Algebraic Type Theory

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1. The data for a *category-with-families* is that of a small category $\mathbb{C}$ together with a functor $\mathbb{C}^\circ \longrightarrow \boldsymbol{\mathcal{F}am}$.

Since $\boldsymbol{\mathcal{F}am} = \boldsymbol{\mathcal{S}et}^{\rightarrow}$, this is to give a small category $\mathbb{C}$ together with an indexed family $\tau : T \longrightarrow S$ in $\widehat{\mathbb{C}}$. Here T intuitively stands for *terms* and S for *types* (or *sorts*), both in context. The map τ assigns types to terms.

Yet another view of this structure is given by a small category $\mathbb{C}$ and presheaves $S \in \widehat{\mathbb{C}}$ and $\mathcal{T} \in \widehat{\mathbb{S}}$, where $\mathbb{S}$ is the category of elements of S.

One advantage of these alternative points of view is that they allow one to discuss the structure of contexts and types independently to that of terms.

2. Notation. For $\Gamma \in \mathbb{C}$, I write $\Gamma \vdash A$ to mean that A is an element of $S(\Gamma)$. Further, for $\rho : \Delta \longrightarrow \Gamma$ in $\mathbb{C}$ and $\Gamma \vdash A$, I write $A\rho$ for $S(\rho)(A)$ so that $\Delta \vdash A\rho$. Thus the projection functor $p : \mathbb{S} \longrightarrow \mathbb{C}$ maps $\Gamma \vdash A$ to Γ.

The notation $\Gamma \vdash t : A$ states that $\Gamma \vdash A$ and that t is an element of $\mathcal{T}(\Gamma \vdash A)$. In this situation, for $\rho : \Delta \longrightarrow \Gamma$ in $\mathbb{C}$, I write $t[\rho]$ for $\mathcal{T}\big(\rho : (\Delta \vdash A\rho) \longrightarrow (\Gamma \vdash A)\big)(t)$, so that $\Delta \vdash t[\rho] : A\rho$.

3. Let $\mathbb{C}$ be a small category and S a presheaf in $\widehat{\mathbb{C}}$. The intended meaning that we will have for this structure is that of $\mathbb{C}$ being a category of *contexts and renamings*, and S a presheaf of *types in context*. However, this can only be considered so if:

(*i*) the model is non-trivial in that there is an empty context;

(*ii*) types can be internalised as contexts, and these come equipped with structure for context manipulation.

These requirements can be axiomatised by specifying:

(*i*) a terminal object $[\,]$ in $\mathbb{C}$;

(*ii*) a functor

$$\begin{array}{ccc} \mathbb{S} & \xrightarrow{\ \pi\ } & \mathbb{C}^{\rightarrow} \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle \mathrm{cod}} & \\ & \mathbb{C} & \end{array}$$

whose image consists of pullback squares:

$$\begin{array}{ccc} [\Delta, A\rho] & \xrightarrow{\ \rho^*\ } & [\Gamma, A] \\ {\scriptstyle \pi_{\Delta \vdash A\rho}}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{\Gamma \vdash A}} \\ \Delta & \xrightarrow[\ \rho\ ]{} & \Gamma \end{array} \qquad (\Gamma \vdash A, \rho : \Delta \longrightarrow \Gamma \text{ in } \mathbb{C})$$

I will refer to such $(\mathbb{C}, S, [\,], \pi)$ as *(dependently-typed) context structures.*

4. Example. For S terminal in $\widehat{\mathbb{C}}$, a context structure $(\mathbb{C}, S, [\,], \pi)$ amounts to giving a terminal object $[\,]$ in $\mathbb{C}$ and an object $c \in \mathbb{C}$ together with a product diagram

$$\begin{array}{ccc} & [\Gamma, c] & \\ {\scriptstyle \pi_\Gamma}\swarrow & & \searrow{\scriptstyle \Gamma^*} \\ \Gamma & & c \end{array}$$

for every $\Gamma \in \mathbb{C}$.

5. Conceptually, it will be crucial to work not only with the category of contexts $\mathbb{C}$, but also with the categories of *indexed contexts* $\mathbb{C}_{/\Gamma}$ for all $\Gamma \in \mathbb{C}$. For instance, note that the consideration of variables of type A for $\Gamma \vdash A$ only makes sense in Γ-indexed contexts.

6. It is convenient to organise the categories of indexed contexts as the indexed category

$$\mathbb{C} \longrightarrow \boldsymbol{Cat} : \Gamma \longmapsto \mathbb{C}_{/\Gamma} \ ,$$

and derive from this the indexed category

$$\mathbb{S}^\circ \longrightarrow \boldsymbol{CAT} : (\Gamma \vdash A) \longmapsto \widehat{\mathbb{C}_{/\Gamma}} \ .$$

I let $\mathscr{C}$ be the induced category of elements, with objects $(\Gamma \vdash A, P \in \widehat{\mathbb{C}_{/\Gamma}})$ and morphisms $(\Gamma \vdash A, P) \longrightarrow (\Delta \vdash B, Q)$ given by maps $\rho : \Gamma \longrightarrow \Delta$ in $\mathbb{C}$ such that $B\rho = A$ and $\widehat{\mathbb{C}_{/\rho}}(Q) = P$.

7. The type-indexed variable-set of terms $\mathcal{T} : \mathbb{S}^\circ \longrightarrow \boldsymbol{Set}$ carries enough information to yield a type-indexed variable-presheaf of terms

$$\mathscr{T} : \mathbb{S} \longrightarrow \mathscr{C} : (\Gamma \vdash A) \longmapsto \big(\Gamma \vdash A\,,\ \mathscr{T}_A : (\rho : \Delta \to \Gamma) \mapsto \mathcal{T}(\Delta \vdash A\rho)\big)$$

over $\mathbb{S}$, and hence also over $\mathbb{C}$. In fact, we have the following result.

Proposition. For $S \in \widehat{\mathbb{C}}$,

$$\boldsymbol{CAT}[\mathbb{S}^\circ, \boldsymbol{Set}] \ \cong\ \boldsymbol{CAT}_{/\mathbb{C}}[\mathbb{S}, \mathscr{C}] \ .$$

8. Let $(\mathbb{C}, S, [\,], \pi)$ be a context structure. For $\Gamma \vdash A$, I define the presheaf of *variables* of type A, $\mathscr{V}_A \in \widehat{\mathbb{C}_{/\Gamma}}$, as the representable $\mathcal{Y}(\pi : [\Gamma, A] \longrightarrow \Gamma)$. This definition yields a functor $\mathscr{V} : \mathbb{S} \longrightarrow \mathscr{C}$ over $\mathbb{C}$, and hence a presheaf of variables $\mathcal{V}$ in $\widehat{\mathbb{S}}$, with $\mathcal{V}(\Gamma \vdash A) = \mathbb{C}_{/\Gamma}(\mathrm{id}_\Gamma, \pi_{\Gamma \vdash A})$ the set of sections of $\pi : [\Gamma, A] \longrightarrow \Gamma$.

Proposition. For $\Gamma \vdash A$, the points $\mathscr{V}_A \longrightarrow \mathscr{T}_A$ in $\widehat{\mathbb{C}_{/\Gamma}}$ are in bijective correspondence with the elements of $\mathcal{T}([\Gamma, A] \vdash A\pi)$.

Corollary. The points $\mathcal{V} \longrightarrow \mathcal{T}$ in $\widehat{\mathbb{S}}$ are in bijective correspondence with type-indexed families $\big\{x_A \in \mathcal{T}([\Gamma, A] \vdash A\pi)\big\}_{\Gamma \vdash A}$ such that $x_A[\rho] = x_{A\rho}$ for all $\Gamma \vdash A$ and $\rho : \Delta \longrightarrow \Gamma$ in $\mathbb{C}$.

9. We have the following *fundamental lemma.*

Lemma. For $P \in \widehat{\mathbb{C}_{/\Gamma}}$,

$$P^{\mathscr{V}_A}(\Delta \longrightarrow \Gamma) \ \cong\ P([\Delta, A\rho] \longrightarrow \Gamma) \ .$$

Thus, intuitively, the elements of type $P^{\mathcal{V}_A}$ in context Δ are the elements of type P in the extension of the context Δ with a variable of type A.

Corollary. For $F \in \widehat{\mathbb{S}}$,

$$F^{\mathcal{V}}(\Gamma \vdash A) \;\cong\; F([\Gamma, A] \vdash A\pi)\ .$$

The lemma is a *local* version of the *global* corollary.

10. The presheaf of types $S \in \widehat{\mathbb{C}}$ can be locally regarded as a presheaf

$$S_\Gamma = \big((\mathbb{C}_{/\Gamma})^\circ \longrightarrow \mathbb{C}^\circ \xrightarrow{\ S\ } \boldsymbol{Set}\big) \text{ in } \widehat{\mathbb{C}_{/\Gamma}}\ ,$$

or globally regarded as a presheaf

$$\mathcal{S} = (\mathbb{S}^\circ \longrightarrow \mathbb{C}^\circ \xrightarrow{\ S\ } \boldsymbol{Set}) \text{ in } \widehat{\mathbb{S}}\ .$$

11. An operation $\mathsf{O} : S^n \longrightarrow S$ in $\widehat{\mathbb{C}}$ amounts to giving mappings

$$\big(\Gamma \vdash A_1, \ldots, \Gamma \vdash A_n\big) \longmapsto \big(\Gamma \vdash \mathsf{O}(A_1, \ldots, \vdash A_n)\big)$$

such that $\mathsf{O}(A_1, \ldots, A_n)\rho = \mathsf{O}(A_1\rho, \ldots, A_n\rho)$ for all $\rho : \Delta \longrightarrow \Gamma$ in $\mathbb{C}$.

In particular, the case $n = 0$ reduces to specifying a type in the empty context.

12. Locally, for $\Gamma \vdash A$, an operation

$$\Pi : {S_\Gamma}^{\mathcal{V}_A} \longrightarrow S_\Gamma \text{ in } \widehat{\mathbb{C}_{/\Gamma}} \tag{1}$$

amounts to giving mappings

$$\big([\Delta, A\rho] \vdash B\big) \longmapsto \big(\Delta \vdash \Pi^\rho_A(B)\big) \text{ for all } \rho : \Delta \longrightarrow \Gamma \text{ in } \mathbb{C}$$

such that $\big(\Pi^\rho_A(B)\big)\delta = \Pi^{\rho\delta}_A(B\delta^*)$ for all $\delta : \Delta' \longrightarrow \Delta$ in $\mathbb{C}$.

Globally, an operation

$$\Pi : \mathcal{S}^{\mathcal{V}} \longrightarrow \mathcal{S} \text{ in } \widehat{\mathbb{S}}$$

amounts to giving mappings

$$\big([\Gamma, A] \vdash B\big) \mapsto \big(\Gamma \vdash \Pi_A(B)\big) \text{ for all } \Gamma \vdash A$$

such that $(\Pi_A(B))\rho = \Pi_{A\rho}(B\rho^*)$ for all $\rho : \Delta \longrightarrow \Gamma$ in $\mathbb{C}$.

13. Recall that every functor $f : \mathbb{X} \longrightarrow \mathbb{Y}$ between small categories induces the adjoint situation

$$\widehat{\mathbb{X}} \;\begin{array}{c} \xrightarrow{\ f_*\ } \\ \top \\ \xleftarrow{\ \widehat{f}\ } \\ \top \\ \xrightarrow[\ f_!\ ]{} \end{array}\; \widehat{\mathbb{Y}}$$

Thus, for $\Gamma \vdash A$, the adjunction

$$\mathbb{C}_{/\Gamma} \;\begin{array}{c} \xrightarrow{\ \pi^*\ } \\ \top \\ \xleftarrow[\ \mathbb{C}_{/\pi}\ ]{} \end{array}\; \mathbb{C}_{/[\Gamma, A]}$$

induces the adjoint situation

$$\widehat{\mathbb{C}_{/\Gamma}} \quad \begin{array}{c} \xrightarrow{\quad (\pi^*)_* \quad} \\ \top \\ \xleftarrow{\quad \delta_A \quad} \\ \top \\ \xrightarrow{\quad \epsilon_A \quad} \\ \top \\ \xleftarrow{\quad (\mathbb{C}_{/\pi})_! \quad} \end{array} \quad \widehat{\mathbb{C}_{/[\Gamma,A]}} \tag{2}$$

where $\delta = \widehat{\pi^*} = (\mathbb{C}_{/\pi})_*$ and $\epsilon = \widehat{\mathbb{C}_{/\pi}} = (\pi^*)_!$ are given by

$$\begin{array}{rcl} (\delta_A Q)(\rho : \Delta \longrightarrow \Gamma) & = & Q(\rho^* : [\Delta, A\rho] \longrightarrow [\Gamma, A]) \\ (\epsilon_A P)(\rho : \Delta \longrightarrow [\Gamma, A]) & = & P(\pi\rho : \Delta \longrightarrow \Gamma) \end{array}$$

Thus, we have the following *decomposition lemma.*

LEMMA. For $\Gamma \vdash A$, the monad $(-)^{\mathscr{V}_A}$ on $\widehat{\mathbb{C}_{/\Gamma}}$ is induced by the adjunction $\epsilon_A \dashv \delta_A : \widehat{\mathbb{C}_{/[\Gamma,A]}} \longrightarrow \widehat{\mathbb{C}_{/\Gamma}}$.

14. For $\Gamma \vdash A$ and $[\Gamma, A] \vdash B$, an operation

$$\lambda : \delta_A(\mathscr{T}_B) \longrightarrow \mathscr{T}_{\Pi_A(B)} \text{ in } \widehat{\mathbb{C}_{/\Gamma}} \tag{3}$$

amounts to giving mappings

$$\big([\Gamma, A] \vdash t : B\big) \longmapsto \big(\Gamma \vdash \lambda(t) : \Pi_A(B)\big)$$

such that

$$\big(\lambda(t)\big)[\rho] = \lambda\big(t[\rho^*]\big)$$

for all $\rho : \Delta \longrightarrow \Gamma$ in $\mathbb{C}$.

15. Through the isomorphism

$$\widehat{\mathbb{C}_{/\Delta}} \cong \widehat{\mathbb{C}}_{/\mathcal{Y}\Delta}$$

the adjoint situation (2) amounts to the following one

$$\widehat{\mathbb{C}}_{/\mathcal{Y}\Gamma} \quad \begin{array}{c} \xrightarrow{\qquad\quad} \\ \top \\ \xleftarrow{\quad \prod_A \quad} \\ \top \\ \xrightarrow{\quad (\mathcal{Y}\pi)^* \quad} \\ \top \\ \xleftarrow{\quad \sum_A \quad} \end{array} \quad \widehat{\mathbb{C}}_{/\mathcal{Y}[\Gamma,A]}$$

conceptually explaining δ_A as exponentiation; in fact exponentiation by the *atom* A, as indicated by the existence of a right adjoint to $\prod_A$.

16. As a further consequence, the algebraic structure internalises within $\widehat{\mathbb{C}}$.
For instance,

1. an operation as in (1) amounts to a map $\prod_A \big(\mathcal{Y}_{[\Gamma,A]}{}^* S\big) \longrightarrow S$ in $\widehat{\mathbb{C}}$, and

2. an operation as in (3) amounts to a map $\prod_A(\tau_B) \longrightarrow T$ in $\widehat{\mathbb{C}}$ such that

$$\begin{array}{ccc} \prod_A(\tau_B) & \longrightarrow & T \\ \downarrow & & \downarrow{\scriptstyle \tau} \\ \mathcal{Y}\Gamma & \xrightarrow[\Pi_A(B)]{} & S \end{array}$$

where, for $\Delta \vdash C$, the diagram

$$\begin{array}{ccc} \tau_C & \longrightarrow & T \\ \downarrow & & \downarrow{\scriptstyle \tau} \\ \mathcal{Y}\Delta & \xrightarrow[C]{} & S \end{array}$$

is a pullback.

17. Term-in-term substitution is given by a natural family of operations

$$\prod_A \left(\mathcal{Y}_{[\Gamma,A]}{}^*T\right) \underset{\mathcal{Y}\Gamma}{\times} \tau_A \longrightarrow T$$

forming a *substitution algebra* with neutral element given by a natural family of operations for *variables*

$$\begin{array}{ccc} \mathcal{Y}_{[\Gamma,A]} & \longrightarrow & T \\ & {\scriptstyle A\pi} \searrow \quad \swarrow {\scriptstyle \tau} & \\ & S & \end{array}$$

Term-in-type substitution is given by a natural family of *actions*

$$\prod_A \left(\mathcal{Y}_{[\Gamma,A]}{}^*S\right) \underset{\mathcal{Y}\Gamma}{\times} \tau_A \longrightarrow S$$

These substitution structures are required to be compatible in the sense of satisfying the following typing discipline:

$$\begin{array}{ccc} \prod_A \left(\mathcal{Y}_{[\Gamma,A]}{}^*T\right) \underset{\mathcal{Y}\Gamma}{\times} \tau_A & \longrightarrow & T \\ \downarrow & & \downarrow \\ \prod_A \left(\mathcal{Y}_{[\Gamma,A]}{}^*S\right) \underset{\mathcal{Y}\Gamma}{\times} \tau_A & \longrightarrow & S \end{array}$$