Everything about Spectral Graph Theory

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Contents

Chapter 1

Introduction

The aim of this book is to understand the spectral grpah theory. We combine all classic sources, e.g. Fan Chung's book, Dan Spielman and Luca Trevisan's graduate courses. Also some other important sources.

There is not much to say in the introduction. If later I have some insightful ideas about this entire field I would probably write some rubbish here.

Chapter 2

Linear Algebra Preliminaries

2.1 Views about a matrix for a graph

We have two important views about a matrix **A**. First it is a function that maps a vector **x** to the **vector Ax**. Second, it is a function that maps a vector **x** to a **number** $\mathbf{x^T Ax}$, which is a **quadratic** form. Note the name quadratic form is very fancy but it just means

$$x^T Ax = \sum_{i,j} x(i)A(i,j)x(j)$$

The adjacency matrix is

$$A_G(u,v) = \begin{cases} 1 & if (u,v) \in E \\ 0 & otherwise \end{cases}$$

Though the adjacency matrix is natural to see a graph, it is the least useful. The eigenvectors and eigenvalues are the most meaningful when trying to understand a natural operator or a natural quadratic form.

2.2 Similar matrices and diagonalization

The definition of the similarity between two matrices is:

Definition 2.1. *A matrix A is similar to a matrix B if there is a non-singular matrix M such that $M^{-1}AM = B$. In this case, A and B have the same eigenvalues.*

Definition 2.2. *A square matrix A is **diagonalizable** or nondefective it is similar to a diagonal matrix, i.e. if there is an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.*

2.3 Spectral Theory

Recall that an eigenvector of a matrix M with eigenvalue λ is defined as

$$Mf = \lambda f$$

where $\lambda I - M$ should be a **singular** matrix. The eigenvalues are the roots of the **characteristic** polynomial of M:

$$\det(xI - M)$$

Theorem 2.3 (The Spectral Theorem)**.** *If M is an $n \times n$, real, symmetric matrix, then there exist real numbers $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$ and n mutually orthogonal unit vectors $f_1, ..., f_n$ and such that f_i is an eigenvector of M of eigenvalue λ_i for each i.*

This is why we like symmetric matrices. If the matrix is not symmetric it might not have n eigenvalues. Even if it has n eigenvalues, their eigvenvectors will not be orthogonal (prove by contradiction).

If the matrix is not symmetric, we may even not be interested in their eigenvalues/eigenvectors.

Definition 2.4 (The Rayleigh quotient)**.** *The Rayleigh quotient of a vector x with respect to a matrix $<$ is the ratio*

$$\frac{x^T M x}{x^T x}$$

Observe that if f is an eigenvector, then

$$\frac{f^T M f}{f^T f} = \frac{\lambda f^T f}{f^T f} = \lambda$$

Theorem 2.5. *Let M be a symmetric matrix and x be a non-zero vector that maximizes the Rayleigh quotient w.r.t. M. Then x is an eigenvectro of M with eigenvalue equal to the Rayleigh quotient. Moreover, this eigenvalue is the largest eigenvalue of M.*

Proof. The common trick we would use to prove stuff in spectral graph theory is to decompose the vector into n eigenvectors directions.

$$x = \sum_i (f_i^T x) f_i$$

The intuition here is that, we first compute the projection length of x onto f_i which is just the inner product $x^T f_i$. Then we multiply it with the eigenvector of that direction. Assuming x is a unit vector, then

$$\begin{aligned}
\frac{x^T M x}{x^T x} &= x^T M x \\
&= \left(\sum_i (f_i^T x) f_i \right)^T M \left(\sum_j (f_j^T x) f_j \right) \\
&= \left(\sum_i (f_i^T x) f_i \right)^T \left(\sum_j (f_j^T x) \lambda_j f_j \right) \\
&= \sum_{i,j} (f_i^T x)(f_j^T x) \lambda_j f_i^T f_j && \text{because } f_i^T x \text{ and } f_j^T x \text{ are constants} \\
&= \sum_j (f_j^T x)^2 \lambda_j && \text{because the eigenvectors are orthogonal, their products are 0} \\
&\leq \lambda_n \sum_j (f_j^T x)^2 \\
&= \lambda_n && \text{because } x \text{ is assumed to be a unit vector}
\end{aligned}$$

It is easy to show that the equality can be reached. Hence the Rayleigh quotient is never greater than λ_n. □

Note that this theorem shows how the Rayleigh quotient plus the spectral information characterize the matrix M. When we want to study a symmetric matrix, we study this quotient and it should reveal some information we need.
Similarly we have a very useful conclusion about the relationship between the eigenvalues and the Rayleigh quotient.

$$\lambda_i = \min_{x \perp f_1, \ldots, f_{i-1}} \frac{x^T M x}{x^T x}$$

So the ith smallest eigenvalue of M is the minimum Rayleigh quotient of M when x is an vector from the perpendicular space of the subspece formed by f_1 to f_{i-1}. Note that when $i = n$, we only have one dimension in our matrix, so x is trivial.

Similarly we have

$$f_i = \arg \min_{x \perp f_1, \ldots, f_{i-1}} \frac{x^T M x}{x^T x}$$

2.4 Positive (Semi)definite Matrices

For a **symmetric** matrix A, we write

$$A \succcurlyeq 0$$

if A is **positive semidefinite**, which means all the eigenvalues are **nonnegative**. This is equivalent to

$$v^T A v \geq 0$$

for all v.

An extention is

$$A \succcurlyeq B$$

if

$$A - B \succcurlyeq 0$$

which is equivalent to

$$v^T A v \geq v^T B v$$

for all v.

2.5 Matrix Norm

The operator norm of a matrix M also called the 2-norm is defined by

$$\|M\| = \max_v \frac{\|Mv\|}{\|v\|}$$

It measures how much a vector can increase in size when it is multiplied by M. When M is symmetric, the 2-norm is just the largest absolute value of an eigenvalue of M. Also we have

$$\|M_1 M_2| \ \leq \|M_1\| \|M_2\|$$

Chapter 3

Graph Preliminaries

Definition 3.1. *A graph is 3-connected if there is no set of two vertices whose removal disconnects the graph.*

Definition 3.2. *A **planar** graph is a special graph that can be drawn in the plane without crossing edges.*

Definition 3.3. *A planar graph divides the plane into connected regions called $faces$. Each face is identified with the vertices and edges on its boarder.*

Theorem 3.4. *Let $G = (V, E)$ and F be the set of faces. Then*

$$|V| - |E| + |F| = 2$$

Definition 3.5. *A **hypercube** is the graph with vertex set $\{0, 1\}^d$ with edges between vertices whose labels differ in exactly one bit.*

Chapter 4

The Matrices of Interest

We give some basic information about the matrices we like to discuss and some simple facts about their eigenvalues.

4.1 Laplacian matrix

This is the most basic matrix in the spectral graph theory.

$$L := D - A$$

where D is the diagonal matrix in which

$$D(u, u) = \deg(u)$$

and the rest are all 0s.

Why is this design important? If we look at the Laplacian quadratic form, we see its relationship with the Rayleigh quotient

$$x^T Lx = \sum_{(u,v)\in E} (x(u) - x(v))^2$$

because we would have a $-x(u)x(v)$ and a $-x(v)x(u)$ and also one $x(u)^2$ and one $x(v)^2$. The thing is we should notice that for one edge (u, v) we only compute it once. We do not consider the edge (u, v) and (v, u).

> This form measures the **smoothness** of the function x. It is small if the function x does not jump too much over any edge. Intuitively, if $x(u) - x(v)$ is large, then it contributes a lot to the quadratic form. Note that if the edge is not in the graph, then it is ok to have large difference between the two nodes.

If the graph is weighted then we can have $G = (V, E, w)$ where $w : E \to \mathbb{R}^+$. Then the Laplacian quadratic form is

$$x^T Lx = \sum_{(u,v)\in E} w_{u,v}(x(u) - x(v))^2$$

For two vertices in the graph G, the corresponding entries of them u, v, in the graph G in the Laplacian is

$$L_{G_{u,v}} = (\delta_u - \delta_v)(\delta_u - \delta_v)^T$$

4.2 Adjacency Matrix

In this part we study the features of the adjacency matrix. We denote its eigenvalues by

$$\mu_1 \geq \mu_2 ... \geq \mu_n$$

here we make it a bit different from what we do for the Laplacian. First we order them in a decreasing order, and we use μ instead of λ. There is a reason why we do such things. Note that in the Laplacian we use $L = D - A$. Hence μ_i in fact corresponds to λ_i. If the graph is d-regular, then

$$L = Id - A$$

hence

$$\lambda_i = d - \mu_i$$

Note that if it is not regular, then we do not see this relationship immediately. So the largest eigenvalue of the adjacency matrix of a d-regular graph is d because we know the smallest eigenvalue of the Laplacian is 0. The eigenvector is the uniform eigenvector.

4.3 Non-regular graphs

Now we look at a more interesting case, graphs that are not necessarily regular and we study its eigenvalues.

Lemma 4.1. *For a graph G, we let $d_{\max}$ be the maximum degree and let d_{ave} be the average degree. Then we have*

$$d_{ave} \leq \mu_1 \leq d_{\max}$$

Proof. The lower bound proof simply follows the Rayleigh quotient. The upper bound proof requires using the eigenvector of eigenvalue. □

Lemma 4.2. *If G is connected and $\mu_1 = d_{\max}$ then G is $d_{\max}$-regular.*

> The eigenvector corresponding to the largest eigenvalue is usually not a constant vector. It is always a positive vector if the graph is connected.

Theorem 4.3 (Perron-Frobenius)**.** *Let G be a graph and A its adjacency matrix, then*

- $\mu_1 \geq -\mu_n$
- $\mu_1 > \mu_2$
- *The eigenvalue μ_1 has a strictly positive eigenvector*

Proposition 4.4. *If G is connected, then $\mu_n = -\mu_1$ if and only if G is bipartite.*

Lemma 4.5. *Let A be symmetric and S be a subset of its row and column indices then we have*

$$\lambda_{max}(A) \geq \lambda_{max}(A(S)) \geq \lambda_{min}(A(S)) \geq \lambda_{min}(A)$$

Chapter 5

Spectral Information

We have seen some interesting facts in the first preliminary chapter. Here we basically focus more on the eigenvalues stuff.

5.1 Eigenvectors of the Laplacian

The matrix M in previous chapter is general. We need the theorems because we want to study the Laplacian matrix L and we want to know its eigenvalues.

By observing the above theorem, we should notice that the quadratic form $x^T Lx$ is non-negative. Hence the smallest eigenvalue λ_1 is 0. We can also show that $\lambda_2 > 0$ if and only if the graph is connected. If the graph is disconnected, then we can construct two orthogonal vectors with eigenvalue zero: consider we have two components, then one eigenvector can be constant on one component and 0 everywhere else for the other component.

> λ_2 has a name: algebraic connectivity of a graph [1]. When we relate λ_2 to how well a graph is connected, we are converting qualitative statements to quantitative statements.

An interesting thing is that if we compute all the eigenvectors of the Laplacian, and for each node u, we use two of the eigenvectors as its cordinates say $f_i(u), f_j(u)$ we can have very nice embedding drawings.

5.2 The Courant-Fischer Theoerm

We have seen this theorem briefly before, now we give the complete verison of it.

Theorem 5.1 (Courant-Fischer Theorem)**.** *Let L be a symmetric matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq ... \leq \lambda_n$. Then,*

$$\lambda_k = \min_{\substack{S \subset \mathbb{R}^n \\ \dim(S)=k}} \max_{x \in S} \frac{x^T Lx}{x^T x} = \max_{\substack{T \subset \mathbb{R}^n \\ \dim(T)=n-k+1}} \min_{x \in T} \frac{x^T Lx}{x^T x}$$

In previous chapters, we denote S as the eigenspace formed by the first k eigenvectors. Also we denote T as the complement space of the eigenspace formed by the first $k-1$ eigenvectors.

To see it more intuitively, when $k = 1$, then S is just the span of f_1 and T is all of $\mathbb{R}^n$. For general k, the optima will be achieved when S is the span of $f_1, ..., f_k$ and T is the span of $f_k, ..., f_n$.

Proof. We only prove the first equlity because they are similar.

For $x \in S_k$ which is the span of $f_1, ..., f_k$, we decompose x:

$$x = \sum_{i=1}^{k} c_i f_i$$

so we apply the old trick,

$$\frac{x^T L x}{x^T x} = \frac{\sum_{i=1}^{k} \lambda_i c_i^2}{\sum_{i=1}^{k} c_i^2} \leq \frac{\sum_{i=1}^{k} \lambda_k c_i^2}{\sum_{i=1}^{k} c_i^2} = \lambda_k$$

Now we should explain the $\min_{\substack{S \subset \mathbb{R}^n \\ \dim(S)=k}}$ part. Essentially this says that we should pick a subspace S of dimension k, among all candidates, we pick the S in which the maximum Rayleigh quotient that $x \in S$ can make is minimized.

We pick T_k be the span of $f_k, ..., f_n$. Then we know S must have an intersection with T_k because their dimensions sum to $n + 1$.

$$\max_{x \in S} \frac{x^T L x}{x^T x} \geq \max_{y \in S \cap T_k} \frac{y^T L y}{y^T y}$$

We decompose y as

$$y = \sum_{i=k}^{n} c_i f_i$$

and so

$$\frac{y^T L y}{y^T y} = \frac{\sum_{i=k}^{n} \lambda_i c_i^2}{\sum_{i=k}^{n} c_i^2} \geq \frac{\sum_{i=k}^{n} \lambda_k c_i^2}{\sum_{i=k}^{n} c_i^2} = \lambda_k$$

This shows that all k-dimensional subspaces should have a lower bound on the Rayleigh quotient λ_k. The equality holds trivially. □

5.3 Bounds on λ_2

We apply the Courant-Fischer theorem then we have

$$\lambda_2 = \min_{v : v^T 1 = 0} \frac{v^T L v}{v^T v}$$

Chapter 6

Graph Coloring

Definition 6.1. *A **coloring** of a graph is an assignment of one color to every vertex in a graph so that each edge attaches vertices of different colors.*

It is easy to see that we are interested in using as few colors as possible.

Definition 6.2. *The **chromatic** number of a graph, χ_G, is the least k for which G is k-colorable.*

Lemma 6.3. *A graph is 2-colorable if and only if it is bipartite.*

6.1 Wilf's theorem

Theorem 6.4. $\chi_G \leq \lfloor \mu_1 \rfloor + 1$.

Proof. We order all the vertices. Then we look at the number κ where

$$\forall u, |\{v : v < u, (u, v) \in E\}| \leq \kappa$$

Once we have the smallest κ we can color the graph because for i we can always pick a color different from its previous less than κ neighbours. Now we need to show $\kappa \leq \lfloor \mu_1 \rfloor$.

Since we know the average degree is less than μ_1 we put a node with that degree as n. For the subgraph, meaning we remove the row/column corresponding to that vertex, the largest eigenvalue of the new adjacency matrix is at most μ_1, so we put another node to $n - 1$ and we keep doing so. □

Note that the graph's max eigenvalue can be smaller than d_{max} so the chromatic number can be smaller than $d_{max} + 1$ where we can easily show that all graphs can be colored by $d_{max} + 1$ colors.

6.2 Hoffman's bound

Hoffman proved a bound on the chromatic number of a graph in terms of its adjacency matrix eigenvalues that is tight for bipartite graphs.

Chapter 7

Conductance

7.1 Isoperimetry and λ_2

Here λ_2 is the second **smallest** eigenvalue of the Laplacian.

Definition 7.1. *For a subset S of the vertices of a graph. One way of measuring how well S can be separated from the graph is to count the number of edges connecting S to the rest of the graph. It is the **boundary** of S:*

$$\partial(S) = \{(u, v) \in E : u \in S, v \notin S\}$$

*Then we can define the **isoperimetric ratio** of S.*

$$h(S) = \frac{|\partial(S)|}{|S|}$$

*The **isoperimetric number** of a graph is the minimum isoperimetric number over all sets of at most half the vertices:*

$$h(G) = \min_{|S| \leq n/2} h(S)$$

Theorem 7.2 (Lower bound of Cheeger)**.**

$$h(S) \geq \lambda_2(1 - s)$$

where $s = |S|/|V|$. In particular

$$h(G) \geq \lambda_2/2$$

Proof. By the min-max theorem, we know for all vectors orthogonal to $\mathbf{1}$, we have

$$x^T L x \geq \lambda_2 x^T x$$

Hence we construct an indicator vector χ_S where

$$\chi_S(u) = \begin{cases} 1 & u \in S \\ 0 & otherwise \end{cases}$$

We can observe a nice feature:

$$\chi_S^T L \chi_S = \sum_{(u,v) \in E} (\chi_S(u) - \chi_S(v))^2 = |\partial(S)|$$

To make χ_S orthogonal to $\mathbf{1}$, we make $x = \chi_S - s\mathbf{1}$. Then x satisfies everything. To finish the proof we only need to see

$$x^T x = |S|(1-s)$$

□

The isoperimetric number shows the worst connected part of the graph because it has the least ratio. This is lower bounded by the second-smallest eigenvalue of the Laplacian. Hence λ_2 shows how good a graph is connected: if it is large, then G is well connected. If

7.1.1 Complete Graph K_n

Lemma 7.3. *The Laplacian of K_n has eigenvalue 0 with multiplicity 1 and n with multiplicity $n-1$.*

7.2 Conductance

In previous section we have seen some relationship between the isoperimetric parameter and the second smallest eigenvalue of the Laplacian. To get tighter bounds and cleaner results, we use the **conductance** and the second smallest eigenvalue of the **normalized** Laplacian.

Definition 7.4 (Conductance)**.** *We define*

$$\Phi(S) = \frac{|\partial(S)|}{\min(\mathrm{vol}(S), \mathrm{vol}(V-S))}$$

where $\mathrm{vol}(S)$ *is the sum of the degrees of the vertices in S. Note that many similar, though slightly different definitions appear in the literature, like*

$$\frac{\mathrm{vol}(V)\partial(S)}{\mathrm{vol}(S)\mathrm{vol}(V-S)}$$

Also the conductance of a graph is

$$\Phi_G = \min_{S \subset V} \Phi(S)$$

7.3 The Normalized Laplacian

Definition 7.5. *We define the **normalized** Laplacian:*

$$\mathcal{L} = D^{-1/2} L D^{-1/2}$$

7.4 Cheeger's inequality

Theorem 7.6. *Let $0 = \nu_1 \leq \nu_2 \leq ... \leq \nu_n$ be the eigenvalues of $\mathcal{L}$. Then Cheeger gives us*

$$\frac{\nu_2}{2} \leq \Phi_G \leq \sqrt{2\nu_2}$$

Proof. For now we omit it and if we have time we go back to type in the proof. The harder part, which is the square root part, should be one's favorite theorem in spectral graph theory. □

Chapter 8

Random Walks

The details about random walks can be found in other books on Markov chains. Here we only see its analysis in spectral graph theory.

The matrix form of a lazy random walk is

$$W = \frac{1}{2}(I + D^{-1}A)$$

The distribution of the walk p we use is a row vector, though in many places people use column vectors. The difference is minor. Note that the degrees of different nodes are not necessarily the same, hence the matrix may not be symmetric.

How do we deal with this matrix and how to see its eigenvalues? We borrow the normalized Laplacian:

$$\begin{aligned} W &= I - \frac{1}{2}(I - D^{-1}A) \\ &= I - \frac{1}{2}D^{-1/2}(I - D^{-1/2}AD^{-1/2})D^{1/2} \\ &= I - \frac{1}{2}D^{-1/2}\mathcal{L}D^{1/2} \end{aligned}$$

where $\mathcal{L} = I - D^{-1/2}AD^{-1/2}$. $\mathcal{L}$ is symmetric, hence $D^{-1/2}\mathcal{L}D^{1/2}$ is diagonalizable. Therefore W is also diagonalizable.

Assuming we know f_i and ν_i are the eigenvectors/eigenvalues of $\mathcal{L}$, then we can prove that $f_iD^{1/2}$ is a left-eigenvector of W of eigenvalue $1 - \nu_i/2$.

$$\begin{aligned} (f_iD^{1/2})W &= (f_iD^{1/2})\left(I - \frac{1}{2}D^{-1/2}\mathcal{L}D^{1/2}\right) \\ &= f_iD^{1/2} - \frac{1}{2}f_i\mathcal{L}D^{1/2} \\ &= f_iD^{1/2} - \frac{\nu_i}{2}f_iD^{1/2} \\ &= (1 - \nu_i/2)f_iD^{1/2} \end{aligned}$$

Note that because in Markov chain theory we usually use row vector. So here we adjust the eigenvector to be left row vector. Essentially it is the same as using right column vector. Also

here we use the advantage that $\mathcal{L}$ is symmetric, so the eigenvectors are the same no matter which form we would use. Also note that since W can be asymmetric, so these eigenvectors of W are not necessarily orthogonal.

Lazy random walk has some good features. One is that the eigenvalues of W are all between 1 and 0. Also from the above relationship, we can see the relationship between the eigenvalues of the normalized Laplaican and the eigenvalues of the random walk matrix: $\omega_i = (1-\nu_i/2)$. So the second largest eigenvalue of W corresponds to $1-\nu_2/2$. Then the spectral gap of W is just $\nu_2/2$. So when we use Cheeger to discuss the second smallest eigenvalue of the Laplacian we are talking about the spectral gap of the random walk. They are essentially the **same** thing.

8.1 The rate of convergence

We know the mixing time is related to the relaxation time: $1/\gamma$ where γ is the spectral gap of the random walk matrix.

Theorem 8.1. *For all a, b and t, if $p_0 = e_a$, then*

$$|p_t(b) - \pi(b)| \leq \sqrt{\frac{\deg(b)}{\deg(a)}}\omega_2^t$$

To be consistent with our Markov chain stuff, we use row vectors.

Proof. First of all, we have

$$p_t(b) = p_t e_b^T$$

and we decompose $p_0 D^{-1/2} = \sum_i c_i f_i$.

$$\begin{aligned}
p_t &= p_0 W^t \\
&= p_0 \left(D^{-1/2} \left(I - \frac{1}{2}\mathcal{L} \right) D^{1/2} \right)^t \\
&= p_0 \left(D^{-1/2} \left(I - \frac{1}{2}\mathcal{L} \right)^t D^{1/2} \right) \\
&= \left(\sum_i c_i f_i \right) \left(I - \frac{1}{2}\mathcal{L} \right)^t D^{1/2} \\
&= \left(\sum_i c_i f_i \right) \left(\sum_j (1-\nu_i/2)^t f_i^T f_i \right) D^{1/2} \\
&= \left(\sum_i (1-\nu_i/2)^t c_i f_i \right) D^{1/2} \\
&= c_1 f_1 D^{1/2} + \left(\sum_{i \geq 2} (1-\nu_i/2)^t c_i f_i \right) D^{1/2}
\end{aligned}$$

The last line is true because $\nu_1 = 0$. Also note that $c_1 f_1 D^{1/2} = \pi$, so we have

$$p_t(b) = \pi(b) + \left(\sum_{i \geq 2} (1 - \nu_i/2)^t c_i f_i \right) D^{1/2} e_b^T$$

Since $p_0 = e_a$, hence

$$c_i = e_a D^{-1/2} f_i^T$$

and

$$\left(\sum_{i \geq 2} (1 - \nu_i/2)^t c_i f_i \right) D^{1/2} e_b^T = \left(\sum_{i \geq 2} \omega_i^t e_a D^{-1/2} f_i^T f_i \right) D^{1/2} e_b^T = \sqrt{\frac{\deg(b)}{\deg(a)}} \left(\sum_{i \geq 2} \omega_i^t e_a f_i^T f_i \right) e_b^T$$

We upper bound the right hand side.

$$\left| \left(\sum_{i \geq 2} \omega_i^t e_a f_i^T f_i \right) e_b^T \right| \leq \sum_{i \geq 2} \omega_i^t |e_a f_i^T| |f_i e_b^T| \leq \omega_2^t \sum_{i \geq 2} |e_a f_i^T| |f_i e_b^T| \leq \omega_2^t$$

We know $\omega_2 = 1 - \nu_2/2$. So when converging $t = O(\log n / \nu_2)$. □

Note that this proof is very classic. We should memorize it. How? Essentially we decompose the product of W^t and notice that the first eigenvector is related to π. Then the rest is to bound the rest of the decomposition using the second largest eigenvalue.

Chapter 9

Expanders

Intuitively expanders are graphs that connect well. For a set of vertices within an expander graph, we can expect it to have a lot of neighbours and that is why people like expanders.

Definition 9.1. ***Expander*** *graphs are an incredibly useful family of graphs. An expander is a graph with constant vertex degree and constant conductance.*

An expander family is ***a sequence of graphs of increasing numbers of vertices*** *if there exist constants d and ϕ such that every graph in the family has degree d and conductance at least ϕ.*

The spectral characterization of expanders is:

$$|\lambda_i - d| \leq \epsilon d, \forall i \geq 2$$

Random d-regular graphs are expanders with high probability.

9.1 Approximations of the Complete Graph

By introduing the approximation of graphs, we use this theory to determine good graphs when we cannot achieve complete graphs.

Definition 9.2. *An ϵ-expander is a d-regular graph such that*

$$|\mu_i| \leq \epsilon d$$

for $i \geq 2$ where $\mu_1 \geq ... \geq \mu_n$ are the eigenvalues of the adjacency matrix. The Laplacian eigenvalues are

$$\lambda_i = d - \mu_i$$

which is equivalent to what we have above.

Note that if the spectral gap is a constant, then the mixing time is also of order $\log n$ which is very small.

Definition 9.3. *A graph G is ϵ-approximation of a graph H if*

$$(1-\epsilon)H \preccurlyeq G \preccurlyeq (1+\epsilon)H$$

where $H \preccurlyeq G$ means for all x,

$$x^T L_H x \leq x^T L_G x$$

9.2 Vertex Expansion

Expanders have a good property: for a small set of vertices, it has unusually large number of neighbours.

Theorem 9.4. *Let $G = (V, E)$ be a d-regular graph on n vertices that ϵ-approximates $\frac{d}{n}K_n$. Then for all $S \subset V$,*

$$|\Gamma(S)| \geq \frac{|S|}{\epsilon^2(1-\alpha)+\alpha}$$

where $|S| = \alpha n$

9.3 Ramanujan graphs

The Ramanujan grpahs constructed by Margulis [3] and Lubotzky, Phillips and Sarnak [2] achieve

$$\epsilon \leq \frac{2\sqrt{d-1}}{d}$$

What does it mean? It means the Ramanujan graph can be very close to a complete graph in terms of the adjacent matrices.

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