

# Algebraic models of identity type

I originally wanted to give algebraic models of identity type along the lines of my LICS'08 paper on dependency-sorted syntax so as to be able to connect them more directly to models of  $\omega$ -groupoids based on the globular approach as the sort dependency

$$x, y : A \vdash \underline{A}(x, y),$$

$$x, y : A, f, g : \underline{A}(x, y)$$

$$\vdash \underline{A}(x, y)(f, g),$$

yields exactly the globular category

But I cannot do that, because there is an issue which though it is trivial I had not fully assimilated before: the theory of identity types does not only have sort dependency but also variable binding, that is it is not purely first order. This, I believe, opens a big gap between identity types and  $\omega$ -groups (the theory of which should be first order,

Though I don't know much about this yet)

Following along these lines, I thought of sketching the algebraic interpretation of identity types in the now involved setting for dependent type theory that I worked on (viz. my old 2008 note and my newer TYPES 2011 talk on algebraic modelling of type theory). At the moment, I'm not sure what this can be useful for, besides giving us a mathematical

formalisation of The  $\lambda$ -calculus  
rule-based syntactic formalism

Here, I will work with substitution as formalised in Dybjer's framework of categories with families. I do not regard this as purely algebraic but it simplifies matters, avoiding us to go into the theory of substitution

So the basic set up consists of a small category (of contexts), say  $\mathbb{C}$ , with terminal object

(the empty context), a presheaf  $S \in \hat{\mathbb{C}}$  (of sorts or types), and a presheaf  $T \in \hat{\int S}$  (of terms)

As for notation, I use  $\Gamma, \Delta$ , for the object of  $\mathbb{C}$ ;  $\Gamma, \Gamma'$ , for the morphism of  $\mathbb{C}$ ;  $\Gamma \vdash \sigma$  to indicate that  $\sigma \in S(\Gamma)$  and also for an object of  $\int S$ ; and  $\Gamma \vdash t :: \sigma$  to indicate that  $t \in T(\Gamma \vdash \sigma)$ .

The above data comes with the following structure:

for every  $\Gamma \vdash \sigma$  a (projection) map  $(\Gamma, \sigma) \xrightarrow{p} \Gamma$  in  $\mathbb{C}$ ,

coherent pullback square

$$\begin{array}{ccc} \Delta, \sigma[p] & \longrightarrow & \Gamma, \sigma \\ p \downarrow & \lrcorner & \downarrow p \\ \Delta & \xrightarrow{p} & \Gamma \end{array}$$

where  $\sigma[p]$  stands for  $S(p)(\sigma)$   
 (Thus,  $\mathbb{C}/\Gamma$  is closed under product  
 with  $p: (\Gamma, \sigma) \rightarrow \Gamma$ ), and  
 a representation

$$\gamma_{\mathbb{C}/\Gamma} \left( \begin{array}{c} \Gamma, \sigma \\ \downarrow p \\ \Gamma \end{array} \right) \cong T_{\sigma} \in \widehat{\mathbb{C}/\Gamma}$$

where

$$\begin{array}{ccc} T_{\sigma} & \longrightarrow & \Gamma \\ \downarrow & \lrcorner & \downarrow \\ \gamma(\Gamma) & \xrightarrow{\sigma} & S \end{array}$$

The last condition says that there is a natural bijection correspondence

$$\begin{array}{ccc}
 & \langle p, t \rangle & \\
 \Delta & \longrightarrow & \Gamma, \sigma \\
 p & \searrow & \swarrow p \\
 & \Gamma &
 \end{array}$$


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$$\Gamma \vdash t : \sigma$$

A main effect of this is to allow terms to be internalised as morphisms; Thereby inducing a notion of substitution via composition

This is a reformulation of Dybjer's categories with families

which are a reformulation of  
Cartmell's categories with  
attributes, and it is just  
to set up the wiring of  
dependent type theories

On top of this, I'd like to  
add algebraic structure  
specifying identity types

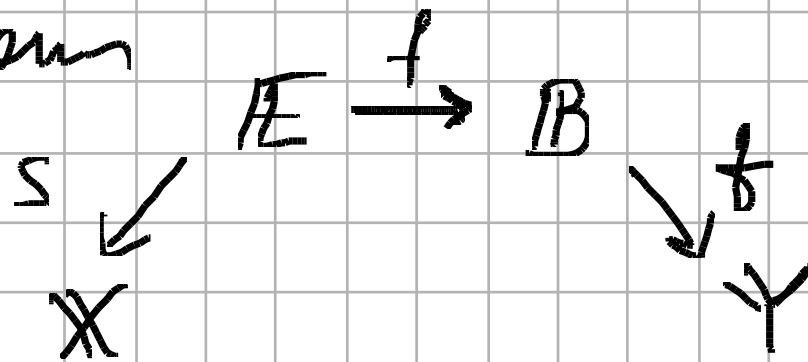
The general philosophy is to  
do this by means of operations

$$\alpha : F A \rightarrow A$$

for suitable polynomial-like  
functors  $F$ , corresponding to the



erty of the operator. In our case, these functions will be generalised polynomials as I've recently considered. By this I mean that they arise from diagrams



$$\text{as } \hat{X} \xrightarrow{S^*} \hat{A} \xrightarrow{f_*} \hat{B} \xrightarrow{t!} \hat{Y}$$

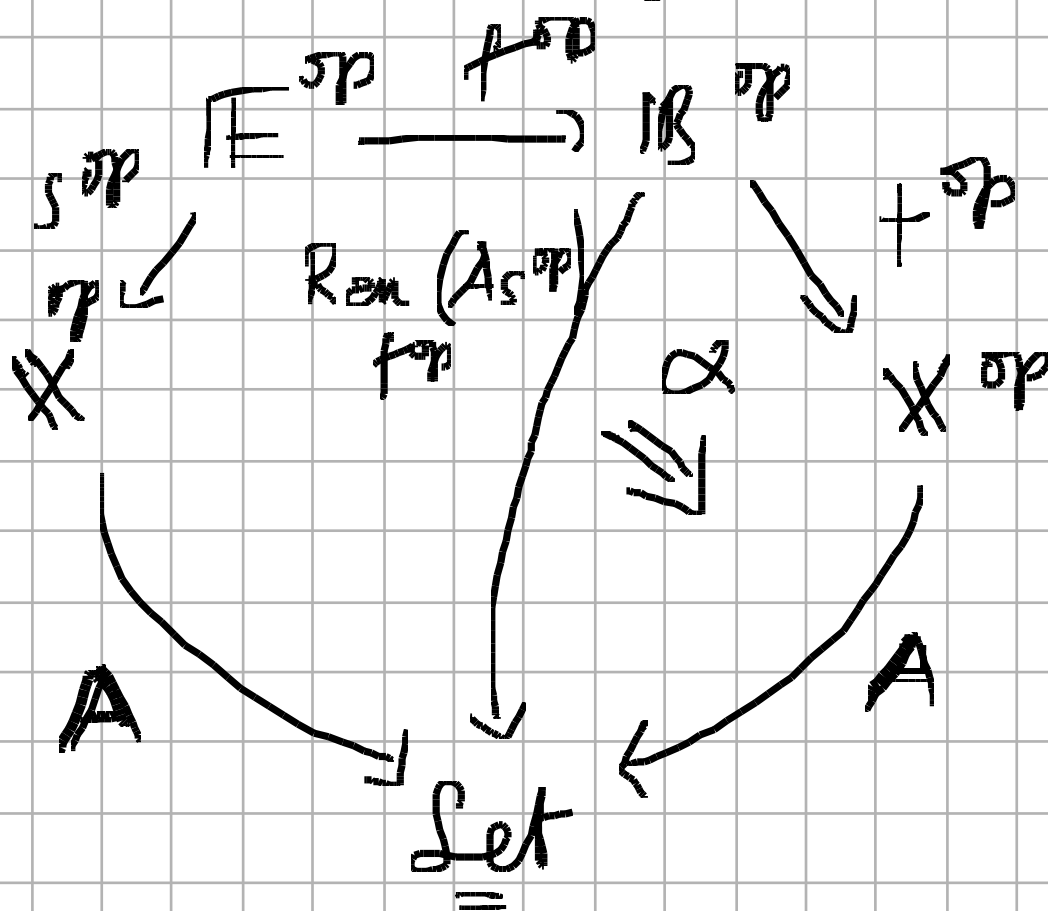
We can easily work with algebras for such endofunctors as follows:

Consider  $S \xrightarrow{f} B$

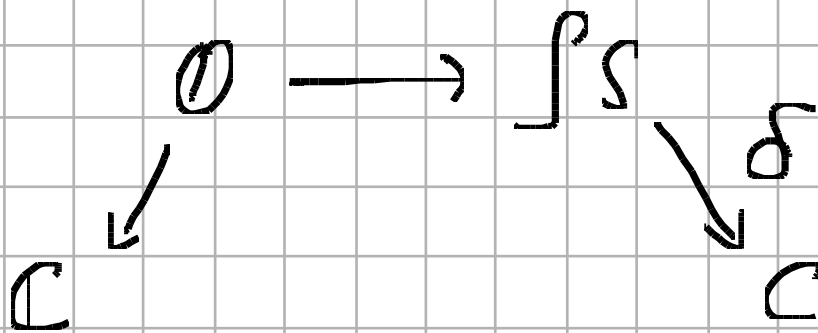
An algebra

$$t_{\parallel} f_x s^x(A) \rightarrow A \text{ in } \hat{X}$$

is equivalently given by a natural transformation  $\alpha$



The identity type constructor  $\text{Id}$  should be specified by such an algebraic structure on the presheaf  $S$ . The polynomial diagram for it is



where  $\delta$  is the function

$$\delta(\Gamma \vdash \sigma) = (\Gamma, \sigma, \sigma[p])$$

An algebra for the induced generalised polynomial function works out to be a natural

transformation

$$\int S^{\sigma} \xrightarrow{\delta^{\sigma}} C^{\sigma}$$

$$\begin{array}{ccc} 1 & \xRightarrow{\text{Id}} & S \\ & \searrow \quad \swarrow & \\ & \text{Set} & \end{array}$$

That is a family

$$\Gamma, \sigma, \sigma[p] \vdash \text{Id}(\Gamma \vdash \sigma)$$

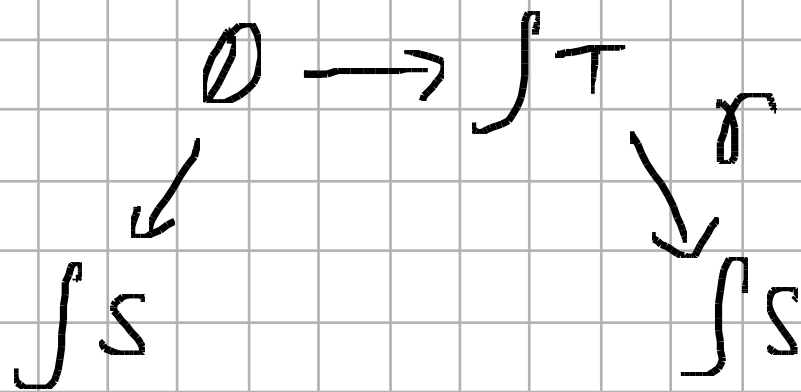
such that

$$\text{for all } p: \Delta \rightarrow \Gamma \sim C:$$

$$\text{Id}(\Gamma \vdash \sigma)[\delta(p)] = \text{Id}(\Delta \vdash \sigma[p]).$$

Let us now give the operation  
for reflexivity on algebraic  
structure on the presheaf  $T$

Then we consider the poly-  
nomial diagram



where

$$\delta(\Gamma \vdash t : \sigma) = \text{Id}(\Gamma \vdash \sigma) [\langle id, t, t \rangle]$$

An algebra for the induced  
polynomial functor is a natural  
transformation

$$\begin{array}{ccc}
 \int_T \sigma & \xrightarrow{\gamma \sigma} & \int_S \sigma \\
 \downarrow & \xrightarrow{r} & \downarrow \\
 \mathbf{1} & & T \\
 & \text{Set} &
 \end{array}$$

That is a family

$$\begin{aligned}
 \Gamma \vdash r(\Gamma \vdash t : \sigma) \\
 = \text{Id}(\Gamma \vdash \sigma) [\langle id, t, t \rangle]
 \end{aligned}$$

such that

$$\begin{aligned}
 r(\Gamma \vdash t : \sigma) [f] \\
 = r(\Delta \vdash t[f] : \sigma[f])
 \end{aligned}$$

for all  $f: \Delta \rightarrow \Gamma \in \mathbb{C}$ .

Note that from

$$\begin{array}{ccc} \Gamma, \sigma & \xrightarrow{\alpha} & \Gamma, \sigma \\ & \searrow \rho & \swarrow \rho \\ & \Gamma & \end{array}$$

we get

$$\Gamma, \sigma \vdash v : \sigma[\rho]$$

and we have that

$$\begin{aligned} & r(\Gamma, \sigma \vdash v : \sigma[\rho]) [\langle \alpha, t \rangle] \\ &= r(\Gamma \vdash t : \sigma) \end{aligned}$$

For the algebraic structure of the  $\mathcal{J}$  operator consider first

$$K : \int S \longrightarrow \mathbb{C}$$

given by

$$K(\Gamma \vdash \sigma) = (\Gamma, \sigma, \sigma[p], Id(\Gamma \vdash \sigma))$$

and define

$$\kappa : \int (S \kappa^{\sigma p}) \longrightarrow \int S$$

as

$$\kappa(\Gamma \vdash \sigma; \Gamma, \sigma, \sigma[p], Id(\Gamma \vdash \sigma) \vdash c)$$

$$= (\Gamma, \sigma, \sigma[p], Id(\Gamma \vdash \sigma) \vdash c)$$

We need also define

$$\lambda : \int (S \kappa^{\sigma p}) \longrightarrow \int S$$

given by

$$\lambda(\Gamma \vdash \sigma; \Gamma, \sigma, \sigma[p], Id(\Gamma \vdash \sigma) \vdash c)$$

$$= (\Gamma, \sigma \vdash c')$$

where



$$c' \equiv C[\langle p, v, v, r(\Gamma, \sigma \vdash v : \sigma[p]) \rangle]$$

This is well defined because

$$\Gamma, \sigma \vdash r(\Gamma, \sigma \vdash v : \sigma[p])$$

$$: Id(\Gamma, \sigma \vdash \sigma[p])$$

$$[\langle d_{(\Gamma, \sigma)}, v, v \rangle]$$

and

$$Id(\Gamma, \sigma \vdash \sigma[p])$$

$$[\langle d_{(\Gamma, \sigma)}, v, v \rangle]$$

$$= Id(\Gamma, \sigma \vdash \sigma[p])$$

$$[\langle p, v, v, v \rangle]$$

$$= Id(\Gamma \vdash \sigma)[\langle p, v, v \rangle]$$

where the last equality is a

consequence of

$$\Gamma, \sigma, \sigma[p] \vdash Id(\Gamma \vdash \sigma)$$

$$\uparrow \delta(p) = \langle p p p, v[p], v \rangle$$

$$\Gamma, \sigma, \sigma[p], \sigma[p][p]$$

$$\vdash Id(\Gamma \vdash \sigma)[\delta(p)]$$

//

$$Id(\Gamma, \sigma \vdash \sigma[p])$$

$$\langle p, v, v, v \rangle$$

$$\Gamma, \sigma \vdash Id(\Gamma, \sigma \vdash \sigma[p])$$

$$[\langle p, v, v, v \rangle]$$

//

$$Id(\Gamma \vdash \sigma)[\langle p, v, v \rangle]$$

$\langle p, v, v, v \rangle$

The polynomial diagram for the  $J$  operator is then

$$\begin{array}{ccc}
 \int (S \kappa^{\text{op}}) & \xrightarrow{J} & \int (S \kappa^{\text{op}}) \\
 \lambda \swarrow & & \searrow \kappa \\
 \int S & & \int S
 \end{array}$$

An algebra for the induced generalised polynomial functor is a natural transformation

$$\begin{array}{ccc}
 \int^{\text{op}} (S \kappa^{\text{op}})^{\text{op}} & & \int S^{\text{op}} \\
 \int^{\text{op}} \swarrow & \xRightarrow{J} & \searrow \kappa^{\text{op}} \\
 \int S^{\text{op}} & & \int S^{\text{op}} \\
 \downarrow T & & \downarrow T \\
 \text{Set} & & \text{Set}
 \end{array}$$

That is, a natural family of maps

$$J(\Gamma \vdash \sigma; \Gamma, \sigma, \sigma[p], \text{Id}(\Gamma \vdash \sigma) \vdash C) \\ : T(\Gamma, \sigma \vdash C[\langle p \ v, v, r(\Gamma, \sigma \vdash v : \sigma[p]) \rangle]) \\ \Rightarrow T(\Gamma, \sigma, \sigma[p], \text{Id}(\Gamma \vdash \sigma) \vdash C)$$

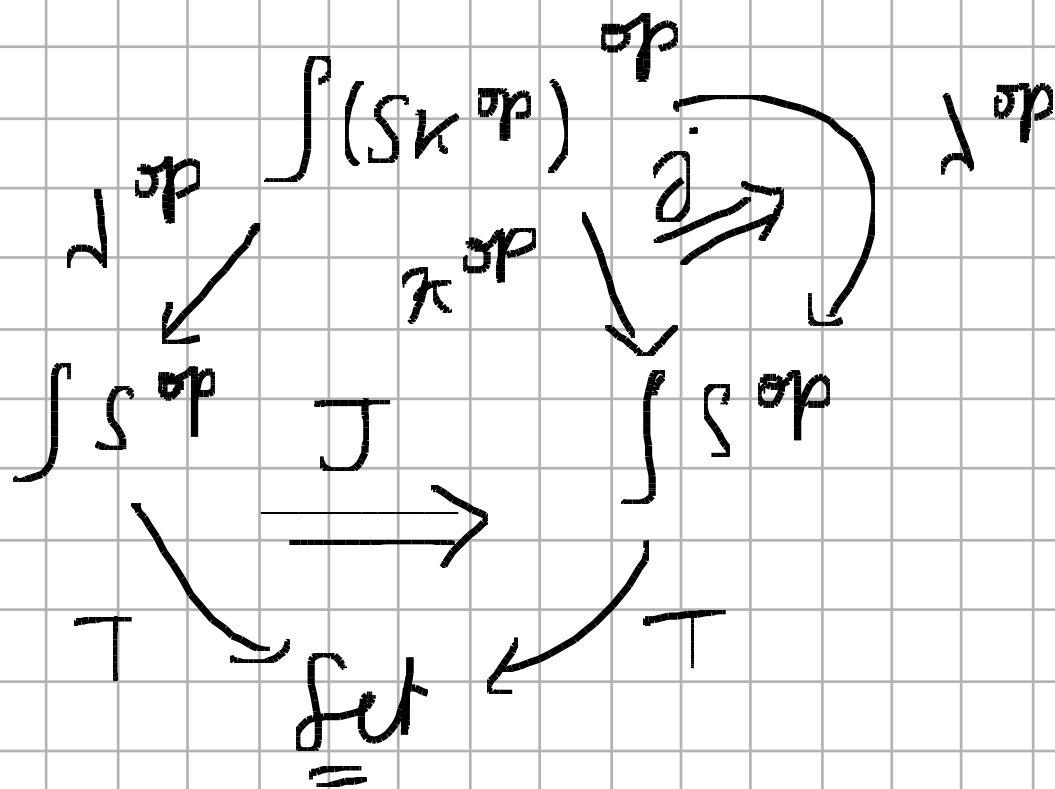
As for the computation rule, the already used map

$$\Gamma, \sigma, \sigma[p], \text{Id}(\Gamma \vdash \sigma)$$

$$\uparrow \langle p, v, v, r(\Gamma, \sigma \vdash v : \sigma[p]) \rangle$$

$$\Gamma, \sigma$$

induces a natural transformation  $j$  as in the diagram below



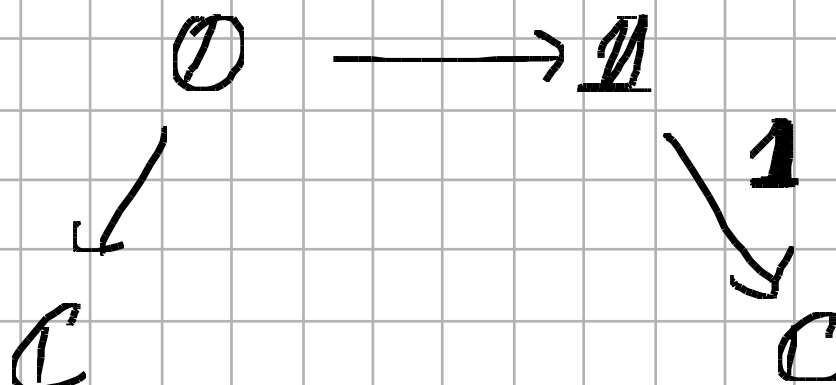
and we need just require that

$$(T \circ j) \circ J = \text{id}$$

closed

To include a generic  $\checkmark$  Type, one asks for an algebra structure on  $S$  with respect to the generalised polynomial functor induced by the polynomial

Diagram



Globular structure on the  
general closed type can also  
be added.