big)\\
&\qquad\qquad \times !\mathbb{C}\,[C_1\otimes C_2, C]\\
&\quad= \big((P\circ R)\cdot(Q\circ R)\big)(C)(b)
\end{aligned}
$$

□

For $P : \mathbb{A} \,!\!\!\longrightarrow \mathbb{B}$, we have $P^\# : \mathbb{A} \,!\!\!\longrightarrow\, !\mathbb{B}$ and Lemma 2.1 gives

$$P^\#{}_{\langle\!\lbrace b \rbrace\!\rangle} \cong P_b \qquad \text{and} \qquad P^\#{}_{B_1 \otimes B_2} \cong P^\#{}_{B_1} \cdot P^\#{}_{B_2}$$

for all $b \in \mathbb{B}^\circ$ and $B_1, B_2 \in !\mathbb{B}^\circ$. Thus

$$P^\#{}_{\langle\!\lbrace b_1,\ldots,b_n \rbrace\!\rangle} \cong P_{b_1} \cdot \ldots \cdot P_{b_n}$$

for all $\langle\!\lbrace b_1, \ldots, b_n \rbrace\!\rangle \in !\mathbb{B}^\circ$.

2.3 Linear structure

We refer to a $\mathbb{C} \,!\!\!\longrightarrow \mathbb{A}^\circ \times \mathbb{B}$ species as an $\mathbb{A} \times \mathbb{B}$-*matrix*. The *transpose* of an $\mathbb{A} \times \mathbb{B}$-matrix $U : \mathbb{C} \,!\!\!\longrightarrow \mathbb{A}^\circ \times \mathbb{B}$ is the $\mathbb{B}^\circ \times \mathbb{A}^\circ$-matrix $U^t : \mathbb{C} \,!\!\!\longrightarrow (\mathbb{B}^\circ)^\circ \times \mathbb{A}^\circ$ defined as

$$U^t(C)(b, a) \; = \; U(C)(a, b)$$

More generally, for a species $P : \mathbb{C} \,!\!\!\longrightarrow \prod_{i=1}^n \mathbb{A}_i$ we define the *transposition* $P^\sigma : \mathbb{C} \,!\!\!\longrightarrow \prod_{i=1}^n \mathbb{A}_{\sigma i}$ according to the permutation $\sigma \in \mathfrak{S}_n$ by

$$P^\sigma(C)(a_1, \ldots, a_n) \; = \; P(C)(a_{\sigma 1}, \ldots, a_{\sigma n})$$

Matrix multiplication. The *matrix multiplication* (or *linear composition*) of the matrices $U : \mathbb{K} \,!\!\!\longrightarrow \mathbb{A}^\circ \times \mathbb{B}$ and $V : \mathbb{K} \,!\!\!\longrightarrow \mathbb{B}^\circ \times \mathbb{C}$ is the matrix $V \bullet_{\mathbb{B}} U : \mathbb{K} \,!\!\!\longrightarrow \mathbb{A}^\circ \times \mathbb{C}$ defined by

$$\big(V \bullet_{\mathbb{B}} U\big)(K)(a, c) \; = \; \int^{b \in \mathbb{B},\, K_1, K_2 \in \mathbb{K}} V(K_1)(b, c) \times U(K_2)(a, b) \times !\mathbb{K}\,[K_1 \otimes K_2, K]$$

(Compare with the composition of profunctors (5) and the multiplication of species (15).) Using (7), we obtain the familiar formula for matrix multiplication

$$\big(V \bullet_{\mathbb{B}} U\big)_{(a,c)} \; = \; \int^{b \in \mathbb{B}} V_{(b,c)} \cdot U_{(a,b)}$$

for $a \in \mathbb{A}$ and $c \in \mathbb{C}$.

The associativity of matrix multiplication and the unit laws with respect to the *identity matrix* $\Delta_{\mathbb{A}} : \mathbb{C} \,!\!\!\longrightarrow \mathbb{A}^\circ \times \mathbb{A}$ defined as

$$\Delta_{\mathbb{A}}(C)(a', a) \; = \; !\mathbb{C}\left[\langle\!\lbrace\,\rbrace\!\rangle, C\right] \times \mathbb{A}\,[a, a']$$

hold

$$\boxed{\begin{array}{c} W \bullet_{\mathbb{B}} (V \bullet_{\mathbb{A}} U) \cong (W \bullet_{\mathbb{B}} V) \bullet_{\mathbb{A}} U \\ U \bullet_{\mathbb{A}} \Delta_{\mathbb{A}} \cong U \cong \Delta_{\mathbb{B}} \bullet_{\mathbb{B}} U \end{array}}$$

where $U : \mathbb{K} \,!\!\!\longrightarrow \mathbb{A}^\circ \times \mathbb{B}$, $V : \mathbb{K} \,!\!\!\longrightarrow \mathbb{B}^\circ \times \mathbb{C}$, and $W : \mathbb{K} \,!\!\!\longrightarrow \mathbb{C}^\circ \times \mathbb{D}$. Further, for $U_i : \mathbb{K} \,!\!\!\longrightarrow \mathbb{A}^\circ \times \mathbb{B}$ $(i \in I)$, and $V_j : \mathbb{K} \,!\!\!\longrightarrow \mathbb{B}^\circ \times \mathbb{C}$ $(j \in J)$, we have

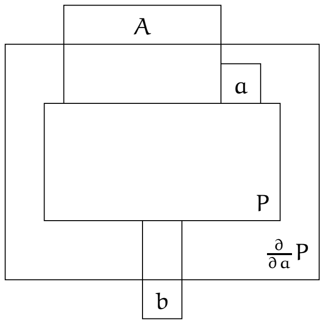
$$\boxed{\big(\textstyle\sum_{j \in J} V_j\big) \bullet_{\mathbb{B}} \big(\sum_{i \in I} U_i\big) \cong \sum_{(j,i) \in J \times I} V_j \bullet_{\mathbb{B}} U_i}$$

2.4 Differential structure

We introduce differentiation in the context of generalised species and establish its basic properties. Higher-order differential operators are further considered in Subsection 2.6.

Differentiation. For $\mathsf{P} : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ and $a \in \mathbb{A}$, the *partial derivative* $\frac{\partial}{\partial a}\mathsf{P} : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ is defined as

$$\big(\tfrac{\partial}{\partial a}\mathsf{P}\big)(A)(b) \;=\; \mathsf{P}(A \otimes \langle\!| a |\!\rangle)(b)$$



For all $\mathsf{P}, \mathsf{Q} : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ and $X \in \widehat{\mathbb{B}}$, we have the following basic properties

$$\begin{array}{c}
\frac{\partial}{\partial a}\big(\frac{\partial}{\partial a'}\mathsf{P}\big) \cong \frac{\partial}{\partial a'}\big(\frac{\partial}{\partial a}\mathsf{P}\big) \\
\frac{\partial}{\partial a}(\mathsf{P} + \mathsf{Q}) \cong (\frac{\partial}{\partial a}\mathsf{P}) + (\frac{\partial}{\partial a}\mathsf{Q}) \\
\frac{\partial}{\partial a}(X\mathsf{P}) = X\big(\frac{\partial}{\partial a}\mathsf{P}\big) \\
\frac{\partial}{\partial a}(\mathrm{E}) = \mathrm{E}
\end{array}$$

and the *Leibniz's rule*

$$\frac{\partial}{\partial a}(\mathsf{P} \cdot \mathsf{Q}) \cong \big(\frac{\partial}{\partial a}\mathsf{P}\big) \cdot \mathsf{Q} + \mathsf{P} \cdot \big(\frac{\partial}{\partial a}\mathsf{Q}\big)$$

PROOF: Use that

$$!\mathbb{A}\left[\textstyle\bigotimes_{i \in I} A_i, A \otimes \langle\!| a |\!\rangle\right] \;\cong\; \textstyle\sum_{i \in I} \int^{A' \in !\mathbb{A}} !\mathbb{A}\left[A_i, A' \otimes \langle\!| a |\!\rangle\right] \times !\mathbb{A}\left[\left(\textstyle\bigotimes_{j \in I \setminus \{i\}} A_j\right) \otimes A', A\right]$$

□

Further, for $\mathsf{P} : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ and $\mathsf{Q} : \mathbb{B} \mathrel{!\!\!\longrightarrow} \mathbb{C}$, we have the *chain rule*

$$\left(\frac{\partial}{\partial a}(\mathsf{Q} \circ \mathsf{P})\right)_c \;\cong\; \int^{b \in \mathbb{B}} \left(\frac{\partial}{\partial b}(\mathsf{Q}) \circ \mathsf{P}\right)_c \cdot \left(\frac{\partial}{\partial a}\mathsf{P}\right)_b$$

where $a \in \mathbb{A}$ and $c \in \mathbb{C}$.

The *differential* application (or *Jacobian* matrix) $\mathrm{d}\mathsf{P} : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{A}^\circ \times \mathbb{B}$ of $\mathsf{P} : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ is defined as

$$(\mathrm{d}\mathsf{P})(A)(a, b) \;=\; \tfrac{\partial}{\partial a}\mathsf{P}(A)(b)$$

The basic properties of partial derivatives translate in terms of differentials; in particular, the *chain rule* amounts to the identity

$$\boxed{\mathrm{d}(Q \circ P) \cong \big(\mathrm{d}(Q) \circ P\big) \bullet_{\mathbb{B}} \mathrm{d}P}$$

For a species $P : \sum_{i \in I} \mathbb{A}_i \mathrel{!\!\!\longrightarrow} \mathbb{B}$, one may introduce j-differentials $\mathrm{d}_j P : \sum_{i \in I} \mathbb{A}_i \mathrel{!\!\!\longrightarrow} \mathbb{A}_j{}^{\circ} \times \mathbb{B}$ $(j \in I)$ as follows

$$(\mathrm{d}_j P)(A)(a, b) \;=\; \tfrac{\partial}{\partial \amalg_j(a)} P(A)(b)$$

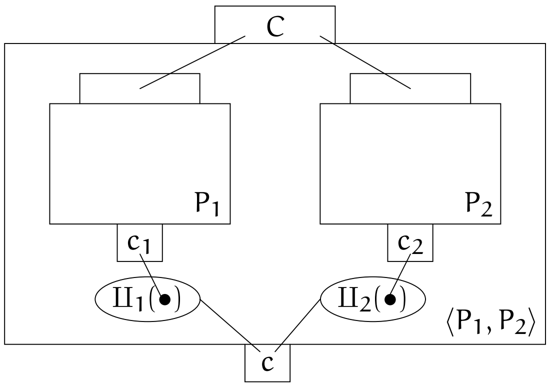
However, as we show below, these are derivable.

2.5 Cartesian closed structure

We informally describe the cartesian closed structure of species.

Pairing and projections. There is exactly one species $\mathbb{C} \mathrel{!\!\!\longrightarrow} \mathbf{0}$. More generally, for $P_i : \mathbb{C} \mathrel{!\!\!\longrightarrow} \mathbb{C}_i$ $(i \in I)$, the *pairing* $\langle P_i \rangle_{i \in I} : \mathbb{C} \mathrel{!\!\!\longrightarrow} \sum_{i \in I} \mathbb{C}_i$ is defined as

$$\begin{aligned} \langle P_i \rangle_{i \in I}(C)(c) \;&=\; \textstyle\sum_{i \in I} \int^{c' \in \mathbb{C}_i} P_i(C)(c') \times \big(\sum_{i \in I} \mathbb{C}_i\big)\,[c, \amalg_i(c')] \\ &\cong\; P_i(C)(c') \quad \text{where } c = \amalg_i(c') \end{aligned} \tag{16}$$

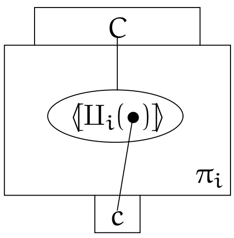


That is,

$$\langle P_i \rangle_{i \in I} = \big([P_i{}^{\perp}]_{i \in I}\big)^{\perp}$$

For $i \in I$, the *projection* species $\pi_i : \sum_{i \in I} \mathbb{C}_i \mathrel{!\!\!\longrightarrow} \mathbb{C}_i$ is defined as

$$\pi_i(C) \;=\; !\Big(\sum_{i \in I} \mathbb{C}_i\Big)\big[\{\!|\amalg_i(\_)|\!\}, C\big] \tag{17}$$



The usual laws of pairing and projections are satisfied up to isomorphism:

$$\boxed{\begin{array}{rcll} \pi_k \circ \langle P_i \rangle_{i \in I} & \cong & P_k & : \mathbb{C} \mathrel{!\!\!\to} \mathbb{C}_k \qquad (k \in I) \\ \langle \pi_i \circ P \rangle_{i \in I} & \cong & P & : \mathbb{C} \mathrel{!\!\!\to} \sum_{i \in I} \mathbb{C}_i \end{array}} \tag{18}$$

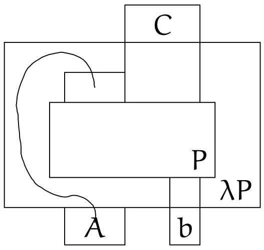
Note that in the presence of cartesian structure the differentials $d_k P : \sum_{i \in I} \mathbb{A}_i \mathrel{!\!\!\to} \mathbb{A}_k^\circ \times \mathbb{B}$ $(k \in I)$ are derivable from the differential $dP : \sum_{i \in I} \mathbb{A}_i \mathrel{!\!\!\to} \sum_{i \in I} \mathbb{A}_i^\circ \times \mathbb{B}$, as

$$\boxed{d_k(P) \cong \pi_k \circ d(P) \qquad (k \in I)}$$

for all $P : \sum_{i \in I} \mathbb{A}_i \mathrel{!\!\!\to} \mathbb{B}$.

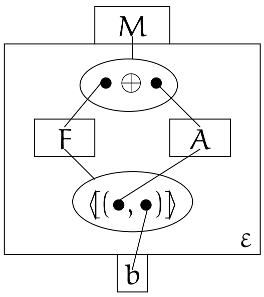
Abstraction and evaluation. For $P : \mathbb{C} + \mathbb{A} \mathrel{!\!\!\to} \mathbb{B}$, the *abstraction* $\lambda_{\mathbb{A}} P : \mathbb{C} \mathrel{!\!\!\to} !\mathbb{A}^\circ \times \mathbb{B}$ is defined as

$$(\lambda_{\mathbb{A}} P)(C)(A, b) \; = \; P(C \oplus A)(b) \tag{19}$$



and the *evaluation* $\varepsilon_{\mathbb{A},\mathbb{B}} : (!\mathbb{A}^\circ \times \mathbb{B}) + \mathbb{A} \mathrel{!\!\!\to} \mathbb{B}$ by

$$\begin{aligned} \varepsilon_{\mathbb{A},\mathbb{B}}(M)(b) \; &= \; \int^{F \in !(!\mathbb{A}^\circ \times \mathbb{B}),\, A \in !\mathbb{A}} !(!\mathbb{A}^\circ \times \mathbb{B})[\, \{\!|(A, b)|\!\}\,, F\,] \times \; !\big((!\mathbb{A}^\circ \times \mathbb{B}) + \mathbb{A}\big)[F \oplus A, M] \\ &\cong \; !(!\mathbb{A}^\circ \times \mathbb{B})\, \big[\, \{\!|(M._2, b)|\!\}\,, M._1 \big] \end{aligned} \tag{20}$$



For $P : \mathbb{C} \mathrel{!\!\!\to} !\mathbb{A}^\circ \times \mathbb{B}$, we write $\upsilon_{\mathbb{A}}(P)$ for the composite $\varepsilon \circ \langle P \circ \pi_1, \pi_2 \rangle : \mathbb{C} + \mathbb{A} \mathrel{!\!\!\to} \mathbb{B}$. The usual laws of abstraction and evaluation are satisfied up to isomorphism:

$$\boxed{\begin{array}{c} \upsilon(\lambda P) \cong P : \mathbb{C} + \mathbb{A} \mathrel{!\!\!\to} \mathbb{B} \\ \lambda(\upsilon P) \cong P : \mathbb{C} \mathrel{!\!\!\to} !\mathbb{A}^\circ \times \mathbb{B} \end{array}} \tag{21}$$

We further note the following interesting commutation property between abstraction and linear composition: for σ the permutation $(12)(3)$,

$$\boxed{\big((\lambda_{\mathbb{A}} Q)^{\sigma} \underset{\mathbb{C}}{\bullet} P\big)^{\sigma} \cong \lambda_{\mathbb{A}}\big(Q \underset{\mathbb{C}}{\bullet} (P \circ \pi_1)\big)} \tag{22}$$

for all $P : \mathbb{K} \mathrel{!\!\!\longrightarrow} \mathbb{B}^{\circ} \times \mathbb{C}$ and $Q : \mathbb{K} + \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{C}^{\circ} \times \mathbb{D}$.

2.6 Higher-order differential structure

We relate the linear and cartesian closed structures, and introduce an operator which is shown to satisfy the basic properties of differentiation.

Linear and cartesian closed structure. For a matrix $U : \mathbb{C} \mathrel{!\!\!\longrightarrow} \mathbb{A}^{\circ} \times \mathbb{B}$ we define the species $\widetilde{U} : \mathbb{C} + \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$ as

$$\begin{aligned} \widetilde{U}(M)(b) &= \int^{a \in \mathbb{A},\, C \in !\mathbb{C},\, A \in !\mathbb{A}} U(C)(a,b) \times !\mathbb{A}\left[\langle\!| a |\!\rangle, A\right] \times !(\mathbb{C}+\mathbb{A})\,[C \oplus A, M] \\ &\cong \int^{a \in \mathbb{A}} U(M_{.1})(a,b) \times !\mathbb{A}\left[\langle\!| a |\!\rangle, M_{.2}\right] \end{aligned}$$

This construction internalises as an embedding of matrices into exponentials as follows

$$\iota_{\mathbb{A},\mathbb{B}} = \lambda_{\mathbb{A}}(\widetilde{I_{\mathbb{A}^{\circ} \times \mathbb{B}}}) : \mathbb{A}^{\circ} \times \mathbb{B} \mathrel{!\!\!\longrightarrow} !\mathbb{A}^{\circ} \times \mathbb{B}$$

That is,

$$\iota_{\mathbb{A},\mathbb{B}}(U)(A,b) = \int^{a \in \mathbb{A}} !(\mathbb{A}^{\circ} \times \mathbb{B})\left[\langle\!| (a,b) |\!\rangle, U\right] \times !\mathbb{A}\left[\langle\!| a |\!\rangle, A\right]$$

Indeed, for all $P : \mathbb{C} \mathrel{!\!\!\longrightarrow} \mathbb{A}^{\circ} \times \mathbb{B}$, we have that

$$\iota_{\mathbb{A},\mathbb{B}} \circ P \cong \lambda_{\mathbb{A}}(\widetilde{P}) : \mathbb{C} \mathrel{!\!\!\longrightarrow} !\mathbb{A}^{\circ} \times \mathbb{B}$$

Further, the embedding commutes with identities and composition; since, for

$$\ell_{\mathbb{A},\mathbb{B},\mathbb{C}} = \pi_2 \underset{\mathbb{B}}{\bullet} \pi_1 : (\mathbb{B}^{\circ} \times \mathbb{C}) + (\mathbb{A}^{\circ} \times \mathbb{B}) \mathrel{!\!\!\longrightarrow} \mathbb{A}^{\circ} \times \mathbb{C}$$

we have that

$$\ell \circ \langle P, Q \rangle \cong P \underset{\mathbb{B}}{\bullet} Q$$

for all $P : \mathbb{K} \mathrel{!\!\!\longrightarrow} \mathbb{A}^{\circ} \times \mathbb{B}$ and $Q : \mathbb{K} \mathrel{!\!\!\longrightarrow} \mathbb{B}^{\circ} \times \mathbb{C}$, and

$$\begin{aligned} \iota_{\mathbb{A},\mathbb{A}} \circ \Delta_{\mathbb{A}} &\cong \lambda_{\mathbb{A}}(I_{\mathbb{A}}) : \mathbf{0} \mathrel{!\!\!\longrightarrow} !\mathbb{A}^{\circ} \times \mathbb{A} \\ \iota_{\mathbb{A},\mathbb{B}} \circ \ell_{\mathbb{A},\mathbb{B},\mathbb{C}} &\cong \langle \iota_{\mathbb{B},\mathbb{C}} \circ \pi_2, \iota_{\mathbb{A},\mathbb{B}} \circ \pi_1 \rangle \circ m_{\mathbb{A},\mathbb{B},\mathbb{C}} \end{aligned}$$

for $m_{\mathbb{A},\mathbb{B},\mathbb{C}} = \lambda_{\mathbb{A}}\big(\varepsilon_{\mathbb{B},\mathbb{C}} \circ \langle \pi_1, \varepsilon_{\mathbb{A},\mathbb{B}} \circ \pi_2 \rangle\big)$ the internal composition $(!\mathbb{B}^{\circ} \times \mathbb{C}) + (!\mathbb{A}^{\circ} \times \mathbb{B}) \mathrel{!\!\!\longrightarrow} !\mathbb{A}^{\circ} \times \mathbb{C}$.

Differentiation operator. We introduce the *differentiation operator*

$$\mathrm{D}_{\mathbb{A},\mathbb{B}} : !\mathbb{A}^\circ \times \mathbb{B} \mathrel{\mid\!\!\to} !\mathbb{A}^\circ \times \mathbb{A}^\circ \times \mathbb{B}$$

defined as

$$\mathrm{D}_{\mathbb{A},\mathbb{B}}(F)(A, a, b) \;=\; !(!\mathbb{A}^\circ \times \mathbb{B})\left[\,\langle\!|(A \otimes \langle\!|a|\!\rangle, b)|\!\rangle, F\,\right]$$

Proposition 2.7 *The differential operator is linear. Indeed,*

$$\mathrm{D} \cong \widetilde{\delta}$$

for δ *the* $(!\mathbb{A}^\circ \times \mathbb{B}) \times (!\mathbb{A}^\circ \times \mathbb{A}^\circ \times \mathbb{B})$*-matrix given by*

$$\delta(U, (A, a, b)) \;=\; !\mathbb{A}^\circ \times \mathbb{B}\left[(A \otimes \langle\!|a|\!\rangle, b), U\right]$$

Further, this operator internalises differential application since

$$\boxed{\mathrm{d}_2 P \cong \upsilon_{\mathbb{A}}(\mathrm{D} \circ \lambda_{\mathbb{A}} P) : \mathbb{C} + \mathbb{A} \mathrel{\mid\!\!\to} \mathbb{A}^\circ \times \mathbb{B}} \tag{23}$$

for all $P : \mathbb{C} + \mathbb{A} \mathrel{\mid\!\!\to} \mathbb{B}$ and is constant on linear maps as

$$\boxed{\mathrm{D} \circ \imath_{\mathbb{A},\mathbb{B}} \cong \lambda_{\mathbb{A}}(\pi_1) : \mathbb{A}^\circ \times \mathbb{B} \mathrel{\mid\!\!\to} !\mathbb{A}^\circ \times \mathbb{A}^\circ \times \mathbb{B}}$$

It follows that

$$\boxed{\mathrm{d}(\mathrm{I}_{\mathbb{A}}) \cong \Delta_{\mathbb{A}} : \mathbb{A} \mathrel{\mid\!\!\to} \mathbb{A}^\circ \times \mathbb{A}}$$

and we have from (22) and (23) above that, for σ the permutation $(12)(3)$,

$$\boxed{\Big((\mathrm{D} \circ \lambda_{\mathbb{A}} P)^{\sigma} \underset{\mathbb{A}}{\bullet} U\Big)^{\sigma} \cong \; \lambda_{\mathbb{A}}\big(\mathrm{d}_2(P) \underset{\mathbb{A}}{\bullet} (U \circ \pi_1)\big)}$$

for all $P : \mathbb{C} + \mathbb{A} \mathrel{\mid\!\!\to} \mathbb{B}$ and $U : \mathbb{C} \mathrel{\mid\!\!\to} \mathbb{D}^\circ \times \mathbb{A}$. This identity corresponds to the β-rule of the differential lambda calculus [ER03].

2.7 Operators on generalised Fock space

Annihilation and creation. Let $\mathbb{A}$ and $\mathbb{B}$ be small categories. For $a \in \mathbb{A}$ define the *annihilation* and *creation* operators as the $(!\mathbb{A}^\circ \times \mathbb{B}) \times (!\mathbb{A}^\circ \times \mathbb{B})$-matrices α_a and γ_a given by

$$\alpha_a(U, (A, b)) \;=\; \delta(U, (A, a, b))$$

$$\gamma_a(U, (A, b)) \;=\; \int^{A' \in !\mathbb{A}} !\mathbb{A}^\circ \times \mathbb{B}\,[\,(A', b), U\,] \times !\mathbb{A}\left[A' \otimes \langle\!|a|\!\rangle, A\right]$$

Further, let α and γ be the $(!\mathbb{A}^\circ \times \mathbb{B}) \times (!\mathbb{A}^\circ \times \mathbb{B})$-matrices

$$\alpha(U, V) \;=\; \int^{a \in \mathbb{A}} \alpha_a(U, V) \qquad \text{and} \qquad \gamma(U, V) \;=\; \int^{a \in \mathbb{A}} \gamma_a(U, V)$$

Proposition 2.8 *Let $\mathbb{A}$ and $\mathbb{B}$ be small categories. For $u, v \in \mathbb{A}$ the following hold:*

$$\boxed{\begin{array}{c} \alpha_u \bullet \gamma_v \cong \gamma_v \bullet \alpha_u + \mathbb{A}[v,u] \;\; \Delta_{!\mathbb{A}^\circ \times \mathbb{B}} \\ \alpha_u \bullet \alpha_v \cong \alpha_v \bullet \alpha_u \qquad\qquad \gamma_u \bullet \gamma_v \cong \gamma_v \bullet \gamma_u \end{array}}$$

Further, for non-empty $\mathbb{A}$, we also have that

$$\boxed{\alpha \bullet \gamma \;\; \cong \;\; \gamma \bullet \alpha + \Delta_{!\mathbb{A}^\circ \times \mathbb{B}}}$$

PROOF: We only show that $\alpha_u \bullet \gamma_v \cong \gamma_v \bullet \alpha_u + \mathbb{A}[v,u] \;\; \Delta_{!\mathbb{A}^\circ \times \mathbb{B}}$.

On the one hand we have that

$$\begin{aligned}
(\gamma_v \bullet \alpha_u)(U,(A,b)) &= \int^{V \in !\mathbb{A}^\circ \times \mathbb{B}} \gamma_v(V,(A,b)) \times \alpha_u(U,V) \\
&= \int^{V \in !\mathbb{A}^\circ \times \mathbb{B}} \Big(\int^{A' \in !\mathbb{A}} [(A',b),V] \times \big[A' \otimes \langle\!\langle v \rangle\!\rangle, A\big] \Big) \times \alpha_u(U,V) \\
&\cong \int^{A' \in !\mathbb{A}} \big[A' \otimes \langle\!\langle v \rangle\!\rangle, A\big] \times \alpha_u(U,(A',b)) \\
&= \int^{A' \in !\mathbb{A}} \big[A' \otimes \langle\!\langle v \rangle\!\rangle, A\big] \times \big[\langle\!\langle (A' \otimes \langle\!\langle u \rangle\!\rangle, b) \rangle\!\rangle, U\big]
\end{aligned}$$

and on the other that

$$\begin{aligned}
(\alpha_u \bullet \gamma_v)(U,(A,b)) &= \int^{V \in !\mathbb{A}^\circ \times \mathbb{B}} \big[(A \otimes \langle\!\langle u \rangle\!\rangle, b), V\big] \times \gamma_v(U,V) \\
&\cong \gamma_v(U,(A \otimes \langle\!\langle u \rangle\!\rangle, b)) \\
&= \int^{A' \in !\mathbb{A}} [(A',b),U] \times \big[A' \otimes \langle\!\langle v \rangle\!\rangle, A \otimes \langle\!\langle u \rangle\!\rangle\big]
\end{aligned}$$

Further, since

$$\big[A' \otimes \langle\!\langle v \rangle\!\rangle, A \otimes \langle\!\langle u \rangle\!\rangle\big] \;\; \cong \;\; \Big(\int^{X \in !\mathbb{A}} \big[A', X \otimes \langle\!\langle u \rangle\!\rangle\big] \times \big[X \otimes \langle\!\langle v \rangle\!\rangle, A\big] \Big) + \big([v,u] \times [A',A] \big)$$

we finally have that

$$\begin{aligned}
(\alpha_u \bullet \gamma_v)(U,(A,b)) &\cong \int^{A' \in !\mathbb{A}} \Big([(A',b),U] \times \int^{X \in !\mathbb{A}} \big[A', X \otimes \langle\!\langle u \rangle\!\rangle\big] \times \big[X \otimes \langle\!\langle v \rangle\!\rangle, A\big] \Big) \\
&\qquad\qquad + \big([(A',b),U] \times [v,u] \times [A',A] \big) \\
&\cong \int^{A' \in !\mathbb{A}, X \in !\mathbb{A}} [(A',b),U] \times \big[A', X \otimes \langle\!\langle u \rangle\!\rangle\big] \times \big[X \otimes \langle\!\langle v \rangle\!\rangle, A\big] \\
&\qquad + \int^{A' \in !\mathbb{A}} [(A',b),U] \times [v,u] \times [A',A] \\
&\cong \Big(\int^{X \in !\mathbb{A}} \big[(X \otimes \langle\!\langle u \rangle\!\rangle, b), U\big] \times \big[X \otimes \langle\!\langle v \rangle\!\rangle, A\big] \Big) + \big([v,u] \times [(A,b),U] \big) \\
&\cong (\gamma_v \bullet \alpha_u)(U,(A,b)) + [v,u] \;\; \Delta(U,(A,b)) \\
&= \big(\gamma_v \bullet \alpha_u + [v,u] \;\; \Delta\big)(F)(A,b)
\end{aligned}$$

□

Let $A_a = \widetilde{\alpha_a}$, $C_a = \widetilde{\gamma_a}$ and $A = \widetilde{\alpha}$, $C = \widetilde{\gamma}$.

Proposition 2.9 *For all* $\mathsf{A}_a, \mathsf{C}_a : !\mathbb{A}^\circ \times \mathbb{B} \mapsto !\mathbb{A}^\circ \times \mathbb{B}$,

$$\mathsf{A}_a(F)(A,b) \cong (\tfrac{\partial}{\partial a}\overline{F})(A)(b) \qquad \textit{and} \qquad \mathsf{C}_a(F)(A,b) \cong \big(\overline{F} \cdot \chi_a\big)(A)(b)$$

where $\overline{F}(A)(b) = \varepsilon_{\mathbb{A},\mathbb{B}}(F \oplus A)(b)$ *and* $\chi_a(A)(b) = I_{\mathbb{A}}(A)(a)$.

Corollary 2.10 *Let* $\mathbb{A}$ *and* $\mathbb{B}$ *be small categories. For* $u, v \in \mathbb{A}$ *the following hold:*

$$\begin{array}{|c|}\hline \mathsf{A}_u \circ \mathsf{C}_v \cong \mathsf{C}_v \circ \mathsf{A}_u + \mathbb{A}[v,u]\ \ I_{!\mathbb{A}^\circ \times \mathbb{B}} \\ \mathsf{A}_u \circ \mathsf{A}_v \cong \mathsf{A}_v \circ \mathsf{A}_u \qquad\qquad \mathsf{C}_u \circ \mathsf{C}_v \cong \mathsf{C}_v \circ \mathsf{C}_u \\ \hline\end{array}$$

Further, for non-empty $\mathbb{A}$*, we also have that*

$$\boxed{\mathsf{A} \circ \mathsf{C} \ \ \cong \ \ \mathsf{C} \circ \mathsf{A} + I_{!\mathbb{A}^\circ \times \mathbb{B}}}$$

3 Remarks

The cartesian closed structure of species.

Conjecture 3.1 *The bicategory of generalised species* $\boldsymbol{\mathcal{ES}}$ *is pseudo-cartesian-closed.*

The only verifications missing to obtain this result are the functoriality of the pairing operation (16) and the abstraction operation (19), and the naturality of the isomorphisms (18) and (21).

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A The bicategory of species

In this technical section I exhibit the bicategorical structure of $\mathcal{ES}$.

Composition. Define the composition functors

$$\circ_{\mathbb{A},\mathbb{B},\mathbb{C}} : \mathcal{ES}\,[\mathbb{B},\mathbb{C}] \times \mathcal{ES}\,[\mathbb{A},\mathbb{B}] \longrightarrow \mathcal{ES}\,[\mathbb{A},\mathbb{C}]$$

by taking the following explicit description of the coends (5) and (6)

$$(Q \circ P)(A)(c) = \Big(\textstyle\sum_{B\in\mathbb{B}} Q(B)(c) \times \sum_{A_b \in !\mathbb{A}\ (b\in B)} \big(\prod_{b\in B} P(A_b)(B_{@b})\big) \times !\mathbb{A}\left[\bigotimes_{b\in B} A_b, A\right] \Big)_{/\approx}$$

where $\approx$ is the equivalence relation generated by

$$\begin{array}{c} \big(\ B\ ,\ q\ ,\ \langle A_b\rangle_{b\in B}\ ,\ \langle\ P(\alpha_b)(\beta_{@b})(p_{\beta b})\ \rangle_{b\in B}\ ,\ \alpha\ \big) \\ \approx \\ \big(\ B'\ ,\ \ Q(\beta)(c)(q)\ ,\ \langle A'_{b'}\rangle_{b'\in B'}\ ,\ \langle p_{b'}\rangle_{b'\in B'}\ ,\ \alpha \circ (\bigotimes_{b\in B}\alpha_b)\circ\sigma\ \big) \end{array}$$

for $\beta : B \longrightarrow B'$ in $!\mathbb{B}$, $q \in Q(B)(c)$, $\langle \alpha_b : A'_{\beta b} \longrightarrow A_b\rangle_{b\in B}$ in $!\mathbb{A}$, $\langle p_{b'} \in P(A'_{b'})(B'_{@b'})\rangle_{b'\in B'}$, and $\alpha : \bigotimes_{b\in B} A_b \longrightarrow A$ in $!\mathbb{A}$, where $\sigma : \bigotimes_{b'\in B'} A'_{b'} \cong \bigotimes_{b\in B} A'_{\beta b}$ in $!\mathbb{A}$.

Clearly, for $\varphi : A \longrightarrow A'$ in $!\mathbb{A}$ and $g : c' \longrightarrow c$ in $\mathbb{C}$, we have that

$$(Q \circ P)(\varphi)(g)\big[B, q, \langle A_b\rangle_{b\in B}, \langle p_b\rangle_{b\in B}, \alpha\big] \ = \ \big[B, Q(B)(g)(q), \langle A_b\rangle_{b\in B}, \langle p_b\rangle_{b\in B}, \varphi\circ\alpha\big]$$

For $\varphi : P_1 \Longrightarrow P_2 : \mathbb{A} \mathrel{\mid\!\!\!\rightarrow} \mathbb{B}$ and $\gamma : Q_1 \Longrightarrow Q_2 : \mathbb{B} \mathrel{\mid\!\!\!\rightarrow} \mathbb{C}$, we have $\gamma \circ \varphi : P_2 \circ P_1 \Longrightarrow Q_2 \circ Q_1 : \mathbb{A} \mathrel{\mid\!\!\!\rightarrow} \mathbb{C}$ given by

$$(\gamma \circ \varphi)_{A,c} : \big[B, q, \langle A_b \rangle_{b \in B}, \langle p_b \rangle_{b \in B}, \alpha\big] \longmapsto \big[B, \gamma_{B,c}(q), \langle A_b \rangle_{b \in B}, \langle \varphi_{A_b, B@b}(p_b) \rangle_{b \in B}, \alpha\big]$$

Unit laws. The natural family of left unit laws

$$\big\{\, l_P : I_{\mathbb{B}} \circ P \overset{\cong}{\Longrightarrow} P \,\big\}_{P \in \mathcal{ES}(\mathbb{A},\mathbb{B})}$$

arising from (11) is explicitly given by

$$\begin{aligned}(l_P)_{A,b} : \ [\, B \,,\ \beta \,,\ \langle A_b \rangle_{b \in B} \,,\ \langle p_b \rangle_{b \in B} \,,\ \alpha \,] &\longmapsto [\, B_{@1} \,,\ \beta_{@1} \,,\ A_1 \,,\ p_1 \,,\ \alpha \,] \\ &\longmapsto [\, B_{@1} \,,\ \beta_{@1} \,,\ P(\alpha)(B_{@1})(p_1) \,] \\ &\longmapsto P(\alpha)(\beta_{@1})(p_1)\end{aligned}$$

whilst the natural family of right unit laws

$$\big\{\, r_P : P \circ I_{\mathbb{A}} \overset{\cong}{\Longrightarrow} P \,\big\}_{P \in \mathcal{ES}(\mathbb{A},\mathbb{B})}$$

arising from (12) is explicitly given by

$$\begin{aligned}(r_P)_{A,b} : \ [\, A' \,,\ p \,,\ \langle X_a \rangle_{a \in A'} \,,\ \langle \alpha_a \rangle_{a \in A'} \,,\ \alpha \,] &\longmapsto [\, A' \,,\ p \,,\ \alpha \circ \textstyle\bigotimes_{a \in A'} \alpha_a \,] \\ &\longmapsto P\big(\alpha \circ \textstyle\bigotimes_{a \in A'} \alpha_a\big)(b)(p)\end{aligned}$$

Associativity law. The natural family of associativity laws

$$\big\{\, a_{R,Q,P} : (R \circ Q) \circ P \overset{\cong}{\Longrightarrow} R \circ (Q \circ P) : \mathbb{A} \mathrel{\mid\!\!\!\rightarrow} \mathbb{D} \,\big\}_{R \in \mathcal{ES}(\mathbb{C},\mathbb{D}),\, Q \in \mathcal{ES}(\mathbb{B},\mathbb{C}),\, P \in \mathcal{ES}(\mathbb{A},\mathbb{B})}$$

arising from (13) and (14) is explicitly given by

$$
\begin{aligned}
(a_{R,Q,P})_{A,d} : \big[\, B \,,\, [\, C \,,\, r \,,\, \langle B_c\rangle_{c\in C} \,,\, \langle q_c\rangle_{c\in C} \,,\, \beta \,] \,,\, \langle A_b\rangle_{b\in B} \,,\, \langle p_b\rangle_{b\in B} \,,\, \alpha \,\big] \\
\longmapsto \Big[\; & C \,,\, r \,, \\
& \langle B_c\rangle_{c\in C} \,,\, \langle q_c\rangle_{c\in C} \,, \\
& \big\langle A_{\beta b}\big\rangle_{b\in(\bigotimes_{c\in C} B_c)} \,,\, \big\langle P(A_{\beta b})(\beta_{@b})(p_{\beta b})\big\rangle_{b\in(\bigotimes_{c\in C} B_c)} \,, \\
& \big(\textstyle\bigotimes_{b\in(\bigotimes_{c\in C} B_c)} A_{\beta b}\big) \xrightarrow[\cong]{\sigma} \big(\textstyle\bigotimes_{b\in B} A_b\big) \xrightarrow{\alpha} A \;\Big] \\
\longmapsto \Big[\; & C \,,\, r \,, \\
& \langle B_c\rangle_{c\in C} \,,\, \langle q_c\rangle_{c\in C} \,, \\
& \big\langle A_{\beta(c.b)}\big\rangle_{c\in C, b\in B_c} \,,\, \big\langle P(A_{\beta(c.b)})(\beta_{@(c.b)})(p_{\beta(c.b)})\big\rangle_{c\in C, b\in B_c} \,, \\
& \big(\textstyle\bigotimes_{c\in C}\bigotimes_{b\in B_c} A_{\beta(c.b)}\big) = \big(\textstyle\bigotimes_{b\in(\bigotimes_{c\in C} B_c)} A_{\beta b}\big) \xrightarrow{\alpha\circ\sigma} A \;\Big] \\
\longmapsto \Big[\; & C \,,\, r \,, \\
& \big\langle \textstyle\bigotimes_{b\in B_c} A_{\beta(c.b)}\big\rangle_{c\in C} \,, \\
& \big[\; \langle B_c\rangle_{c\in C} \,,\, \langle q_c\rangle_{c\in C} \,, \\
& \quad \big[\; \big\langle A_{\beta(c.b)}\big\rangle_{c\in C, b\in B_c} \,,\, \big\langle P(A_{\beta(c.b)})(\beta_{@(c.b)})(p_{\beta(c.b)})\big\rangle_{c\in C, b\in B_c} \,, \\
& \quad\quad \big\langle \mathrm{id}_{\bigotimes_{b\in B_c} A_{\beta(c.b)}}\big\rangle_{c\in C} \;\big]\;\big] \,, \\
& \alpha\circ\sigma \;\Big] \\
\longmapsto \Big[\; & C \,,\, r \,, \\
& \big\langle \textstyle\bigotimes_{b\in B_c} A_{\beta(c.b)}\big\rangle_{c\in C} \,, \\
& \big[\; \langle B_c\rangle_{c\in C} \,,\, \langle q_c\rangle_{c\in C} \,, \\
& \quad \Big\langle\, \big[\; \big\langle A_{\beta(c.b)}\big\rangle_{b\in B_c} \,,\, \big\langle P(A_{\beta(c.b)})(\beta_{@(c.b)})(p_{\beta(c.b)})\big\rangle_{b\in B_c} \,, \\
& \quad\quad \mathrm{id}_{\bigotimes_{b\in B_c} A_{\beta(c.b)}} \;\big]\, \Big\rangle_{c\in C} \;\big] \,, \\
& \alpha\circ\sigma \;\Big] \\
\longmapsto \Big[\; & C \,,\, r \,, \\
& \big\langle \textstyle\bigotimes_{b\in B_c} A_{\beta(c.b)}\big\rangle_{c\in C} \,, \\
& \Big\langle\, \big[\; B_c \,,\, q_c \,, \\
& \quad\quad \big\langle A_{\beta(c.b)}\big\rangle_{b\in B_c} \,,\, \big\langle P(A_{\beta(c.b)})(\beta_{@(c.b)})(p_{\beta(c.b)})\big\rangle_{b\in B_c} \,, \\
& \quad\quad \mathrm{id}_{\bigotimes_{b\in B_c} A_{\beta(c.b)}} \;\big]\, \Big\rangle_{c\in C} \,, \\
& \alpha\circ\sigma \;\Big]
\end{aligned}
$$

for $B \in {!\mathbb{B}}$, $C \in {!\mathbb{C}}$, $r \in R(C)(d)$, $\langle B_c \in {!\mathbb{B}}\rangle_{c\in C}$, $\langle q_c \in Q(B_c)(C_{@c})\rangle_{c\in C}$, $\beta : \bigotimes_{c\in C} B_c \longrightarrow B$ in ${!\mathbb{B}}$, $\langle A_b \in {!\mathbb{A}}\rangle_{b\in B}$, $\langle p_b \in P(A_b)(B_{@b})\rangle_{b\in B}$, and $\alpha : \bigotimes_{b\in B} A_b \longrightarrow A$ in ${!\mathbb{A}}$, where for $Y \in {!\mathbb{Y}}$ and $\langle X_y \in {!\mathbb{X}}\rangle_{y\in Y}$ the index of $x \in X_y$ for $y \in Y$ within $\big(\bigotimes_{y\in Y} X_y\big) \in {!\mathbb{X}}$ is denoted $y.x$ and where $\sigma(b.a) = (\beta b).a$ and $\sigma_{@(b.a)} = \mathrm{id}_{(A_{\beta b})_{@a}}$ for all $b \in \bigotimes_{c\in C} B_c$ and $a \in A_{\beta b}$.

Coherence axioms. I will first check the commutativity of

$$
\begin{array}{ccc}
((Q\circ I_{\mathbb{B}})\circ P)(A)(c) & \xrightarrow{\quad (a_{Q,I_{\mathbb{B}},P})_{A,c} \quad} & (Q\circ(I_{\mathbb{B}}\circ P))(A)(c) \\
& {\scriptstyle (r_Q\circ \mathrm{id}_{I_{\mathbb{B}}})_{A,c}} \searrow \quad \swarrow {\scriptstyle (\mathrm{id}_Q\circ l_P)_{A,c}} & \\
& (Q\circ P)(A)(c) &
\end{array}
$$

for $P : \mathbb{A} \mathrel{!\!\!\longrightarrow} \mathbb{B}$, $Q : \mathbb{B} \mathrel{!\!\!\longrightarrow} \mathbb{C}$, $A \in !\mathbb{A}$, and $c \in \mathbb{C}$.

Note that for $B, B' \in !\mathbb{B}$, $q \in Q(B')(c)$, $\langle B_{b'} \in !\mathbb{B}\rangle_{b' \in B'}$, $\big\langle \beta_{b'} : \llbracket B'_{@b'} \rrbracket \longrightarrow B_{b'} \text{ in } !\mathbb{B}\big\rangle_{b' \in B'}$, $\beta : \bigotimes_{b' \in B'} B_{b'} \longrightarrow B$ in $!\mathbb{B}$, $\langle A_b \in !\mathbb{A}\rangle_{b \in B}$, $\langle p_b \in P(A_b)(B_{@b})\rangle_{b \in B}$, and $\alpha : \bigotimes_{b \in B} A_b \longrightarrow A$ in $!\mathbb{A}$ we have that

$$\big[\ B\ ,\ [\ B'\ ,\ q\ , \langle B_{b'}\rangle_{b' \in B'}\ ,\ \langle \beta_{b'}\rangle_{b' \in B'}\ ,\ \beta\ ]\ ,\ \langle A_b\rangle_{b \in B}\ ,\ \langle p_b\rangle_{b \in B}\ ,\ \alpha\ \big]$$

$$\xmapsto{(r_Q \circ \mathrm{id}_P)_{A,d}} \big[\ B\ ,\ Q\big(\beta \circ (\textstyle\bigotimes_{b' \in B'} \beta_{b'})\big)(c)(q)\ ,\ \langle A_b\rangle_{b \in B}\ ,\ \langle p_b\rangle_{b \in B}\ ,\ \alpha\ \big] \tag{24}$$

and that

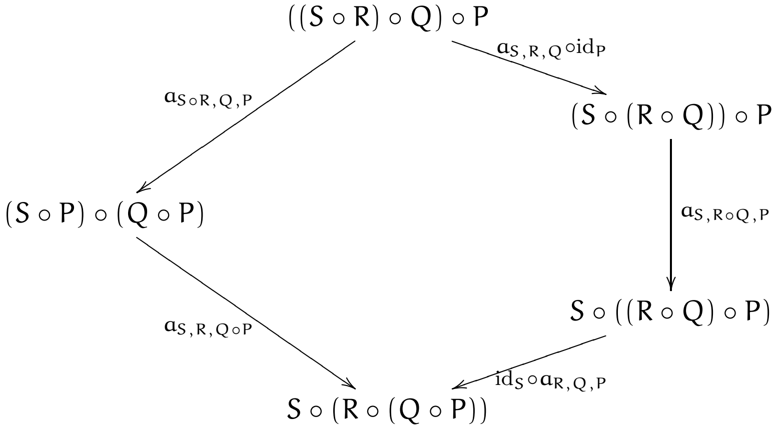
$$\begin{aligned}
&\big[\ B\ ,\ [\ B'\ ,\ q\ , \langle B_{b'}\rangle_{b' \in B'}\ ,\ \langle \beta_{b'}\rangle_{b' \in B'}\ ,\ \beta\ ]\ ,\ \langle A_b\rangle_{b \in B}\ ,\ \langle p_b\rangle_{b \in B}\ ,\ \alpha\ \big] \\
&\xmapsto{(a_{Q,I_{\mathbb{B}},P})_{A,c}} \begin{array}{l} \big[\ B'\ ,\ q\ , \\ \ \big\langle A_{\beta(b'.1)}\big\rangle_{b' \in B'}\ , \\ \ \big\langle\ \big[\ \ B_{b'}\ ,\ \beta_{b'}\ , \\ \qquad \big\langle A_{\beta(b'.1)}\big\rangle_{b' \in B'}\ ,\ \big\langle P(A_{\beta(b'.1)})(\beta_{@b'})(p_{\beta(b'.1)})\big\rangle_{b' \in B'}\ , \\ \qquad \mathrm{id}_{A_{\beta(b'.1)}}\ \big]\ \big\rangle_{b' \in B'}\ , \\ \ \big(\bigotimes_{b' \in B'} A_{\beta(b'.1)}\big) \cong \big(\bigotimes_{b \in B} A_b\big) \xrightarrow{\alpha} A\ \big] \end{array} \\
&\xmapsto{(\mathrm{id}_Q \circ l_P)_{A,c}} \begin{array}{l} \big[\ B'\ ,\ q\ , \\ \ \big\langle A_{\beta(b'.1)}\big\rangle_{b' \in B'}\ , \\ \ \Big\langle P\big(\mathrm{id}_{A_{\beta(b'.1)}}\big)\big((\beta_{b'})_{@1}\big)\big(P(A_{\beta(b'.1)})(\beta_{@(b'.1)})(p_{\beta(b'.1)})\big)\Big\rangle_{b' \in B'} \\ \ \big(\bigotimes_{b' \in B'} A_{\beta(b'.1)}\big) \cong \big(\bigotimes_{b \in B} A_b\big) \xrightarrow{\alpha} A\ \big] \end{array}
\end{aligned} \tag{25}$$

Further, since

$$\begin{aligned}
(25) \ &= \ \begin{array}{l} \big[\ B'\ ,\ q\ , \\ \ \big\langle A_{\beta(b'.1)}\big\rangle_{b' \in B'}\ , \\ \ \big\langle P\big(A_{\beta(b'.1)}\big)\big(\beta_{@(b'.1)} \circ (\beta_{b'})_{@1}\big)\big(p_{\beta(b'.1)}\big)\big\rangle_{b' \in B'}\ , \\ \ \big(\bigotimes_{b' \in B'} A_{\beta(b'.1)}\big) \cong \big(\bigotimes_{b \in B} A_b\big) \xrightarrow{\alpha} A\ \big] \end{array} \\
&= \ \begin{array}{l} \big[\ B'\ ,\ q\ , \\ \ \big\langle A_{\beta(b'.1)}\big\rangle_{b' \in B'}\ , \\ \ \Big\langle P\big(\mathrm{id}_{A_{\beta(b'.1)}}\big)\big((\beta \circ (\bigotimes_{b' \in B'} \beta_{b'}))_{@b'}\big)\big(p_{(\beta \circ (\bigotimes_{b' \in B'} \beta_{b'}))(b')}\big)\Big\rangle_{b' \in B'}\ , \\ \ \big(\bigotimes_{b' \in B'} A_{\beta(b'.1)}\big) \cong \big(\bigotimes_{b \in B} A_b\big) \xrightarrow{\alpha} A\ \big] \end{array} \\
&= \ \begin{array}{l} \big[\ B\ ,\ Q\big(\beta \circ (\bigotimes_{b' \in B'} \beta_{b'})\big)\big(c\big)\big(q\big)\ , \\ \ \langle A_b\rangle_{b \in B}\ , \\ \ \langle p_b\rangle_{b \in B}\ , \\ \ \big(\bigotimes_{b \in B} A_b\big) \cong \big(\bigotimes_{b' \in B'} A_{\beta(b'.1)}\big) \cong \big(\bigotimes_{b \in B} A_b\big) \xrightarrow{\alpha} A\ \big] \end{array} \\
&= \ (24)
\end{aligned}$$

we are done.

To conclude, for $\mathsf{P} : \mathbb{A} \,{!}{\rightarrowtail}\, \mathbb{B}$, $\mathsf{Q} : \mathbb{B} \,{!}{\rightarrowtail}\, \mathbb{C}$, $\mathsf{R} : \mathbb{C} \,{!}{\rightarrowtail}\, \mathbb{D}$, and $\mathsf{S} : \mathbb{D} \,{!}{\rightarrowtail}\, \mathbb{E}$, I will check the commutativity of



instantiated at every $A \in !\mathbb{A}$ and $e \in \mathbb{E}$.

To this end, for $B \in !\mathbb{B}$, $C \in !\mathbb{C}$, $D \in !\mathbb{D}$, $s \in S(D)(e)$, $\langle C_d \in !\mathbb{C}\rangle_{d\in D}$, $\langle r_d \in R(C_d)(D_{@d})\rangle_{d\in D}$, $\gamma : \bigotimes_{d\in D} C_d \longrightarrow C$ in $!\mathbb{C}$, $\langle B_c \in !\mathbb{B}\rangle_{c\in C}$, $\langle q_c \in Q(B_c)(C_{@c})\rangle_{c\in C}$, $\beta : \bigotimes_{c\in C} B_c \longrightarrow B$ in $!\mathbb{B}$, $\langle A_b \in !\mathbb{A}\rangle_{b\in B}$, $\langle p_b \in P(A_b)(B_{@b})\rangle_{b\in B}$, $\alpha : \bigotimes_{b\in B} A_b \longrightarrow A$ in $!\mathbb{A}$, $A \in !\mathbb{A}$, and $e \in \mathbb{E}$, we proceed to evaluate the above two composites at

$$\begin{aligned} &[\ B\ , \\ &\quad [C\ ,\ [\ D\ ,\ s\ ,\ \langle C_d\rangle_{d\in D}\ ,\ \langle r_d\rangle_{d\in D}\ ,\ \gamma\ ]\ ,\ \langle B_c\rangle_{c\in C}\ ,\ \langle q_c\rangle_{c\in C}\ ,\ \beta\ ]\ , \\ &\quad \langle A_b\rangle_{b\in B}\ ,\ \langle p_b\rangle_{b\in B}\ ,\ \alpha\ ] \end{aligned} \tag{26}$$

in $(((S \circ R) \circ Q) \circ P)(A)(e)$.

The evaluation of the left-hand-side composite is as follows

$$(26)\quad \xmapsto{a_{S\circ R,Q,P}} \left[\ C\ ,\ [\ D\ ,\ s\ ,\ \langle C_d\rangle_{d\in D}\ ,\ \langle r_d\rangle_{d\in D}\ ,\ \gamma\ ]\ ,\ \langle u_c\rangle_{c\in C}\ ,\ \alpha'\ \right]$$

where

$$u_c \ = \ \Big[\ B_c\ ,\ q_c\ ,\ \big\langle A_{\beta(c.b)}\big\rangle_{b\in B_c}\ ,\ \big\langle P(A_{\beta(c.b)})(\beta_{@(c.b)})(p_{\beta(c.b)})\big\rangle_{b\in B_c}\ ,\ \mathrm{id}\ \Big]$$

$$\alpha' \ = \ \Big(\big(\bigotimes_{c\in C}\bigotimes_{b\in B_c} A_{\beta(c.b)}\big) = \big(\bigotimes_{b\in(\bigotimes_{c\in C} B_c)} A_{\beta b}\big) \cong \big(\bigotimes_{b\in B} A_b\big) \xrightarrow{\alpha} A\Big)$$

$$\xmapsto{a_{S,R,Q\circ P}} \Big[\ D\ ,\ s\ ,\ \Big\langle \textstyle\bigotimes_{c\in C_d}\bigotimes_{b\in B_{\gamma(d.c)}} A_{\beta(\gamma(d.c).b)}\Big\rangle_{d\in D}\ ,\ \Big\langle\ \big[\ C_d\ ,\ r_d\ ,\ \big\langle \textstyle\bigotimes_{b\in B_{\gamma(d.c)}} A_{\beta(\gamma(d.c).b)}\big\rangle_{c\in C_d}\ ,\ \langle \nu_c\rangle_{c\in C_d}\ ,\ \mathrm{id}\ \big]\ \Big\rangle_{d\in D}\ ,\ \alpha''\ \Big] \tag{27}$$

where

$$\begin{aligned} \nu_c \ &= \ (Q\circ P)\big(\textstyle\bigotimes_{b\in B_{\gamma(d.c)}} A_{\beta(\gamma(d.c).b)}\big)(\gamma_{@(c.d)})(u_{\gamma(d.c)}) \\ &= \ \Big[\ B_{\gamma(d.c)}\ ,\ Q(\mathrm{id})(\gamma_{@(d.c)})(q_{\gamma(d.c)})\ ,\ \big\langle A_{\beta(\gamma(d.c).b)}\big\rangle_{b\in B_{\gamma(d.c)}}\ ,\\ &\qquad \big\langle P(\mathrm{id})(\beta_{@(\gamma(d.c).b)})(p_{\beta(\gamma(d.c).b)})\big\rangle_{b\in B_{\gamma(d.c)}}\ ,\ \mathrm{id}\ \Big] \end{aligned}$$

and

$$\begin{aligned} \alpha'' \ = \ \Big(\ &\big(\textstyle\bigotimes_{d\in D}\bigotimes_{c\in C_d}\bigotimes_{b\in B_{\gamma(d.c)}} A_{\beta(\gamma(d.c).b)}\big) \\ &= \big(\textstyle\bigotimes_{c\in(\bigotimes_{d\in D} C_d)}\bigotimes_{b\in B_{\gamma c}} A_{\beta(\gamma(c).b)}\big) \\ &\cong \big(\textstyle\bigotimes_{c\in C}\bigotimes_{b\in B_c} A_{\beta(c.d)}\big) \xrightarrow{\alpha'} A\ \Big) \end{aligned}$$

whilst the evaluation of the right-hand-side composite is as follows

$$
\begin{aligned}
(26) \xmapsto{a_{S,R,Q}\circ \mathrm{id}_P} \quad & \Big[\ B\ , \\
& \quad \Big[\ D\ ,\ s\ , \big\langle \textstyle\bigotimes_{c\in C_d} B_{\gamma(d.c)} \big\rangle_{d\in D}\ ,\ \langle x_d\rangle_{d\in D}\ ,\ \beta'\ \Big]\ , \\
& \quad \langle A_b\rangle_{b\in B}\ ,\ \langle p_b\rangle_{b\in B}\ ,\ \alpha\ \Big]
\end{aligned}
$$

where

$$
\begin{aligned}
x_d = \Big[\ & C_d\ ,\ r_d\ , \\
& \big\langle B_{\gamma(d.c)} \big\rangle_{c\in C_d}\ ,\ \big\langle Q(B_{\gamma(d.c)})(\gamma_{@(d.c)})(q_{\gamma(d.c)}) \big\rangle_{c\in C_d}\ , \\
& \mathrm{id}\ \Big]
\end{aligned}
$$

and

$$
\beta' = \Big(\big(\bigotimes_{d\in D} \bigotimes_{c\in C_d} B_{\gamma(c.d)} \big) = \big(\bigotimes_{c\in (\bigotimes_{d\in D} C_d)} B_{\gamma c} \big) \cong \big(\bigotimes_{c\in C} B_c \big) \xrightarrow{\beta} B \Big)
$$

$$
\begin{aligned}
\xmapsto{a_{S,R\circ Q,P}} \quad & \Big[\ D\ ,\ s\ , \\
& \quad \big\langle \textstyle\bigotimes_{b\in (\bigotimes_{c\in C_d} B_{\gamma(d.c)})} A_{\beta'(d.b)} \big\rangle_{d\in D}\ , \\
& \quad \Big\langle\ \Big[\ \textstyle\bigotimes_{c\in C_d} B_{\gamma(d.c)}\ ,\ x_d\ , \\
& \qquad\quad \big\langle A_{\beta'(d.b)} \big\rangle_{b\in (\bigotimes_{c\in C_d} B_{\gamma(d.c)})}\ , \\
& \qquad\quad \Big\langle P(\mathrm{id})(\beta'_{@(d.b)})(p_{\beta'(d.b)}) \Big\rangle_{b\in (\bigotimes_{c\in C_d} B_{\gamma(d.c)})}\ , \\
& \qquad\quad \mathrm{id}\ \Big]\ \Big\rangle_{d\in D}\ , \\
& \quad \alpha'\ \Big]
\end{aligned}
$$

where

$$
\begin{aligned}
\alpha' = \Big(\ & \big(\textstyle\bigotimes_{d\in D} \bigotimes_{c\in C_d} \bigotimes_{b\in B_{\gamma(d.c)}} A_{\beta(\gamma(d.c).b)} \big) \\
& = \big(\textstyle\bigotimes_{b\in (\bigotimes_{d\in D} \bigotimes_{c\in C_d} B_{\gamma(d.c)})} A_{\beta' b} \big) \\
& \cong \big(\textstyle\bigotimes_{b\in B} A_b \big) \xrightarrow{\alpha} A\ \Big)
\end{aligned}
$$

$$
= \quad \Big[\ D\ ,\ s\ ,\ \Big\langle \bigotimes_{c\in C_d} \bigotimes_{b\in B_{\gamma(d.c)}} A_{\beta(\gamma(d.c).b)} \Big\rangle_{d\in D}\ ,\ \langle y_d\rangle_{d\in D}\ ,\ \alpha'\ \Big]
$$

where

$$
\begin{aligned}
y_d = \Big[\ & \textstyle\bigotimes_{c\in C_d} B_{\gamma(d.c)}\ ,\ x_d\ , \\
& \big\langle A_{\beta(\gamma(d.c).b)} \big\rangle_{(c.b)\in \left(\bigotimes_{c\in C_d} B_{\gamma(d.c)}\right)}\ , \\
& \big\langle P(\mathrm{id})(\beta_{@(\gamma(d.c).b)})(p_{\beta(\gamma(d.c).b)}) \big\rangle_{(c.b)\in \left(\bigotimes_{c\in C_d} B_{\gamma(d.c)}\right)}\ , \\
& \mathrm{id}\ \Big]
\end{aligned}
$$

$$
\xmapsto{\mathrm{id}_S\circ a_{R,Q,P}} \quad \Big[\ D\ ,\ s\ ,\ \Big\langle \bigotimes_{c\in C_d} \bigotimes_{b\in B_{\gamma(d.c)}} A_{\beta(\gamma(d.c).b)} \Big\rangle_{d\in D}\ ,\ \langle z_d\rangle_{d\in D}\ ,\ \alpha'\ \Big]
$$

where (28)

$$
\begin{aligned}
z_d = \Big[\ & C_d\ ,\ r_d\ , \\
& \big\langle \textstyle\bigotimes_{b\in B_{\gamma(d.c)}} A_{\beta(\gamma(d.c).b)} \big\rangle_{c\in C_d}\ , \\
& \Big\langle\ \Big[\ B_{\gamma(d.c)}\ ,\ Q(\mathrm{id})(\gamma_{@(d.c)})(q_{\gamma(d.c)})\ , \\
& \qquad \big\langle A_{\beta(\gamma(d.c).b)} \big\rangle_{b\in B_{\gamma(d.c)}}\ , \\
& \qquad \big\langle P(\mathrm{id})(\beta_{@(\gamma(d.c).b)})(p_{\beta(\gamma(d.c).b)}) \big\rangle_{b\in B_{\gamma(d.c)}}\ , \\
& \qquad \mathrm{id}\ \Big]\ \Big\rangle_{c\in C_d}\ , \\
& \textstyle\bigotimes_{c\in C_d} \bigotimes_{b\in B_{\gamma(d.c)}} \mathrm{id}_{A_{\beta(\gamma(d.c).b)}}\ \Big]
\end{aligned}
$$

and, since (27) and (28) are equal, we are done.