Algebraic models of identity type

I originally wanted to give algebraic models of identity type along the lines of my LICS '08 paper on dependently-sorted syntax so as to be able to connect them more directly to models of ω-groupoids based on the globular approach as the sort dependency

$$x, y : A \vdash \underline{A}(x, y) ,$$

$$x, y : A, f, g : \underline{A}(x, y) \vdash \underline{A}(x, y)(f, g) .$$

yields exactly the globular cate-
gory.
But I cannot do this, because
there is an issue which though
it is trivial I had not fully
assimilated before: the theory
of identity types does not only
have sort dependency but also
variable binding. That is it is
not purely first order. This,
I believe, opens a big gap bet-
ween identity types and
ω-groupoids (the theory of
which should be first order,

Though I don't know much about this yet)

Following along these lines, I thought of sketching the algebraic interpretation of identity types in the more involved setting for dependent type theory that I worked on (viz. my old 2008 note and my newer TYPES 2011 talk on algebraic modelling of type theory)
At the moment, I'm not sure what this can be useful for, besides giving us a mathematical

formalisation of the vernacular rule-based syntactic formalism.

Here, I will work with substitution as formalised in Dybjer's framework of categories with families. I do not regard this as purely algebraic but it simplifies matters, avoiding us to go into the theory of substitution

So the basic set up consists of a small category (of contexts), say $\mathbb{C}$, with terminal object (the empty context); a presheaf $S \in \hat{\mathbb{C}}$ (of sorts or types), and a presheaf $T \in \widehat{\int S}$ (of Terms)

As for notation, I use $\Gamma, \Delta,$ for the object of $\mathbb{C}$; $\rho, f,$ for the morphisms of $\mathbb{C}$; $\Gamma \vdash \sigma$ to indicate that $\sigma \in S(\Gamma)$ and also for an object of $\int S$; and $\Gamma \vdash t : \sigma$ to indicate that $t \in T(\Gamma \vdash \sigma)$.

The above data comes with the following structure:

for every $\Gamma \vdash \sigma$ a (projection) map $(\Gamma, \sigma) \xrightarrow{p} \Gamma$ in $\mathbb{C}$, coherent pullback squares

$$\begin{tikzcd}
\Delta, \sigma[\rho] \arrow[r] \arrow[d, "p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma, \sigma \arrow[d, "p"] \\
\Delta \arrow[r, "\rho"'] & \Gamma
\end{tikzcd}$$

where $\sigma[\rho]$ stands for $S(\rho)(\sigma)$

(Thus, $\mathbb{C}/\Gamma$ is closed under product with $p : (\Gamma, \sigma) \to \Gamma$); and

a representation

$$y_{\mathbb{C}/\Gamma}\left(\begin{matrix}\Gamma, \sigma \\ \downarrow p \\ \Gamma\end{matrix}\right) \cong T_\sigma \in \widehat{\mathbb{C}/\Gamma}$$

where

$$\begin{tikzcd}
T_\sigma \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d] \\
y(\Gamma) \arrow[r, "\sigma"] & S
\end{tikzcd}$$

The last condition says that there is a natural bijective correspondence

$$\begin{tikzcd} \Delta \arrow[r, "\langle \rho, t \rangle"] \arrow[dr, "\rho"'] & \Gamma.\sigma \arrow[d, "p"] \\ & \Gamma \end{tikzcd}$$

$$\Gamma \vdash t : \sigma$$

A main effect of this is to allow terms to be internalised as morphisms, thereby inducing a notion of substitution via composition.

This is a reformulation of Dybjer's categories with families which are a reformulation of Cartmell's categories with attributes, and it is just to set up the wiring of dependent type theories

On top of this, I'd like to add algebraic structure specifying identity types

The general philosophy is to do this by means of operators

$$\alpha : FA \longrightarrow A$$

for suitable polynomial-like functors F, corresponding to the

-ority of the operator. In our case, these functors will be generalised polynomial as I've recently considered. By this I mean that they arise from diagrams

$$\mathbb{X} \xleftarrow{s} \mathbb{E} \xrightarrow{f} \mathbb{B} \xrightarrow{t} \mathbb{Y}$$

as

$$\hat{\mathbb{X}} \xrightarrow{s^*} \hat{\mathbb{E}} \xrightarrow{f_*} \hat{\mathbb{B}} \xrightarrow{t_!} \hat{\mathbb{Y}}$$

We can easily work with algebras for such endofunctors as follows:

Consider

$$\begin{tikzcd} & \mathbb{E} \arrow[dl, "s"'] \arrow[r, "f"] & \mathbb{B} \arrow[dr, "t"] & \\ \mathbb{X} & & & \mathbb{X} \end{tikzcd}$$

An algebra

$$t_! f_* s^*(A) \longrightarrow A \quad \text{in } \hat{\mathbb{X}}$$

is equivalently given by a natural transformation α

$$\begin{tikzcd} & \mathbb{E}^{op} \arrow[dl, "s^{op}"'] \arrow[r, "f^{op}"] & \mathbb{B}^{op} \arrow[dr, "t^{op}"] \arrow[ddl, "\mathrm{Ran}_{f^{op}}(As^{op})"'] \arrow[d, phantom, "\Downarrow \alpha"] & \\ \mathbb{X}^{op} \arrow[dr, "A"'] & & {} & \mathbb{X}^{op} \arrow[dll, "A"] \\ & \underline{\underline{Set}} & & \end{tikzcd}$$

The identity type constructor Id should be specified by such an algebraic structure on the presheaf S. The polynomial diagram for it is

$$\begin{tikzcd} \mathbb{O} \arrow[r] \arrow[d] & \int S \arrow[d, "\delta"] \\ \mathbb{C} & \mathbb{C} \end{tikzcd}$$

where δ is the functor

$$\delta(\Gamma \vdash \sigma) = (\Gamma, \sigma, \sigma[p])$$

An algebra for the induced generalised polynomial functor works out to be a natural transformation

$$\begin{tikzcd} \int S^{op} \arrow[rr, "\delta^{op}"] \arrow[dr, "\mathbb{1}"'] & {} \arrow[d, phantom, "\overset{Id}{\Longrightarrow}"] & \mathbb{C}^{op} \arrow[dl, "S"] \\ & \underline{\underline{Set}} & \end{tikzcd}$$

That is a family

$$\Gamma, \sigma, \sigma[p] \vdash Id(\Gamma \vdash \sigma)$$

such that
for all $\rho : \Delta \to \Gamma$ in $\mathbb{C}$:

$$Id(\Gamma \vdash \sigma)[\delta(\rho)] = Id(\Delta \vdash \sigma[\rho]) .$$

Let us now give the operator for reflexivity as algebraic structure on the presheaf T

Fo Thus we consider the poly-nomial diagram

$$
\begin{tikzcd}
 & 0 \arrow[r] \arrow[dl] & \int T \arrow[dr, "\gamma"] & \\
\int S & & & \int S
\end{tikzcd}
$$

where

$$\gamma(\Gamma \vdash t : \sigma) = \mathrm{Id}(\Gamma \vdash \sigma)[\langle id, t, t \rangle]$$

An algebra for the induced polynomial functor is a natural transformation

$$\begin{tikzcd}
\int T^{op} \arrow[rr, "\gamma^{op}"] \arrow[dr, bend right, "\mathbf{1}"'] & \arrow[d, phantom, "\overset{r}{\Longrightarrow}"] & \int S^{op} \arrow[dl, bend left, "T"] \\
& \mathbf{Set} &
\end{tikzcd}$$

That is a family

$$\Gamma \vdash r(\Gamma \vdash t : \sigma)$$
$$: \mathrm{Id}(\Gamma \vdash \sigma)[\langle \mathrm{id}, t, t \rangle]$$

such that

$$r(\Gamma \vdash t\ \sigma)[\rho]$$
$$= r(\Delta \vdash t[\rho] : \sigma[\rho])$$

for all $\rho : \Delta \to \Gamma$ in $\mathbb{C}$.

Note that from

$$\begin{tikzcd} \Gamma, \sigma \arrow[rr, "d"] \arrow[dr, "p"'] & & \Gamma, \sigma \arrow[dl, "p"] \\ & \Gamma & \end{tikzcd}$$

we get

$$\Gamma, \sigma \vdash v : \sigma[p]$$

and we have that

$$r(\Gamma, \sigma \vdash v : \sigma[p])[\langle d, t \rangle]$$
$$= r(\Gamma \vdash t : \sigma)$$

For the algebraic structure of the $\int$ operator consider first

$$K : \int S \longrightarrow \mathbb{C}$$

given by

$$K(\Gamma \vdash \sigma) = (\Gamma, \sigma, \sigma[p], \mathrm{Id}(\Gamma \vdash \sigma))$$

and define

$$\kappa : \int (S K^{op}) \longrightarrow \int S$$

as

$$\kappa(\Gamma \vdash \sigma;\ \Gamma, \sigma, \sigma[p], \mathrm{Id}(\Gamma \vdash \sigma) \vdash C)$$

$$= (\Gamma, \sigma, \sigma[p], \mathrm{Id}(\Gamma \vdash \sigma) \vdash C)$$

We need also define

$$\lambda : \int (S K^{op}) \longrightarrow \int S$$

given by

$$\lambda(\Gamma \vdash \sigma;\ \Gamma, \sigma, \sigma[p], \mathrm{Id}(\Gamma \vdash \sigma) \vdash C)$$

$$= (\Gamma, \sigma \vdash C')$$

where

$C' = C[\langle p, v, v, r(\Gamma, \sigma \vdash v \; \sigma[p])\rangle]$

This is well defined because

$$\Gamma, \sigma \vdash r(\Gamma, \sigma \vdash v : \sigma[p])$$
$$: \mathrm{Id}(\Gamma, \sigma \vdash \sigma[p])$$
$$[\langle \mathrm{id}_{(\Gamma, \sigma)}, v, v\rangle]$$

and

$$\mathrm{Id}(\Gamma, \sigma \vdash \sigma[p])$$
$$[\langle \mathrm{id}_{(\Gamma, \sigma)}, v, v\rangle]$$
$$= \mathrm{Id}(\Gamma, \sigma \vdash \sigma[p])$$
$$[\langle p, v, v, v\rangle]$$
$$= \mathrm{Id}(\Gamma \vdash \sigma)[\langle p\; v, v\rangle]$$

where the last equality is a

Consequence of

$$\Gamma, \sigma, \sigma[p] \vdash Id(\Gamma \vdash \sigma)$$

$$\uparrow \; \delta(p) = \langle ppp, v[p], v \rangle$$

$$\Gamma, \sigma, \sigma[p], \sigma(p)[p] \vdash Id(\Gamma \vdash \sigma)[\delta(p)]$$

$$\uparrow \quad //$$

$$Id(\Gamma, \sigma \vdash \sigma[p])$$

$$\langle p, v, v, v \rangle$$

$$\langle p, v, v \rangle$$

$$\Gamma, \sigma \vdash Id(\Gamma, \sigma \vdash \sigma[p])[\langle p, v, v, v \rangle]$$

$$//$$

$$Id(\Gamma \vdash \sigma)[\langle p, v, v \rangle]$$

The polynomial diagram for the J operator is then

$$\begin{tikzcd}
\int(Sk^{op}) \arrow[r, "\mathrm{id}"] \arrow[d, "\lambda"'] & \int(Sk^{op}) \arrow[d, "\kappa"] \\
\int S & \int S
\end{tikzcd}$$

An algebra for the induced generalised polynomial functor is a natural transformation

$$\begin{tikzcd}
& \int(Sk^{op})^{op} \arrow[dl, "\lambda^{op}"'] \arrow[dr, "\kappa^{op}"] & \\
\int S^{op} \arrow[rr, Rightarrow, "J"] \arrow[dr, bend right, "T"'] & & \int S^{op} \arrow[dl, bend left, "T"] \\
& \mathrm{Set} &
\end{tikzcd}$$

That is, a natural family of maps

$$J(\Gamma\vdash\sigma;\ \Gamma,\sigma,\sigma[p],Id(\Gamma\vdash\sigma)\vdash C)$$

$$: T\big(\Gamma,\sigma\vdash C[\langle p, v, v, r(\Gamma,\sigma\vdash v:\sigma[p])\rangle]\big)$$

$$\to T(\Gamma,\sigma,\sigma[p],Id(\Gamma\vdash\sigma)\vdash C)$$

As for the computation rule, the already used map

$$\begin{tikzcd}
\Gamma,\sigma,\sigma[p],Id(\Gamma\vdash\sigma) \\
\Gamma,\sigma \arrow[u, "{\langle p, v, v, r(\Gamma,\sigma\vdash v\ \sigma[p])\rangle}"']
\end{tikzcd}$$

induces a natural transformation j as in the diagram below

$$
\begin{tikzcd}
 & \int (Sk^{op})^{op} \arrow[dl, "d^{op}"'] \arrow[dr, "\pi^{op}"] \arrow[dr, bend left=60, "d^{op}"] \arrow[dr, phantom, shift left=4ex, "\xRightarrow{j}"] & \\
\int S^{op} \arrow[dr, "T"'] \arrow[rr, phantom, "\xRightarrow{J}"] & & \int S^{op} \arrow[dl, "T"] \\
 & \underline{Set} &
\end{tikzcd}
$$

and we need just require that

$$(T \bullet j) \circ J = id$$

To include a generic closed type, one asks for an algebra structure on S with respect to the generalised polynomial functor induced by the polynomial diagram

$$
\begin{tikzcd}
\mathbb{0} \arrow[r] \arrow[d] & \mathbb{1} \arrow[d, "\mathbf{1}"] \\
C & C
\end{tikzcd}
$$

Globular structure on the generic closed Type can also be added.