On Differences between the Real and Physical Plane

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Abstract. When formalising diagrammatic systems, it is quite common to situate diagrams in the *real plane*, $\mathbb{R}^2$. However this is not necessarily sound unless the link between formal and physical diagrams is examined. We explore some issues relating to this, and potential mistakes that can arise. This demonstrates that the effects of drawing resolution and the limits of perception can change the meaning of a diagram in surprising ways. These effects should therefore be taken into account when giving formalisations based on $\mathbb{R}^2$.

1 Introduction

When formalising diagrammatic systems, it is quite common to situate diagrams in the *real plane*, $\mathbb{R}^2$.. Curves and points in the diagram are associated with curves and points in the real plane. Results from real analysis – most commonly the *Jordan Curve Theorem*[1] – can then be used to prove various properties of the representation system.

However this does not guarantee these properties unless the link between diagrams 'drawn' in $\mathbb{R}^2$ and actual physical diagrams is examined. Familiarity makes it easy to forget that $\mathbb{R}^2$ is a technical mathematical construction, and not the same as a physical plane. Caution is suggested by the fact that the Jordan Curve Theorem does not hold for $\mathbb{Q}^2$ – which is a better approximation to $\mathbb{R}^2$ than any physical drawing surface. We must take into account the limited precision of drawing tools, and the limits to which people using the diagram can accurately identify the objects drawn. In $\mathbb{R}^2$, it is possible to draw infinitely thin curves and distinguish between arbitrarily close points. This is, of course, not possible for any physical surface on which a diagram might be drawn. Another discrepancy is that $\mathbb{R}^2$ is not bounded.

To analyse the possible effects of these discrepancies, let us suppose that diagrams are produced by a drawing function that converts diagrams in $\mathbb{R}^2$into physical diagrams (either on paper or a computer screen). We may plausibly assume that given a diagram consisting of curves and points in $\mathbb{R}^2$, its physical drawing is a 'blurred' version of the original (where points have an area, lines have a width, and the drawing process can introduce some small errors). Measurement errors will also occur when reading the diagram, adding another level of blurring.

At least two problems can occur in drawing objects from $\mathbb{R}^2$:

[1] "All non-intersecting closed curves in $\mathbb{R}^2$ are homeomorphic to the unit circle" - and hence have a well-defined inside.

A. Blackwell et al. (Eds.): Diagrams 2004, LNAI 2980, pp. 29–31, 2004.

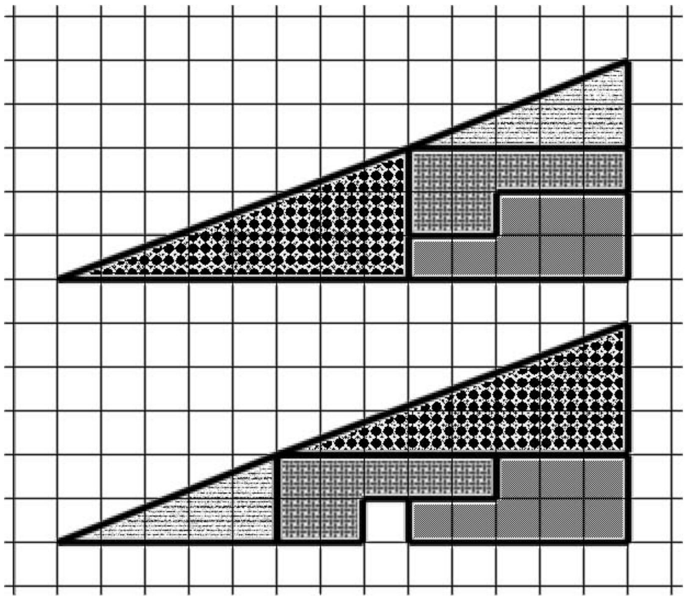

Fig. 1. Example of a mistake arising from the appearance of equality. Spotting the error behind this paradoxical diagram is left as a puzzle for the reader

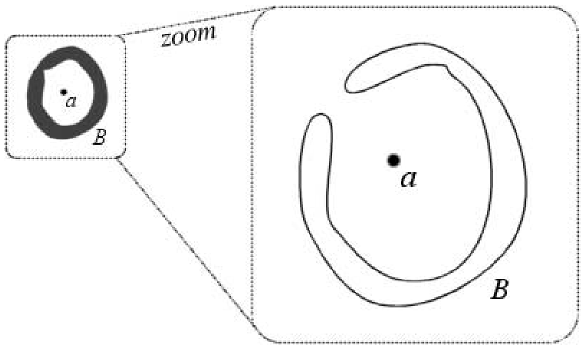

Fig. 2. Example of how using *inside* to represent $\in$ can produce representation errors: the left hand diagrams shows $a \in B$, which turns out to be false on magnification

1. False equality statements can be generated, and in several different ways: If two points are close but not equal, two lines almost but not quite parallel, etc. these distinctions will be lost in the physical diagram. Figure 1 is a classic example[2] of this, which has implications for proofs such as the diagrammatic proof of Pythagoras' Theorem.
2. Many diagram systems use loops to represent sets (with *inside* used for $\subset / \in$). Clearly, drawing at too low a resolution can obscure such relations, but that only results in lost information. More worrying is the possibility that *false* relations might appear as 'artifacts' of the drawing process. Figure 2 shows how this can happen.

[2] Original source unknown.

2 Solutions

Note that using diagram viewers with a 'zoom' ability does not solve these problems, since the user cannot know what level of magnification is required to reveal any errors. However the problems raised here can be dealt with in a rigorous fashion, and in several ways, including:

- We can restrict what diagrams are allowed to say (i.e. what information can legitimately be read from a diagram).
- We can restrict ourselves to diagrams where problems cannot arise. This can be done by identifying classes of diagrams which are immune to problems. For example, it can be proved – subject to very plausible assumptions – that the 'closing eye' structure shown in figure 2 is the only way in which errors involving $\subset / \in$ can occur (see [1]), and that restricting diagrams to using convex curves prevents this.
- Where diagrams are computer-generated, the computer could detect that such errors have occurred in drawing the diagram (by analysing the bitmap produced or otherwise), and warn the user. This is the approach we have taken in our Dr.Doodle system for analysis theorem proving [2].

3 Conclusion

We have shown that some caution is necessary when applying results from real analysis to diagrams. The issues raised here do not threaten diagrammatic reasoning though: these problems are unlikely to occur in 'normal use' of most systems, and can be seen as merely technical difficulties in formalising diagrammatic reasoning. Moreover, where diagrams are computer generated (which is surely the future of diagrammatic reasoning), such drawing errors can be automatically detected. We have, though, only examined *drawing* errors here. Ultimately, we would also like a theory of diagram *reading* errors, which would also cover effects such as optical illusions. This is a much harder requirement, and any such theory must be based in a cognitive science understanding of how people process diagrammatic representations.

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References

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[2] D.Winterstein, A.Bundy, C.Gurr & M.Jamnik "Using Animation in Diagrammatic Theorem Proving" in *Diagrammatic Representation and Inference*, Springer-Verlag, 2002. 31