Algebraic Simple Type Theory

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1 Algebraic Simple Type Theory

Outline of an Algebraic Theory of Simple Types.

1.1 Signatures

Example 1.1. The signature of the simply-typed λ-calculus with sums given below will be our running example.

Type operators $\mathsf{Unit}, \mathsf{Empty} : \to *$

$\mathsf{Prod}, \mathsf{Fun}, \mathsf{Sum} : *, * \to *$

Term operators $\rhd\, \mathsf{u} : \mathsf{Unit}$

$$\begin{aligned}
&\alpha, \beta : * \rhd \mathsf{pair} : \alpha, \beta \to \mathsf{Prod}(\alpha, \beta)\\
&\alpha, \beta : * \rhd \mathsf{proj}_1 : \mathsf{Prod}(\alpha, \beta) \to \alpha\\
&\alpha, \beta : * \rhd \mathsf{proj}_2 : \mathsf{Prod}(\alpha, \beta) \to \beta\\
&\alpha, \beta : * \rhd \mathsf{app} : \mathsf{Fun}(\alpha, \beta), \alpha \to \beta\\
&\alpha, \beta : * \rhd \mathsf{abs} : (\alpha)\beta \to \mathsf{Fun}(\alpha, \beta)\\
&\alpha, \beta : * \rhd \mathsf{in}_1 : \alpha \to \mathsf{Sum}(\alpha, \beta)\\
&\alpha, \beta : * \rhd \mathsf{in}_2 : \beta \to \mathsf{Sum}(\alpha, \beta)\\
&\alpha, \beta : * \rhd \mathsf{case} : \mathsf{Sum}(\alpha, \beta), (\alpha)\gamma, (\beta)\gamma \to \beta
\end{aligned}$$

Remark. Simply-typed theories with type-binding type operators, as that of recursive types below, are at the moment outside the scope of the note, but may be incorporated.

Type operator	$\mathsf{Rec} : (*)* \to *$
Term operators	$\mathrm{T} : [*]* \rhd \mathsf{intro} : \mathrm{T}\big[\mathsf{Rec}(\alpha.\mathrm{T}[\alpha])\big] \to \mathsf{Rec}\big(\alpha.\mathrm{T}[\alpha]\big)$ $\mathrm{T} : [*]* \rhd \mathsf{elim} : \mathsf{Rec}\big(\alpha.\mathrm{T}[\alpha]\big) \to \mathrm{T}\big[\mathsf{Rec}(\alpha.\mathrm{T}[\alpha])\big]$

1.2 Algebraic models

1.2.1 Types

The algebraic structure of types is as in universal algebra, consisting of a set equipped with a polynomial algebra structure induced by the type signature. For our running example, it is

$$\big[[\![\mathsf{Unit}]\!], [\![\mathsf{Empty}]\!], [\![\mathsf{Prod}]\!], [\![\mathsf{Fun}]\!], [\![\mathsf{Sum}]\!]\big] : \coprod_{\mathsf{Unit},\mathsf{Empty}} 1 + \coprod_{\mathsf{Prod},\mathsf{Fun},\mathsf{Sum}} S^2 \to S$$

1.2.2 Terms

Definition 1.2. A *cartesian typed-context structure* consists of a small category $\mathbb{C}$ and a set S together with a specified terminal object ϵ in $\mathbb{C}$ and, for every $\Gamma \in \mathbb{C}$ and $\sigma \in S$, a product diagram

$$\begin{tikzcd} & \Gamma \cdot \sigma \arrow[dl, "\pi_1^{\Gamma,\sigma}"'] \arrow[dr, "\pi_2^{\Gamma,\sigma}"] & \\ \Gamma & & \langle \sigma \rangle \end{tikzcd}$$

where $\langle \sigma \rangle = \epsilon \cdot \sigma$.

Notation. We use the same notation for a set and for the presheaf that is constantly that set; analogously, we do so for a function and the natural transformation that is constantly that function.

Definition 1.3. A *term-typing structure* (T, τ) on a cartesian typed-context structure $(\mathbb{C}, S)$ is a presheaf $T \in \widehat{\mathbb{C}}$ together with a natural transformation $\tau : T \to S$ in $\widehat{\mathbb{C}}$.

The algebraic structure of terms is given along the lines of [**?**], that extended the approach of [**?**] from the mono-sorted to the multi-sorted setting. This is however expressed here in the language of polynomial diagrams. We are thus moving from the indexed viewpoint to the fibred one. Technically, via the equivalence

$$\widehat{\mathbb{C}}/S \simeq \widehat{\mathbb{C}}^S$$

mapping $(X, X \to S)$ to $(X_s)_{\sigma \in S}$ where

$$\begin{tikzcd} X_\sigma \arrow[r] \arrow[d] \arrow[dr, phantom, "\mathrm{pb}"] & X \arrow[d] \\ 1 \arrow[r, "\sigma"'] & S \end{tikzcd}$$

and $(X_s)_{\sigma \in S}$ to $\coprod_{\sigma \in S} X_s$ equipped with the canonical projection.

The polynomial diagram D_{u} in $\widehat{\mathbb{C}}$ is

$$\begin{tikzcd} S & 0 \arrow[l] \arrow[r] & 1 \arrow[r, "{[\![\mathsf{Unit}]\!]}"] & S \end{tikzcd}$$

An algebra $P_{\mathsf{u}}(T,\tau) \to (T,\tau)$ in $\widehat{\mathbb{C}}/S$ amounts to giving

$$\begin{tikzcd}
1 \arrow[r, "\llbracket \mathsf{u} \rrbracket"] \arrow[d, equal] & T \arrow[d, "\tau"] \\
1 \arrow[r, "\llbracket \mathsf{Unit} \rrbracket"'] & S
\end{tikzcd}$$

The polynomial diagram D_{pair} in $\widehat{\mathbb{C}}$ is

$$\begin{tikzcd}
S + S & (S \times S) + (S \times S) \arrow[l, "\pi 1 + \pi 2"'] \arrow[r, "\nabla_2"] & S \times S \arrow[r, "\llbracket \mathsf{Prod} \rrbracket"] & S
\end{tikzcd}$$

An algebra $P_{\mathsf{pair}}(T+T, \tau+\tau) \to (T,\tau)$ in $\widehat{\mathbb{C}}/S$ amounts to giving

$$\begin{tikzcd}
T \times T \arrow[r, "\llbracket \mathsf{pair} \rrbracket"] \arrow[d, "\tau \times \tau"'] & T \arrow[d, "\tau"] \\
S \times S \arrow[r, "\llbracket \mathsf{Prod} \rrbracket"'] & S
\end{tikzcd}$$

The polynomial diagram D_{proj_i} in $\widehat{\mathbb{C}}$ is

$$\begin{tikzcd}
S & S \times S \arrow[l, "\llbracket \mathsf{Prod} \rrbracket"'] \arrow[r, "\mathrm{id}"] & S \times S \arrow[r, "\pi_i"] & S
\end{tikzcd}$$

An algebra $P_{\mathsf{proj}_i}(T,\tau) \to (T,\tau)$ in $\widehat{\mathbb{C}}/S$ amounts to giving

$$\begin{tikzcd}
T_{\llbracket \mathsf{Prod} \rrbracket} \arrow[r, "\llbracket \mathsf{proj}_i \rrbracket"] \arrow[d] & T \arrow[d, "\tau"] & & T \arrow[d, "\tau"'] \arrow[dr, phantom, "\mathrm{pb}"] & T_{\llbracket \mathsf{Prod} \rrbracket} \arrow[l] \arrow[d] \\
S \times S \arrow[r, "\pi_i"'] & S & \text{where} & S & S \times S \arrow[l, "\llbracket \mathsf{Prod} \rrbracket"]
\end{tikzcd}$$

The polynomial diagram D_{app} in $\widehat{\mathbb{C}}$ is

$$\begin{tikzcd}
S + S & (S \times S) + (S \times S) \arrow[l, "\llbracket \mathsf{Fun} \rrbracket + \pi_1"'] \arrow[r, "\nabla_2"] & S \times S \arrow[r, "\pi_2"] & S
\end{tikzcd}$$

An algebra $P_{\mathsf{app}}(T+T, \tau+\tau) \to (T,\tau)$ in $\widehat{\mathbb{C}}/S$ amounts to giving

$$\begin{tikzcd}
\coprod_{\alpha,\beta \in S} T_{\llbracket \mathsf{Fun} \rrbracket(\alpha,\beta)} \times T_\alpha \arrow[r, "\llbracket \mathsf{app} \rrbracket"] \arrow[d] & T \arrow[d, "\tau"] \\
S \times S \arrow[r, "\pi_2"'] & S
\end{tikzcd}$$

Definition 1.4. We let $V = \coprod_{\alpha \in S} \mathrm{y}\langle \alpha \rangle$ in $\widehat{\mathbb{C}}$.

Proposition 1.5. *The canonical projection* $\nu : V \to S$ *in* $\widehat{\mathbb{C}}$ *is representable:*

$$\begin{tikzcd}
\mathrm{y}(\Gamma \cdot \sigma) \arrow[r, "\iota_\sigma \circ \pi_2^{\Gamma,\sigma}"] \arrow[d, "\pi_1^{\Gamma,\sigma}"'] \arrow[dr, phantom, "pb"] & V \arrow[d, "\nu"] \\
\mathrm{y}(\Gamma) \arrow[r, "\sigma"'] & S
\end{tikzcd}$$

More generally, $\nu^n : V^n \to S^n$ *in* $\widehat{\mathbb{C}}$ *is representable:*

$$\begin{tikzcd}
\mathrm{y}(\Gamma \cdot \sigma_1 \cdot \ldots \cdot \sigma_n) \arrow[r] \arrow[d] \arrow[dr, phantom, "pb"] & \mathrm{y}(\Gamma \cdot \sigma_1) \times \cdots \times \mathrm{y}(\Gamma \cdot \sigma_n) \arrow[r] \arrow[d] \arrow[dr, phantom, "pb"] & V \times \cdots \times V \arrow[d, "\nu \times \cdots \times \nu"] \\
\mathrm{y}(\Gamma) \arrow[r, "\Delta"'] \arrow[rr, bend right, "(\sigma_1, \ldots, \sigma_n)"'] & \mathrm{y}(\Gamma) \times \cdots \times \mathrm{y}(\Gamma) \arrow[r, "\sigma_1 \times \cdots \times \sigma_n"'] & S \times \cdots \times S
\end{tikzcd}$$

The polynomial diagram D_{abs} in $\widehat{\mathbb{C}}$ is

$$\begin{tikzcd}
S & V \times S \arrow[l, "\pi_2"'] \arrow[r, "\nu \times \mathrm{id}"] & S \times S \arrow[r, "\llbracket \mathsf{Fun} \rrbracket"] & S
\end{tikzcd}$$

An algebra $P_{\mathsf{abs}}(T, \tau) \to (T, \tau)$ in $\widehat{\mathbb{C}}/S$ amounts to giving

$$\begin{tikzcd}
\coprod_{\alpha,\beta \in S} (T_\beta)^{\mathrm{y}\langle\alpha\rangle} \arrow[r, "\llbracket \mathsf{abs} \rrbracket"] \arrow[d] & T \arrow[d, "\tau"] \\
S \times S \arrow[r, "\llbracket \mathsf{Fun} \rrbracket"'] & S
\end{tikzcd}$$

Remark. The calculation of $\Pi_{(\nu \times \mathrm{id})}(\mathrm{id} \times \tau : V \times T \to V \times S)$ uses that

$$\begin{tikzcd}
\mathrm{y}(\Gamma \cdot \alpha) \arrow[r, "\pi_1^{\Gamma,\beta}"] \arrow[d, "(\pi_2^{\Gamma,\alpha} \circ \iota_\alpha, \beta)"'] \arrow[dr, phantom, "\mathrm{pb}"] & \mathrm{y}(\Gamma) \arrow[d, "(\alpha,\beta)"] \\
V \times S \arrow[r, "\nu \times \mathrm{id}"'] & S \times S
\end{tikzcd}$$

and that $X^{\mathrm{y}\langle\sigma\rangle} \cong X(- \cdot \sigma)$.

The polynomial diagram D_{in_i} in $\widehat{\mathbb{C}}$ is

$$\begin{tikzcd}
S & S \times S \arrow[l, "\pi_i"'] \arrow[r, "\mathrm{id}"] & S \times S \arrow[r, "\llbracket \mathsf{Sum} \rrbracket"] & S
\end{tikzcd}$$

Algebras $P_{\mathsf{in}_i}(T, \tau) \to (T, \tau)$ in $\widehat{\mathbb{C}}/S$ amount to giving

$$\begin{tikzcd}
T \times S \arrow[r, "\llbracket \mathsf{in}_1 \rrbracket"] \arrow[d, "\tau \times \mathrm{id}"'] & T \arrow[d, "\tau"] & S \times T \arrow[r, "\llbracket \mathsf{in}_2 \rrbracket"] \arrow[d, "\mathrm{id} \times \tau"'] & T \arrow[d, "\tau"] \\
S \times S \arrow[r, "\llbracket \mathsf{Sum} \rrbracket"'] & S & S \times S \arrow[r, "\llbracket \mathsf{Sum} \rrbracket"'] & S
\end{tikzcd}$$

The polynomial diagram D_{case} in $\widehat{\mathbb{C}}$ is

$$\begin{tikzcd}
S & (S \times S) + (V \times S) + (V \times S) \arrow[l, "{[\llbracket \mathsf{Sum} \rrbracket, \pi_2, \pi_2]}"'] \arrow[r, "{[\mathrm{id}, \nu \times \mathrm{id}, \nu \times \mathrm{id}]}"] & S \times S \arrow[r, "\pi_2"] & S
\end{tikzcd}$$

1.2.3 Formal theory

An *A-sorted first-order arity* for a set A is an element of

$$\mathsf{ar}(A) = A^\star \times A \ \ ,$$

where $A^\star$ denotes the set of finite sequences on A. An operator o of arity $(\alpha_1 \dots \alpha_\ell, \alpha)$ is indicated as follows

$$\mathsf{o} : \alpha_1, \dots, \alpha_\ell \to \alpha \ \ .$$

An *A-sorted second-order arity* for a set A is an element of

$$\mathsf{ar}^2(A) = (A^\star \times A)^\star \times (A^\star \times A) \ \ .$$

An operator o of arity $\big((\alpha_1^1 \dots \alpha_{k_1}^1, \alpha_1) \dots (\alpha_1^\ell \dots \alpha_{k_n}^\ell, \alpha_\ell), (\beta_1 \dots \beta_k, \beta)\big)$ is indicated as follows

$$\mathsf{o} : (\alpha_1^1, \dots, \alpha_{k_1}^1)\alpha_1, \dots, (\alpha_1^\ell, \dots, \alpha_{k_\ell}^\ell)\alpha_\ell \to (\beta_1, \dots, \beta_k)\beta \ \ . \tag{1}$$

Its intended meaning is that of an operator of sort β parametrized by sorts $\beta_1, \dots, \beta_k$ that takes ℓ arguments where the i^{th} argument, of sort α_i, binds ℓ_i-variables of sorts $\alpha_1^i, \dots, \alpha_{\ell_i}^i$. In type-theoretic style this could be written as a rule as follows:

$$\frac{\Gamma, y_1 : \beta_1, \dots, y_k : \beta_k, x_1^i : \alpha_1^i, \dots, x_{k_i}^i : \alpha_{k_i}^i \vdash t_i : \alpha_i \quad (1 \le i \le \ell)}{\Gamma, y_1 : \beta_1, \dots, y_k : \beta_k \vdash \mathsf{o}\big(\ x_1^1 : \alpha_1^1 \dots x_{k_1}^1 : \alpha_{k_1}^1 . t_1 \ , \dots, \ x_1^\ell : \alpha_1^\ell \dots x_{k_\ell}^\ell : \alpha_{k_\ell}^\ell . t_\ell \ , \ y_1 \ , \dots, \ y_k \ \big) : \beta}$$

In most examples, including all the ones here, $k = 0$.

A *signature of types* Σ is specified by mono-sorted first-order operators. We let Σ^* be the free Σ-algebra construction in $\boldsymbol{\mathcal{S}et}$.[1]

A *signature of terms* over Σ is specified by $(\Sigma^*[n])$-sorted second-order operators for $n \in \mathbb{N}$ where $[n] = \{1, \dots, n\}$.

We give examples.

Example 1.6. The arrows of functional programming have signature

Type operator	$\mathsf{Arrow} : *, * \to *$
Term operators	$\alpha : *, \beta : * \ \triangleright \ \mathsf{arr} : \mathsf{Fun}(\alpha, \beta) \to \mathsf{Arrow}(\alpha, \beta)$ $\alpha_1, \alpha_2, \beta : * \triangleright \ \mathsf{first} : \mathsf{Arrow}(\alpha_1, \alpha_2) \to \mathsf{Arrow}\big(\mathsf{Prod}(\alpha_1, \beta), \mathsf{Prod}(\alpha_2, \beta)\big)$

Example 1.7. The λ-calculus has signature

Type operator	$\mathsf{D} : \to *$
Term operators	$\triangleright \ \mathsf{app} : \mathsf{D}, \mathsf{D} \to \mathsf{D}$ $\triangleright \ \mathsf{abs} : (\mathsf{D})\mathsf{D} \to \mathsf{D}$

Example 1.8. The computational monads of programming semantics have signature

Type operator	$\mathsf{T} : * \to *$
Term operators	$\alpha : * \ \triangleright \ \mathsf{eta} : \alpha \to \mathsf{T}(\alpha)$ $\alpha, \beta : * \ \triangleright \ \mathsf{let} : T(\alpha), (\alpha)\mathsf{T}(\beta) \to \mathsf{T}(\beta)$

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[1] It seems to make sense to generalize the type signature to be multi-sorted, and perhaps this is to do with universes, but I'm refraining from exploring this direction for now.

Term operators binding many variables can be modelled by means of the second part of Proposition 1.5. More generally, we now show how to associate a polynomial diagram to every term signature.

Definition 1.9. An operator as in (1) with types in $\Sigma^*[n]$ where $k = 0$ is modelled by the following composite of polynomial diagrams in $\widehat{\mathbb{C}}$:

$$\begin{tikzcd}
 & \coprod_{1\le i\le\ell} S^n \arrow[r, "\nabla_\ell"] \arrow[d, "\coprod_{1\le i\le\ell}\langle\langle\llbracket\alpha^i_1\rrbracket,\ldots,\llbracket\alpha^i_{k_i}\rrbracket\rangle,\mathrm{id}\rangle"'] & S^n \arrow[d, "\llbracket\beta\rrbracket"] \\
\coprod_{1\le i\le\ell} V^{k_i}\times S^n \arrow[r, "\coprod_{1\le i\le\ell}\nu^{k_i}\times\mathrm{id}"'] \arrow[d, "{\left[\llbracket\alpha_i\rrbracket\circ\pi_2\right]_{1\le i\le\ell}}"'] & \coprod_{1\le i\le\ell} S^{k_i}\times S^n & S \\
S & &
\end{tikzcd}$$

where $\llbracket - \rrbracket : \Sigma^*(\cdot) \to (S^{(\cdot)} \Rightarrow S)$ denotes the semantics induced by a given algebra $\Sigma(S) \to S$.

The general case, is as follows:

$$\begin{tikzcd}
 & \coprod_{1\le i\le\ell} S^n \arrow[r, "\nabla_\ell"] \arrow[d, "\coprod_{1\le i\le\ell}\langle\langle\llbracket\alpha^i_j\rrbracket\rangle_{1\le j\le k_i},\mathrm{id}\rangle"'] & S^n \arrow[dr, "\langle\langle\llbracket\beta_j\rrbracket_{1\le j\le k}\rangle,\mathrm{id}\rangle"'] & & V^k\times S^n \arrow[dl, "\nu^k\times\mathrm{id}"'] \arrow[dr, "\llbracket\beta\rrbracket\circ\pi_2"] & \\
\coprod_{1\le i\le\ell} V^{k_i}\times S^n \arrow[r, "\coprod_{1\le i\le\ell}\nu^{k_i}\times\mathrm{id}"'] \arrow[d, "{\left[\llbracket\alpha_i\rrbracket\circ\pi_2\right]_{1\le i\le\ell}}"'] & \coprod_{1\le i\le\ell} S^{k_i}\times S^n & & S^k\times S^n & & S \\
S & & & & &
\end{tikzcd}$$

Remark. In elementaty terms, an algebra $P_V(T,\tau) \to (T,\tau)$ in $\widehat{\mathbb{C}}/S$ for the polynomial endofunctor P_V on $\widehat{\mathbb{C}}/S$ induced by a diagram as above amounts to giving:

- for the case $k = 0$,

$$\begin{tikzcd}
\coprod_{\vec\sigma\in S^n}\prod_{1\le i\le\ell} T_{\llbracket\alpha_i\rrbracket(\vec\sigma)}\big(-\cdot\llbracket\alpha^i_1\rrbracket(\vec\sigma)\cdot\ldots\cdot\llbracket\alpha^i_{k_i}\rrbracket(\vec\sigma)\big) \arrow[r] \arrow[d] & T \arrow[d, "\tau"] \\
S^n \arrow[r, "\llbracket\beta\rrbracket"'] & S
\end{tikzcd}$$

  or, equivalently,

$$\Big\{\Big(\prod_{1\le i\le\ell} T_{\llbracket\alpha_i\rrbracket(\vec\sigma)}\big(-\cdot\llbracket\alpha^i_1\rrbracket(\vec\sigma)\cdot\ldots\cdot\llbracket\alpha^i_{k_i}\rrbracket(\vec\sigma)\big)\Big) \longrightarrow T_{\llbracket\beta\rrbracket(\vec\sigma)}(-)\Big\}_{\vec\sigma\in S^n}$$

  in $\widehat{\mathbb{C}}$, and

- for the general case,

  $$\Big\{ \Big(\textstyle\prod_{1\le i\le \ell} T_{[\![\alpha_i]\!](\vec\sigma)}\big(- \cdot [\![\alpha^i_1]\!](\vec\sigma) \cdot \ldots \cdot [\![\alpha^i_{k_i}]\!](\vec\sigma)\big)\Big) \longrightarrow T_{[\![\beta]\!](\vec\sigma)}\big(- \cdot [\![\beta_1]\!](\vec\sigma)\cdot \ldots \cdot [\![\beta_k]\!](\vec\sigma)\big)\Big\}_{\vec\sigma\in S^n}$$

  in $\widehat{\mathbb{C}}$.

Remark. Note that in the abscence of binders and parameters in the operator, the above simplifies to

$$\begin{tikzcd}
\coprod_{1\le i\le\ell} S^n \arrow[r, "\nabla_\ell"] \arrow[d, "{\big[[\![\alpha_i]\!]\big]_{1\le i\le\ell}}"'] & S^n \arrow[d, "{[\![\beta]\!]}"] \\
S & S
\end{tikzcd}$$

and to

$$\begin{tikzcd}
\coprod_{\vec\sigma\in S^n} \prod_{1\le i\le\ell} T_{[\![\alpha_i]\!](\vec\sigma)} \arrow[r] \arrow[d] & T \arrow[d, "\tau"] \\
S^n \arrow[r, "{[\![\beta]\!]}"'] & S
\end{tikzcd}$$

as expected.

Definition 1.10. A *model* consists of a cartesian typed-context structure $(\mathbb{C}, S)$, a term-typing structure (T, τ), and algebra structures on $S \in \boldsymbol{Set}$ for type operators and on $(T, \tau) \in \widehat{\mathbb{C}}/S$ for term operators.

1.3 Morphisms

An *homomorphism* $(H, h) : (\mathbb{C}, S) \to (\mathbb{C}', S')$ between cartesian typed-context structures $(\mathbb{C}, S)$ and $(\mathbb{C}', S')$ consists of a functor $H : \mathbb{C} \to \mathbb{C}'$ and a function $h : S \to S'$ such that the canonical maps

$$H(\epsilon) \to \epsilon' \quad \text{and} \quad H(\Gamma \cdot \sigma) \to H(\Gamma) \cdot (h\sigma)$$

are identities.

Moreover, when S and S' come equipped with Σ-algebra structures as in the case of models, the function h is further required to be an homomorphism; that is,

$$\begin{tikzcd}
\Sigma(S) \arrow[r, "\Sigma(h)"] \arrow[d] & \Sigma(S') \arrow[d] \\
S \arrow[r, "h"'] & S'
\end{tikzcd}$$

An *homomorphism* $(H, h, f) : \big(\mathbb{C}, S, (T, \tau)\big) \to \big(\mathbb{C}', S', (T', \tau')\big)$ between term-typing structures (T, τ) and (T', τ') respectively over the cartesian typed-context structures $(\mathbb{C}, S)$ and $(\mathbb{C}', S')$ consists of an homomorphism $(H, h) : (\mathbb{C}, S) \to (\mathbb{C}', S')$ and a natural transformation $f : T \to T'H$ such that

$$\begin{tikzcd}
T \arrow[r, "f"] \arrow[d, "\tau"'] & T'H \arrow[d, "\tau' H"] \\
S \arrow[r, "h"'] & S'
\end{tikzcd}$$

Remark. We have an homomorphism $(H,h,v) : \big(\mathbb{C}, S, (V,\nu)\big) \to \big(\mathbb{C}', S', (V',\nu')\big)$ for $v : V \to V'H$ given by the action of H:

$$\begin{tikzcd}
\coprod_{\alpha\in S}\mathbb{C}\big(-,\langle\alpha\rangle\big) \arrow[r, "v"] & \coprod_{\alpha'\in S'}\mathbb{C}'\big(H(-),\langle\alpha'\rangle\big) \\
\big(\alpha, x : \Gamma \to \langle\alpha\rangle\big) \arrow[r, mapsto] & \big(h\alpha, Hx : H\Gamma \to \langle h\alpha\rangle\big)
\end{tikzcd}$$

For models, in which $(T,\tau) \in \widehat{\mathbb{C}}/S$ and $(T',\tau') \in \widehat{\mathbb{C}}'/S'$ come equipped with operator algebra structures, an homomorphism requirement needs to be imposed on the natural transformation $f : T \to T'H$ in $\widehat{\mathbb{C}}$. To make this precise, we need analyze the relationship between the polynomial endofunctor P_V on $\widehat{\mathbb{C}}/S$ of an operator in the model $\big(\mathbb{C}, S, (T,\tau)\big)$ and the polynomial endofunctor $P'_{V'}$ on $\widehat{\mathbb{C}}'/S'$ of the same operator in the model $\big(\mathbb{C}', S', (T',\tau')\big)$. We have the following.

Lemma 1.11.

$$\begin{tikzcd}
\widehat{\mathbb{C}}'/S' \arrow[r, "H^\star"] \arrow[d, "P'_{V'}"'] & \widehat{\mathbb{C}}/S' \arrow[r, "h^*"] \arrow[d, phantom, "\cong"] & \widehat{\mathbb{C}}/S \arrow[d, "P_V"] \\
\widehat{\mathbb{C}}'/S' \arrow[r, "H^\star"'] & \widehat{\mathbb{C}}/S' \arrow[r, "h^*"'] & \widehat{\mathbb{C}}/S
\end{tikzcd}$$

The crucial reason for the above, besides the fact that h is an homomorphism and $H^\star$ is continuous and cocontinuous, is that

$$\begin{tikzcd}
\widehat{\mathbb{C}}'/S'^n \arrow[r, "H^\star"] \arrow[d, "{\pi_2}^*"'] & \widehat{\mathbb{C}}/S'^n \arrow[r, "(h^n)^*"] & \widehat{\mathbb{C}}/S^n \arrow[d, "{\pi_2}^*"] \\
\widehat{\mathbb{C}}'/(V'^k \times S'^n) \arrow[d, "\Pi_{\nu'^k\times\mathrm{id}}"'] & \cong & \widehat{\mathbb{C}}/(V^k \times S^n) \arrow[d, "\Pi_{\nu^k\times\mathrm{id}}"] \\
\widehat{\mathbb{C}}'/(S'^k \times S'^n) \arrow[r, "H^\star"'] & \widehat{\mathbb{C}}/(S'^k \times S'^n) \arrow[r, "(h^k\times h^n)^*"'] & \widehat{\mathbb{C}}/(S^k \times S^n)
\end{tikzcd}$$

which essentially comes from the identity

$$H(-\cdot\alpha_1\cdot\ldots\cdot\alpha_k) \;=\; H(-)\cdot h(\alpha_1)\cdot\ldots\cdot h(\alpha_k)\ .$$