The Reflection Theorem

Formalizing Meta-Theoretic Reasoning

Lawrence C. Paulson
Computer Laboratory
UNIVERSITY OF CAMBRIDGE

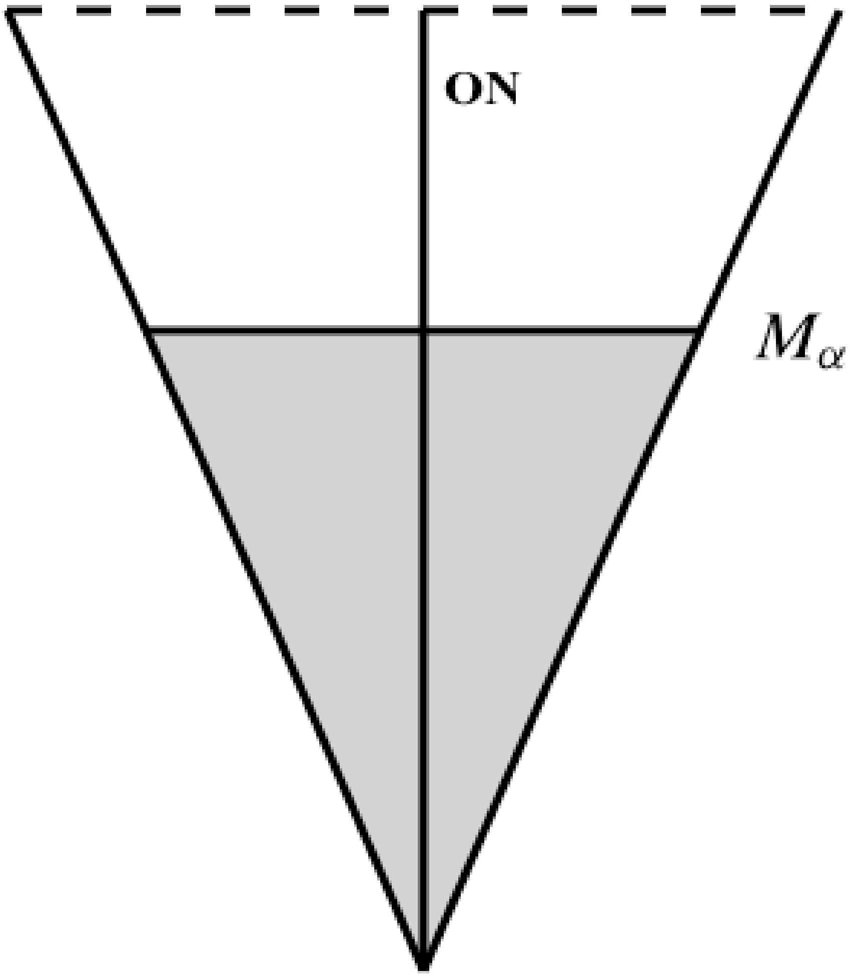
Lecture Overview

- **Motivation for the Reflection Theorem**
- Proving the Theorem in Isabelle
- Applying the Reflection Theorem

Why Do Proofs By Machine?

- *Claim*: too many been done already!
  - Gödel’s incompleteness theorem (Shankar)
  - thousands of Mizar proofs
- *Reply*: many types of reasoning are hard to formalize.
  - Algebraic structures (e.g. group theory)
  - Meta-level reasoning (e.g. about own proof)

Idea of the Reflection Theorem



$$\mathbf{M} = \bigcup_{\alpha \in \mathbf{ON}} M_\alpha$$

□ has the same meaning in $\mathbf{M}$ as in $M_□$, for arbitrarily large □

The Reflection Theorem

Define the class $\mathbf{M} = \bigcup_{\alpha \in \mathbf{ON}} M_\alpha$

where $\{M_\alpha\}$ is a monotonic and continuous family of sets.

For each formula $\phi(x_1, \ldots, x_n)$ *there are arbitrarily large ordinals* □ *such that* □ *holds in* $\mathbf{M}$ *iff* □ *holds in* $M_{□}$

Why is it Hard to Formalize?

- "ϕ holds in $\mathbf{M}$" is not definable in ZF!
  - Because $\mathbf{M}$ is a proper class
  - Tarski: the nondefinability of truth
- ϕ could take any number of arguments
- There is a different proof for each ϕ
  - Reflection is a *meta-theorem*
  - … and not a *theorem scheme*

Must Define Truth Syntactically

$$
\begin{aligned}
(x = y)^{\mathbf{M}} &\mapsto x = y \\
(x \in y)^{\mathbf{M}} &\mapsto x \in y \\
(\phi \wedge \psi)^{\mathbf{M}} &\mapsto \phi^{\mathbf{M}} \wedge \psi^{\mathbf{M}} \\
(\neg \phi)^{\mathbf{M}} &\mapsto \neg(\phi^{\mathbf{M}}) \\
(\exists x\, \phi)^{\mathbf{M}} &\mapsto \exists x\, (x \in \mathbf{M} \wedge \phi^{\mathbf{M}})
\end{aligned}
$$

The *relativization* of ϕ to $\mathbf{M}$

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Isabelle/ZF

- Same code base as Isabelle/HOL
- Higher-order metalogic, ideal for
  - Theorem schemes
  - Classes
  - Class functions
- Develops set theory from the Zermelo-Fraenkel axioms to transfinite cardinals

Proving the Reflection Theorem in Isabelle/ZF

- Use a clean proof from Mostowski, 1969
  - *closed unbounded classes* of ordinals
  - *normal functions* (continuous, increasing)
- One lemma for each logical connective
- Isabelle automatically uses the lemmas to prove *instances* of the theorem

Closed/Unbounded Classes

- *Closed* means closed under unions (limits) of ordinals
- If $\mathbf{M}$, $\mathbf{N}$ are C.U. then so is $\mathbf{M} \cap \mathbf{N}$
- The fixedpoints of a continuous, increasing function form a C.U. class
- E.g. the many solutions of $\aleph_\alpha = \alpha$

Essence of Proof

- “Skolemize” each □ quantifier, obtaining a normal function, F
  - The fixedpoints of F give the desired class of ordinals
- For □ □ □ simply intersect the classes
- Negation and atomic cases are trivial

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- **Applying the Reflection Theorem**

The Axiom of Choice and the Generalized Continuum Hypothesis

- Gödel (1940) proved them consistent with set theory
  - A deep and important theorem
  - Addressed Hilbert's First Problem
- Modern treatments (in ZF) require the Reflection theorem

Sketch of Gödel's Proof

- Define the *constructible universe*, $\mathbf{L}$
  - $L_{\square+1}$ adds subsets that can be defined from existing elements (in $L_{\square}$) by a formula
  - $\mathbf{L}$ contains only sets that must exist
- Show that $\mathbf{L}$ satisfies the ZF axioms
  - Comprehension uses Reflection Theorem:

    $\square$ *holds in* $\mathbf{L}$ iff $\square$ *holds in* some $L_{\square}$
- Show that $\mathbf{L}$ satisfies AC and GCH

Showing That **L** “Thinks” All Sets are Constructible

- Amounts to showing that the construction of **L** is idempotent
- Relies on the concept of *absoluteness*:
  - □ is absolute if it’s preserved in all models
  - Not absolute: powersets, function spaces, transfinite cardinals
- Requires analysing **L**’s definition down to the last detail

Applying Reflection to $\mathbf{L}$

- Define a ZF datatype of FOL formulas
- Define a vocabulary for Reflection
  - No function symbols; purely relational!
  - All concepts from the empty set to “constructible”
  - Repeat for the formula datatype
- For each instance of Comprehension, prove an instance of Reflection (automatically)
- Giant terms describe the classes of ordinals

Finish the Consistency Proof?

- Gödel, 1940: *if a contradiction from AC and GCH could be derived, it could be transformed into a contradiction from the axioms of set theory alone.*
- Theorem statement lies outside the language of set theory!
- It is an even better example of meta-theoretic reasoning.