Large Cardinals and Lebesgue Measurability
Abstract

This essay explores the connection between large cardinals and regularity properties of sets of real numbers - specifically, their Lebesgue measurability. It is well known that, assuming the Axiom of Choice, one can construct non-measurable sets, but it transpires that, under suitable large cardinal assumptions, one can prove the regularity of more and more sets of reals. In fact, the converse holds, in that the regularity of sets of reals entails large cardinal assumptions in inner models of set theory.

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0 Preliminaries

0.1 Introduction and Background

The question of assigning ‘sizes’ to certain subsets of Euclidean space was the motivation behind the development of measure theory in the late 19th and early 20th centuries. Lebesgue [Leb02] developed the Lebesgue measure as part of his thesis in 1902, but Vitali [Vit05] showed in 1905 that, assuming the Axiom of Choice (AC), there exist sets which do not have a well-defined Lebesgue measure. It transpired that all known examples of non-measurable sets constructed used some form of choice in an essential way, and so how much choice was necessary to construct a non-measurable set became an open question.

In fact, it would become apparent that AC and Lebesgue measure were uneasy bedfellows: the Banach-Tarski paradox showed that, assuming AC, it was in fact possible to decompose a sphere into finitely many pieces and rearrange them using Euclidean motions to obtain two spheres of the same volume.

In 1970, Robert Solovay published a construction of a model of Zermelo-Fraenkel set theory (ZF) in which the Principle of Dependent Choices (DC) holds and every set of real numbers is Lebesgue measurable [Sol70], proving that one requires strong choice principles to construct non-measurable sets. Notably, the Banach-Tarski paradox is not a theorem of ZF+DC. However, Solovay had to assume the existence of a strongly inaccessible cardinal, which is stronger than plain ZF. The question of whether this inaccessible was necessary was settled by Shelah [She84], who showed that if every projective set of reals is measurable, then $\omega_1^\text{V}$ is inaccessible in $L$. Shelah’s original proof involved a complicated forcing argument, a technique for constructing models of ZFC which Cohen developed to prove the independence of the Continuum Hypothesis. A simpler combinatorial proof was first published by Raisonnier [Rai84]. This argument was later refined; we shall give Todorcevic’s argument, communicated in [Bek91].

Shelah’s theorem showed that assumptions about the higher echelons of the set-theoretic universe have a direct influence on the properties of sets much lower down in the hierarchy, and that the height and width of the universe are inextricably linked. By collapsing large cardinals to $\omega_1$ using forcing, one turns large cardinal assumptions into properties of sets of reals, and conversely, assumptions about properties of sets of reals transmute into large cardinal assumptions in a suitable inner model.

Stronger and stronger large cardinal assumptions entail the measurability of more sets of reals in larger inner models. Solovay’s result, as we shall see, shows the measurability of all sets of reals definable using countably many ordinals, and assumes the existence of an inaccessible cardinal, a relatively weak large cardinal assumption. In contrast, by assuming the existence of a supercompact cardinal, we shall show that every set of reals in $L(\mathbb{R})$, the smallest inner model containing all the reals, is measurable [SW90]. This includes every ‘reasonably definable’ set of reals, such as all projective sets.
0.2 The Baire Space, the Cantor Space, and the Real Numbers

A reference for the below is given in [Kan09]

We adopt the convention that, as $|\mathbb{R}| = |2^{\omega}| = |\omega^\omega|$, we shall loosely refer to all three sets as ‘the real numbers’, with which interpretation we mean given by the context.

The Baire space $\omega^\omega$ is the set of all functions $f : \omega \to \omega$. We can also identify it as the set of all $\omega$-length sequences of natural numbers. This naturally has the product topology induced by the discrete topology on $\omega$. A basis for this topology is given by declaring the basic open sets to be those of the form $O(s) = \{ f \in \omega^\omega \mid s \subseteq f \}$ for all $s \in \omega^<\omega$, where $s \subseteq f$ means that $f$ extends $s$, i.e. $f$ has $s$ as an initial segment. We note that the basic open sets are also closed (for a fixed $s \in \omega^<\omega$, $\omega^\omega \setminus O(s) = \bigcup_t O(t)$, where the union runs over all $t \in \omega^<\omega$ which are incomparable with $s$, which means that $t$ and $s$ do not contain each other as an initial segment). Also, note that $\omega^\omega$ is second-countable, as $\omega^<\omega$ is a countable set (assuming DC).

In fact, this topology is induced by a metric. Given $f, g : \omega^\omega \to \omega^\omega$, declare $d(f, g) = 0$ if $f = g$, or $d(f, g) = 2^{-i}$, where $i = \min\{ j \mid f(j) \neq g(j) \}$ for $f \neq g$. It is easy to see that this does define a metric, and that the topology induced by this metric is homeomorphic to the one given above.

The Baire space has some very desirable properties - we list some of them below:

- $O(s) \cap O(t)$ is either $\emptyset, O(s)$ or $O(t)$ depending on whether $s \perp t$, $s \subseteq t$, or $t \subseteq s$ respectively.
- Every open set is the disjoint union of basic open sets
- $(\omega^\omega)^k$ is homeomorphic to $\omega^\omega$, and this homeomorphism is given by any bijection between $\omega$ and $\omega \cdot k$
  - In fact, $(\omega^\omega)^\omega$ is homeomorphic to $\omega^\omega$

Now, we seek to put a measure on $\omega^\omega$. First, we put a measure on $\omega$: let $m$ be the measure on $\omega$ such that $m(\{i\}) = 2^{-i(i+1)}$, it is straightforward to verify that $m$ is a probability measure on $\omega$. Let $\mu$ be the product measure on $\omega^\omega$ induced by $m$. With this measure, $\mu(O(s)) = \prod_{i < |s|} 2^{-s(i)(i+1)}$, where $|s|$ means the length of $s$, and $\mu$ is the unique measure on $\omega^\omega$ with this property.

We now have a Borel measure on $\omega^\omega$. To extend this to a Lebesgue measure on $\omega^\omega$, first declare the null sets: a set $N \subseteq \omega^\omega$ is null if there exists a Borel set $X$ such that $\mu(X) = 0$ and $N \subset X$. Then, we say an arbitrary set $A \subseteq \omega^\omega$ is Lebesgue measurable if there exists a Borel set $B$ such that $A \Delta B$ is null, and then define the Lebesgue measure of $A$, $\mu(A)$ to be the Borel measure of $B$ (we conflate the Lebesgue measure with the Borel measure, but this is safe, as the two measures agree on the sets for which both are defined).
One very important fact about the Lebesgue measure is that it is a regular measure: for any Lebesgue measurable set $X$ and any $\varepsilon$, there is a closed set $F$ and an open set $U$ such that $F \subseteq X \subseteq U$ and $\mu(U \setminus F) < \varepsilon$. From this, we can deduce that for any measurable set $X$, there exists an $F_\sigma$ set $A$ and a $G_\delta$ set $B$ such that $A \subseteq X \subseteq B$ and $\mu(A) = \mu(X) = \mu(B)$.

The above considerations apply to the Cantor space, $2^\omega$, *mutatis mutandis*. In particular, the measure on 2 is given simply by declaring that $\mu(0) = \mu(1) = 1/2$, and then extending to $2^\omega$ by the usual product measure. However, one should note that the Cantor space, unlike the Baire space, is compact (by, say, Tychonoff’s theorem).

### 0.3 Constructibility

*References for this section are given in [Jec02] or [Kan09]*

Given a structure $\mathcal{M}$, a set $y$ is *definable over* $\mathcal{M}$ if there is a first-order formula $\varphi(v_0, \ldots, v_n)$ in the language of $\mathcal{M}$ and parameters $x_1, \ldots, x_n$ in the domain of $\mathcal{M}$ such that $z \in y$ iff $\mathcal{M} \models \varphi[z, x_1, \ldots, x_n]$. Let the definable power set operator, Def, be $\text{Def}(X) = \{ Y \subseteq X \mid Y$ is definable over $\langle X, \in \rangle \}$.

Now, recalling the definition of the von Neumann hierarchy, let us build the *constructible universe*, $L$:

- $L_0 = \emptyset$
- $L_{\alpha+1} = \text{Def}(L_\alpha)$
- $L_\lambda = \bigcup_{\delta < \lambda} L_\delta$
- $L = \bigcup_{\delta \in \text{On}} L_\delta$

$L$ is thus a subclass, the class of *constructible sets*. $L$ forms a model of $\text{ZF} + \text{AC}$ (even if the original universe was only a model of $\text{ZF}$), and also the generalised continuum hypothesis, GCH, holds in $L$. Moreover, $L$ is absolute between transitive models of set theory, and hence $L$ is the ‘smallest’ inner model of $\text{ZFC}$: if $M$ is an inner model of $\text{ZFC}$, then $L^M = L^V$. Hence, $L$ satisfies the Axiom of Constructibility: $V = L$.

We also have two versions of relative constructibility.

The first provides, for a given set $A \in V$, the *constructible closure of* $A$, $L(A)$. This is the smallest inner model $M$ such that $A \in M$. We can consider the elements of $A$ as the atoms of our inner model, starting with the transitive closure of $\{A\}$ (to ensure that our resulting inner model is transitive), and then iterating the definable power set operator as above. It will not automatically satisfy AC, as $L$ does, but if there is a well-ordering of $\text{tc}(\{A\})$, the transitive closure of $\{A\}$, in $L(A)$, then $L(A)$ will satisfy AC.
The second is the notion of relative constructibility. For a given set $A$, we can relativise the definability operator to $A$ and form

$$\text{Def}_A(X) = \{Y \subseteq X \mid Y \text{ is definable over } \langle X, \in, A \cap X \rangle\}$$

where we make $A \cap X$ available as a unary relation for definitions (we augment our language with a predicate for membership of $A$). Iterate this operator in a similar fashion to the above:

- $L[A]_0 = \emptyset$
- $L[A]_{\alpha+1} = \text{Def}_A(L[A]_\alpha)$
- $L[A]_\lambda = \bigcup_{\delta<\lambda} L[A]_\delta$
- $L[A] = \bigcup_{\delta \in \text{On}} L[A]_\delta$

$L[A]$ is the smallest inner model $M$ such that, for all $x \in M$, $x \cap A \in M$. Unlike $L(A)$, $L[A]$ will always satisfy AC. In addition, in $L[A]$ there exists some $\alpha_0$ depending on $A$ such that GCH holds for all $\alpha \geq \alpha_0$: $L[A] \models \forall \alpha \exists \kappa \geq \alpha \left(2^{\kappa_\alpha} = \kappa_{\alpha+1}\right)$.

### 0.4 The Levy Collapse

A property $\varphi(x)$ of cardinals is a large cardinal property if whenever $\varphi[\kappa]$ holds, then $L_\kappa$ is a model of ZFC. To begin with, we will state a simple proposition about large cardinals and small forcings (where small means forcing with a poset of size $< \kappa$):

**Proposition 0.1.** Suppose $\kappa$ satisfies a large cardinal property and suppose $P$ is a forcing poset such that $|P| < \kappa$. Then, in $V^P$, $\kappa$ is still the corresponding large cardinal notion (so, for example, if $\kappa$ is inaccessible, $V^P \models \text{`}\kappa\text{ is inaccessible}'$).

**Proof.** See, e.g. [Jec02, p. 390] \hfill \Box

We will mainly use the above proposition in the cases of $\kappa$ inaccessible and $\kappa$ supercompact. Let us turn now to the Levy collapse forcing.

Let $\lambda$ be regular, let $S \subset \text{On}$ be a subset of the ordinals. Then the Levy collapse forcing $\text{Col}(\lambda, S)$ is the set

$$\{p \mid p \text{ is a partial function } \lambda \times S \to S \land |p| < \lambda \land \forall (\alpha, \xi) \in \text{dom}(p) (p(\alpha, \xi) = 0 \lor p(\alpha, \xi) \in \alpha)\}$$

ordered by reverse inclusion: $p \leq q$ iff $p \supseteq q$. In the generic extension, we have surjections $\lambda \to \alpha$ for every $\alpha \in S$. $\text{Col}(\omega_1, \{2^\omega\})$ is used to demonstrate that CH is consistent and $\text{Col}(\omega, \{\omega\})$ adjoins a Cohen real. Below we list some basic properties about the
Proposition 0.2.

1. $\text{Col}(\lambda, S)$ is $\lambda$-closed.

2. If $\kappa$ is regular, $\kappa > \lambda$, and either $\kappa$ is inaccessible, or $\lambda = \omega$, then $\text{Col}(\lambda, \kappa)$ has the $\kappa$-c.c.

3. If $\text{Col}(\lambda, \kappa)$ has the $\kappa$-c.c., then it preserves cardinals $\leq \lambda$ and $> \kappa$.

Many forcings are equivalent to the Levy collapse, specifically:

Proposition 0.3. Suppose that $P$ is a separative poset such that $|P| \leq |\alpha|$ and

$$\Vdash_P \exists f (f : \omega \rightarrow \alpha \text{ is surjective } \land f \notin \check{V})$$

Then there is an injective, dense embedding of a dense subset of $\text{Col}(\omega, \{\alpha\})$ into $P$.

We say a forcing $P$ is weakly homogeneous if for any $p, q \in P$, there is an automorphism $e$ of $P$ such that $e(p) \parallel q$. The next proposition shows why weak homogeneity is important:

Proposition 0.4.

1. If a forcing poset $P$ is weakly homogeneous, then for any formula $\varphi(v_1, \ldots, v_n)$ in the forcing language and $x_1, \ldots, x_n \in V$, either $\Vdash_P \varphi(\check{x}_1, \ldots, \check{x}_n)$ or else $\Vdash_P \neg \varphi(\check{x}_1, \ldots, \check{x}_n)$.

2. $\text{Col}(\lambda, S)$ is weakly homogeneous.

Finally, it is seen that we can absorb any function $x : \omega \rightarrow \text{On}$ in $V[G]$ into our base model:

Proposition 0.5. Suppose that $\kappa > \omega$ is regular, and $G$ is $\text{Col}(\omega, \kappa)$-generic. Then for any $x : \omega \rightarrow \text{On}$ in $V[G]$, there is an $H$ which is $\text{Col}(\omega, \kappa)$-generic over $V[x]$ such that $V[G] = V[x][H]$.

0.5 The Principle of Dependent Choice

The Principle of Dependent Choice is a weak form of choice principle. It states that for any set $X$, and any entire relation $R$ on $X$ (i.e. $\forall x \in X \exists y \in X$ such that $\langle x, y \rangle \in R$), there is a function $f : \omega \rightarrow X$ such that, for all $n \in \omega$, $\langle f(n), f(n+1) \rangle \in R$. While weaker than the full Axiom of Choice, it is still sufficient to develop much of real analysis.
1 Solovay’s Model

In this section, we give a brief overview of Solovay’s construction of a model of ZF in which every set of reals is Lebesgue measurable, and DC holds. This largely follows [Kan09].

1.1 The Borel Algebra on $\omega^\omega$

Let $B$ be the Boolean algebra of Borel sets on $\omega^\omega$, let $B^* = \{X \in B \mid \mu(X) > 0\}$. Put an order on $B^*$: $p \leq q \iff p \subseteq q$.

Proposition 1.1.

1. $p \perp q \iff p \cap q$ is null
2. $B^*$ has the countable chain condition (c.c.c)

Proof.

1. For the forwards direction, if $p \perp q$, that means $\not\exists r$ such that $r \leq p, r \leq q$. Thus, all $r \subseteq p \cap q$ are null $\implies$ $p \cap q$ null. For the converse, if $p \cap q$ null, then any $r \subseteq p \cap q$ is also null, so there are no $r$ such that $r \leq p, r \leq q$, so $p \perp q$.

2. Let $T \subseteq B^*$ be uncountable. We show that there are $p, q \in T$ such that $p \parallel q$. Note that for some $n$, $\{p \in T \mid \mu(p) > \frac{1}{n}\}$ must be uncountable, so WLOG $\mu(p) > \frac{1}{n}$ for all $p \in T$. Suppose $\mu(p \cap q) = 0$ for all $p, q \in T$. Then, by taking any $n$ distinct elements of $T$, say $p_1, \ldots, p_n$, we have that $\mu(p_1 \cup \cdots \cup p_n) > n \cdot \frac{1}{n} = 1$ by inclusion-exclusion, contradicting the fact that $\mu$ is a probability measure.

We observe that $B^*$ is not separative: suppose $p \not\leq q$ (so $p$ is not contained in $q$), but $\mu(p \triangle q) = 0$. Then any $r \subseteq p \setminus q$ is null, so $\not\exists r \leq p$ such that $r \perp q$. This suggests quotienting out by an equivalence relation: $p \sim q$ iff $\mu(p \triangle q) = 0$. Thus, $B^*/\sim$ is the separative quotient of $B^*$.

1.2 Codes

For subsequent sections, fix some enumeration $\langle s_i \mid i \in \omega \rangle$ of the members of $\omega^{<\omega}$ so that if $s_i \subseteq s_j$ then $i \leq j$ (this is easy to arrange - take, for example, the diagonal enumeration).

Let $c \in \omega^\omega$. Define $A_c \in B$ to be:
\[ A_c = \begin{cases} \bigcup \{O(s_i) \mid c(i + 1) = 1\} & \text{if } c(0) = 0 \\ \omega^\omega \setminus \bigcup \{O(s_i) \mid c(i + 1) = 1\} & \text{if } c(0) = 1 \\ \bigcap_n \left( \bigcup \{O(s_i) \mid c(2^n3^{i+1}) = 1\} \right) & \text{if } c(0) > 1 \end{cases} \]

We say \( c \) is an open code if \( c(0) = 0 \), a closed code if \( c(0) = 1 \), and a \( G_\delta \) code if \( c(0) > 1 \). While the exact extension of the set coded by a real may change between models of set theory, their Boolean and measure-theoretic properties are absolute; more precisely:

**Proposition 1.2.** Let \( M \) be a transitive \( \in \)-model of ZF(C). Then:

1. Being a code is absolute
2. The following are absolute for \( M \):
   
   \[ x \in A_c; \ A_c \neq \emptyset; \ A_c \subseteq A_d; \ A_c \subseteq \omega^\omega \setminus A_c; \ A_c \cap A_d = A_e \]
   
   (i.e. if \( x, c \in M \), \( c \) a code then \( x \in A^M_c \iff x \in A_c \) and so on).
3. If \( c \in M \) is a code, then \( \mu^M(A^M_c) = \mu(A_c) \).

**Proof.**

1. Note that a code is a function \( f : \omega \to \omega \), and both ‘being a function’ and \( \omega \) are absolute.
2. As \( \omega \) is absolute, so all the \( s_i \)'s are absolute. Thus, the basic open sets they label are absolute, and so the codes, which determine how the basic open sets interact, are also absolute.
3. Suppose first that \( c \) is an open code. As every open set is a disjoint union of (countably many) basic open sets, we can define a sequence \( \langle i_j \mid j \in \omega \rangle \) such that \( A_c = \bigcup_{j \in \omega} O(s_{i_j}) \), and this union is disjoint. Hence, \( \mu^M(A^M_c) = \mu(A_c) \), as the measure of a basic open set is absolute (the measure of a basic open set depends only on the initial string defining it, and finite strings are absolute).

   If \( c \) is a closed code, let \( d \) code the complement of \( A_c \) and apply the above (\( d \in M \) because we can obtain \( d \) from \( c \) inside \( M \)).

   If \( c \) is a \( G_\delta \) code, then we have a sequence \( \langle c_n \mid n \in \omega \rangle \) of open codes such that \( A_{c_n} \subseteq A_c \). Then \( \mu^M(A^M_c) = \inf \{ \mu^M(A^M_{c_n}) \mid n \in \omega \} \) and is absolute.

**1.3 Random Reals**

Forcing with \( B^* \) can be characterised by adding a real (the random real):
Theorem 1.3. Let \( G \) be \( B^* \)-generic. Then there exists a unique \( x \in \omega^\omega \) such that for any closed code \( c \in \omega^\omega \), \( x \in A^V_c[G] \iff A_c \in G \). Further, \( V[x] = V[G] \).

Proof. Recall that the Lebesgue measure on \( \omega^\omega \) is regular. This implies that, for all \( n \in \omega \),
\[
\{ C \in B^* \mid C \text{ closed} \land \exists k(C \subseteq \{ f \in \omega^\omega \mid f(n) = k \}) \}
\]  
(1)
is dense in \( B^* \): let \( p \in B^* \), \( n \in \omega \) be arbitrary; then the sets \( q_k = \{ f \in p \mid f(n) = k \} \) are open and their disjoint union gives \( p \), a non-null set. Thus, for some particular \( k' \), \( \mu(q_{k'}) > 0 \), and by the regularity of the measure, we can find some closed set \( C \subseteq q_{k'} \) which is again non-null.

Also, for all \( A \subseteq \omega^\omega \), \( A \in M \), \( A \) Lebesgue measurable, the set
\[
\{ C \in B^* \mid C \text{ closed} \land (C \subseteq A \lor C \cap A = \emptyset) \}
\]  
(2)
is dense in \( B^* \): let \( p \in B^* \) be arbitrary. If \( \mu(p \cap A) > 0 \), pick \( C \subseteq p \cap A \) non-null; otherwise \( \mu(p \setminus A) > 0 \) and so pick \( C \subseteq p \setminus A \) non-null.

Now argue in \( V[G] \). We claim that
\[
\bigcap \{ A^V_c[G] \mid c \in V \text{ a closed code} \land A^V_c \in G \}
\]has a single element \( x \). The intersection is non-empty because this collection has the finite intersection property (because \( G \) is a filter, so \( A^V_c[G] \cap A^V_d[G] \in G \) for any closed codes \( c, d \)), and it contains a single element because the family of dense sets in (1) forces a unique value at each index.

Let \( c \) be a closed code. If \( A^V_c \in G \), then trivially \( x \in A^V_c[G] \) by the definition of \( x \). Conversely, if \( x \in A^V_c[G] \), then we need to show \( A^V_c \in G \). By (2), it suffices to show for any closed code \( d \) with \( A^V_d \in G \), \( A^V_c \cap A^V_d \neq \emptyset \). But we know \( x \in A^V_c[G] \cap A^V_d[G] \) and by absoluteness of codes we are done.

We can also define \( G \) in \( V[x] \):
\[
G = \{ p \in B^* \mid \exists c \in V \text{ a closed code} \land x \in A^V_c[x] \land A^V_c[x] \subseteq p \}
\]So \( V[x] = V[G] \).

Let \( M \) be a transitive \( \in \)-model for ZFC. We say \( x \in \omega^\omega \) is random over \( M \) if \( x \) is as above for some \( (B^*)^M \)-generic \( G \) over \( M \). We have the following characterisation of random reals:

Theorem 1.4. Let \( M \) be a transitive \( \in \)-model for ZFC. Then \( x \in \omega^\omega \) is random over \( M \) iff \( x \notin A_c \) for every \( c \in M \) which is a \( G_\delta \) code for a null set.
Remark. Recall that for any $A$ measurable there is a $G_δ$ set $Y$ such that $A \subseteq Y$ and $\mu(A) = \mu(Y)$. So avoiding $G_δ$ null sets means we avoid every null set.

Proof. Suppose first $x$ is random over $M$, with $G$ the corresponding $B^*$-generic. Let $c \in M$ be a $G_δ$ code for a null set. Then:

$$M \models \forall D \in \{C \in B^* \mid C \text{ closed } \land C \subseteq \omega^\omega \setminus A_c\} \text{ is dense in } B^*.$$  

Thus, there is a closed code $d \in M$ such that $A_d^M \in G \cap D$ and so $x \in A_d^M[G]$. But by the absoluteness of codes, we have that $x \in A_d$ so $x \not\in A_c$.

Conversely, suppose $x \not\in A_c$ whenever $c \in M$ is a $G_δ$ code for a null set. To show $x$ random over $M$ it suffices to show whenever $D \in M$ is dense in $(B^*)^M$ there is a closed code $c \in M$ and a $p \in D$ such that $A_c^M \subseteq p$ and $x \in A_c$, recalling how we can define $G$ using $x$.

To this end, let $D \in M$ be $(B^*)^M$-dense and first argue in $M$: let $T$ be a maximal antichain such that $T \subseteq \{C \mid C \text{ closed } \land \exists p \in D(C \subseteq p)\}$. As $B^*$ has the c.c.c., $T$ is countable, so by closed codes let $\langle A_{c_n} \mid n \in \omega\rangle$ enumerate it. Now, let $c \in \omega^\omega$ be any function such that $c(0) > 1$ and $c(2^n3^{n+1}) = 1 \iff O(s_i) \cap A_{c_n} = \emptyset$. Since each $A_{c_n}$ is closed, $c$ is a $G_δ$ code for $\bigcap_{n \in \omega} (\omega^\omega \setminus A_{c_n})$, which is null as $T$ is maximal.

Stepping out of $M$, by hypothesis $x \not\in A_c$, so $x \in A_{c_n}$ for some $c_n \in M$.

1.4 On$^\omega$-Definability

A set $X$ is On$^\omega$-definable if for some $a \in \text{On}^\omega$ and formula $\varphi(v_1,v_2)$,

$$y \in X \iff \varphi[a,y]$$

While we may superficially run into trouble trying to quantify over all first order formulae inside ZF, we can avoid this issue by reflection: a set $X$ is On$^\omega$-definable iff:

$$\exists a \exists a \exists \varphi(a \in \text{On}^\omega \land V_a \land \forall y(y \in X \iff y \in V_a \land V_a \models \varphi[a,y]));$$

Lemma 1.5. Let $G$ be $\text{Col}(\omega, \kappa)$-generic over $V$ with $\kappa$ inaccessible. For each formula $\varphi(v_1)$, there is a corresponding formula $\bar{\varphi}(v_1)$ such that for any $x \in \text{On}^\omega \cap V[G]$,

$$V[G] \models \varphi[x] \iff V[x] \models \bar{\varphi}[x].$$

Proof. For any $x \in \text{On}^\omega \cap V[G]$, by Proposition 0.4 we have some $\text{Col}(\omega, \kappa)$-generic $H$ over $V[x]$ such that $V[G] = V[x][H]$. As $P$ is weakly homogeneous, we do not need to know what $H$ is to know whether $\varphi[x]$ holds:

$$V[x][H] \models \varphi[x] \iff V[x] \models '\exists P \varphi[x]'$$

Thus, we can take $\bar{\varphi}(v)$ to be this latter statement in the forcing language with one free variable corresponding to $x$.  \qed
This lemma is a key step in proving the following key result, which almost gets us all the way to Solovay’s result:

**Theorem 1.6.** Let $\kappa$ be inaccessible, $G$ be $\text{Col}(\omega, \kappa)$-generic. Then in $V[G]$, every $\text{On}^\omega$-definable set of reals is Lebesgue measurable.

**Proof.** Argue in $V[G]$. Note that for every $a \in \text{On}^\omega$, $\omega^\omega \cap V[a]$ is countable (it cannot be uncountable as $\omega^\omega$ is inaccessible in $V[G]$ because we have introduced surjections $\omega \to \lambda$ for all $\lambda < \kappa$ in the generic extension). Now suppose $A \in \omega^\omega$ is $\text{On}^\omega$-definable; let $a \in \text{On}^\omega$, $\varphi(v_1, v_2)$ define $A$. By the two-variable version of Lemma 1.1, there is some $\tilde{\varphi}(v_1, v_2)$ such that

$$x \in A \iff V[a][x] = \tilde{\varphi}[a, x]$$

We now show $A$ is Lebesgue measurable.

Note first that $\{x \in \omega^\omega \mid x \text{ not random over } V[a]\}$ is null, being equal to $\bigcup\{A_c \mid c \in V[a] \text{ is a } G_3 \text{ code for a null set}\}$ by our earlier characterisation of random reals, and this is a countable union of null sets and is thus null. Thus, to show $A$ measurable, it suffices to find $X$ Borel such that $A \triangle X$ consists of reals not random over $V[a]$.

Still working in $V[G]$, force with $B^*$ over $V[a]$, and let $\dot{r}$ be a canonical name for the random real. If $x$ is random over $V[a]$, recall this means for all closed codes $c$, $x \in A^V[a][x] \iff A^V[a] \in H$ for some $H$ $B^*$-generic over $V[a]$. But notice that

$$x \in A \iff V[a][x] = \tilde{\varphi}[a, x]$$

and by the Forcing Theorem we know that

$$V[a][x] = \tilde{\varphi}[a, x] \iff (\exists p \in H)(p \vdash \tilde{\varphi}[\dot{a}, \dot{r}])$$

for some $H$ which is $B^*$-generic over $V[a]$. Hence, let $Y \in V[a]$ be a maximal antichain consisting of closed sets each deciding $\tilde{\varphi}[\dot{a}, \dot{r}]$, all in the sense of $V[a]$. Then, if $x$ random, by the genericity of $G$

$$x \in A \iff x \in \bigcup\{A_c \mid A^V[a] \in Y \land A^V[a] \vdash \tilde{\varphi}(\dot{a}, \dot{r})\}$$

Since $Y$ is countable by the c.c.c. of $B^*$, the set on the right is a Borel (in fact an $F_\sigma$) set and can serve as our desired set $X$. 

### 1.5 A model of set theory in which every set of reals is Lebesgue measurable

Here, we diverge slightly from [Sol70]. In his original paper, Solovay passed to an inner model of the class of hereditarily $\text{On}^\omega$-definable sets. We instead pass to $L(\mathbb{R})$, the constructible closure of the reals (cf. §0.3). First we check that $L(\mathbb{R})$ inherits DC from $V$: 

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**Theorem 1.7.** \((ZF + DC). (DC)^L(\mathbb{R})\).

**Proof.** Recall that, similar to how we can well-order \(L\), we can construct a class-surjection \(\Phi : On \times \mathbb{R} \to \mathbb{R}\) such that, for any \(\alpha \in On\), \(\Phi|_{\alpha \times \mathbb{R}} \in L(\mathbb{R})\).

Suppose now \(X \in L(\mathbb{R})\), \(X \neq \emptyset\), and \(R\) is a total relation on \(X\) in \(L(\mathbb{R})\). Argue in \(V[G]\). By DC, there is an \(f \in X^\omega\) such that \(\langle f(n), f(n+1) \rangle \in R\) for all \(n \in \omega\). Let \(\delta\) be such that \(\text{ran}(f) \in \Phi^\omega(\delta \times \mathbb{R})\), and, using countable choice (which is implied by DC), let \(\langle a_n \mid n \in \omega \rangle\) be such that \(\forall n \in \omega \exists \xi < \delta\) such that \(\Phi(\xi, a_n) = f(n)\).

Define a relation \(\prec\) on \(\delta \times \omega\) by

\[ \langle \eta, i \rangle \prec \langle \zeta, j \rangle \iff i = j + 1 \land \langle \Phi(\zeta, a_j), \Phi(\eta, a_i) \rangle \in R \]

\(\Phi(\delta \times \mathbb{R}) \in L(\mathbb{R})\) and \(\langle a_n \mid n \in \omega \rangle \in L(\mathbb{R})\) by coding (any \(\omega\)-sequence of reals can be coded by a single real). Thus, \(\prec \in L(\mathbb{R})\).

**Claim.** \(\prec\) is ill-founded in \(L(\mathbb{R})\).

To prove the claim, recall that ‘\(R\) is well-founded’ is absolute between transitive \(\in\)-models of \(ZF\), so to show that \(\prec\) is ill-founded in \(L(\mathbb{R})\), it suffices to find an infinite descending \(\prec\)-chain in \(V\). Pick some \(i \in \omega\). We know that for some \(\xi < \delta\), \(\Phi(\xi, a_i) = f(i)\). Then, as \(\langle f(i), f(i+1) \rangle \in R\), \(\exists \nu\) such that \(\Phi(\nu, a_{i+1}) = f(i+1)\), and so \(\langle \nu, a_{i+1} \rangle \prec \langle \xi, a_i \rangle\). We can keep going using DC to pick an infinite descending \(\prec\)-chain.

Let \(E \subseteq \delta \times \omega\), \(E \in L(\mathbb{R})\) be non-empty and without an \(\prec\)-minimal element. Start with any \(\langle \eta, i_0 \rangle \in E\). Define a sequence \(\langle \eta_n \mid n \in \omega \rangle\) such that \(\eta_0 = \eta\) and

\[ \langle \Phi(\eta_n, a_{i_0+n}), \Phi(\eta_{n+1}, a_{i_0+n+1} \rangle \in R \]

for each \(n \in \omega\), specifying \(\eta_{n+1}\) to be the minimal possible given \(\eta_n\).

As we can define the above, \(\langle \Phi(\eta_n, a_{i_0+n}) \mid n \in \omega \rangle \in L(\mathbb{R})\), so \((DC)^L(\mathbb{R})\).

Finally, the result that we have been building up to:

**Theorem 1.8.** Suppose \(ZFC + \{\text{There exists an inaccessible cardinal } \kappa\} \) is consistent. Then so is \(ZF + DC + \{\text{every set of reals is Lebesgue measurable}\}\).

**Proof.** Building from the above, continue to work in \(V[G]\). Note \(\omega^\omega \cap L(\mathbb{R}) = \omega^\omega\), so by the absoluteness of codes every open, closed, \(G_\delta\) and \(F_\sigma\) (and in fact every Borel set, if we code carefully) set is in \(L(\mathbb{R})\) and satisfies these properties in \(L(\mathbb{R})\).

Suppose now that \(A \subseteq \omega^\omega\) is in \(L(\mathbb{R})\). Then \(A\) is On\(\omega\)-definable, and so there is an \(F_\sigma\) set \(X\) such that \(A \Delta X\) is null. It follows that there exists a sequence of open codes \(\langle c_n \mid n \in \omega \rangle\), with \(A \Delta X \subseteq A_{c_0}\), such that \(\mu(A_{c_0}) < \frac{1}{n+1}\). By the previous paragraph, and by the absoluteness results established earlier, all of this still holds in \(L(\mathbb{R})\), so \((A\text{ is Lebesgue measurable})^L(\mathbb{R})\).
2 Descriptive Set Theory

We mostly follow [Jec02] in our development of the fundamentals of descriptive set theory, although [Kan09] is followed in the exposition of the lightface hierarchy.

In order to understand the next result, we must first introduce some descriptive set theory. Descriptive set theory is the study of subsets of the reals and other Polish spaces. The starting point is the classification of the Borel sets of, say, $\omega^\omega$ into a hierarchy according to their complexity:

2.1 The Boldface Hierarchy

The boldface hierarchy classifies all the Borel sets of a Polish space $X$ according to their complexity in the Borel hierarchy. It consists of classes $\Sigma^0_\alpha$, $\Pi^0_\alpha$, $\Delta^0_\alpha$ for each countable ordinal $\alpha$. Let $A$ be a subset of $X$

- $A$ is $\Sigma^0_1$ if it is open
- $A$ is $\Pi^0_\alpha$ if it is the complement of a $\Sigma^0_\alpha$ set
- $A$ is $\Sigma^0_\alpha$ if $A = \bigcup_{i<\omega} A_i$ with $A_i \in \Pi^0_{\alpha_i}$ with $\alpha_i < \alpha$.
- $A$ is $\Delta^0_\alpha$ if it is both $\Sigma^0_\alpha$ and $\Pi^0_\alpha$

We can prove by induction that every Borel set is in one of the above classes, and that the classes really do form a strict hierarchy: $\Sigma^0_\alpha \not\subseteq \Pi^0_\alpha$ for any countable $\alpha > 0$.

However, if $f : X \to X$ is a continuous function and $A$ is Borel, it is not necessarily the case that $f(A)$ is a Borel set. We say a set $A$ is analytic if there is a Borel set $B$ and $f : X \to X$ such that $f(B) = A$. We thus define the hierarchy of projective sets for each $n \in \omega$ as follows:

- $A$ is $\Sigma^1_1$ if it is analytic
- $A$ is $\Pi^1_n$ if it is the complement of a $\Sigma^1_n$ set
- $A$ is $\Sigma^1_{n+1}$ if $\exists B \in \Pi^1_n$, $f : X \to X$ continuous such that $f(B) = A$.
- $A$ is $\Delta^1_n$ if it is both $\Sigma^1_n$ and $\Pi^1_n$

Fact: $\Delta^1_1$ sets are precisely the Borel sets (that is, the analytic sets whose complements are also analytic are precisely the Borel sets).

2.2 The Lightface Hierarchy

The lightface hierarchy is more motivated by the definability of subsets of the reals. For the sake of example, we will work in the space $\omega^\omega$. There are two flavours: absolute,
and relativised to some real parameter $a$.

Second-order arithmetic is the two-sorted structure

$$A^2 = \langle \omega, \omega^\omega, ap, +, \times, \text{exp}, <, 0, 1 \rangle$$

where $\omega$ and $\omega^\omega$ are taken to be our two domains and $ap : \omega^\omega \times \omega \to \omega$, connecting the two domains, is given by $ap(x, m) = x(m)$. We will implicitly use variables $m_1, m_2, \ldots$ to range over members of $\omega$ and $x_1, x_2, \ldots$ to range over members of $\omega^\omega$.

Let $\langle m_1, \ldots, m_i, x_1, \ldots, x_j \rangle \in \omega^i \times (\omega^\omega)^j$. We will write $A(m_1, \ldots, m_i, x_1, \ldots, x_j)$ to mean $\langle m_1, \ldots, m_i, x_1, \ldots, x_j \rangle \in A$ (conflating the extension of a predicate with its intension). We say that $A$ is arithmetical if $A$ is definable by a formula in $A^2$ whose only function quantifiers (i.e. quantifiers over $\omega^\omega$) are bounded.

As we can shift bounded quantifiers to the right and contract like quantifiers, we have the following arithmetic hierarchy: for $w \in \omega^i \times (\omega^\omega)^j$,

- $A \in \Sigma^n_0 \iff \forall w (A(w) \Leftrightarrow \exists m_1 \forall m_2 \ldots Q m_n R(m_1, \ldots, m_n, w))$
- $A \in \Pi^n_0 \iff \forall w (A(w) \leftrightarrow \forall m_1 \exists m_2 \ldots Q m_n R(m_1, \ldots, m_n, w))$

where $Q$ is such that we have alternating quantifiers and $R$ has only bounded quantifiers. It can be easily seen that $\Sigma^n_0$ and $\Pi^n_0$ are dual collections.

Next, we say a set $A$ is analytical if it can be defined in $A^2$. We can shift number quantifiers to the right and function quantifiers to the left, and so we have the following hierarchy of analytical sets: first set $\Sigma^1_0 = \Sigma^0_1$ and $\Pi^0_1 = \Pi^1_0$, and then

- $A \in \Sigma^1_n \iff \forall w (A(w) \leftrightarrow \exists x_1 \forall x_2 \ldots Q x_n R(x_1, \ldots, x_n, w))$
- $A \in \Pi^1_n \iff \forall w (A(w) \leftrightarrow \forall x_1 \exists x_2 \ldots Q x_n R(x_1, \ldots, x_n, w))$

where $Q$ is as above and $R$ is arithmetical.

We now turn to the relativised lightface hierarchy. Given a parameter $a \in \omega^\omega$, second-order arithmetic relative to $a$ is the structure

$$A^2(a) = \langle \omega, \omega^\omega, ap, +, \times, \text{exp}, <, 0, 1, a \rangle$$

where $a$ is taken to be a binary relation (i.e. $m a n \iff a(m) = n$). We have thus augmented our structure with an oracle for $a$, and so we can define the arithmetical and analytical hierarchies relative to $a$ (written as $\Sigma^n_k(a)$ for example) exactly as above. In fact, the relativised hierarchies link the lightface and boldface hierarchies through the following proposition:

**Proposition 2.1.** Suppose that $A \subseteq \omega^\omega$ and $n > 0$. Then:

1. $A \in \Sigma^n_0 \iff A \in \Sigma^n_k(a)$ for some $a \in \omega^\omega$, and similarly for $\Pi^n_0$.
2. $A \in \Sigma^n_1 \iff A \in \Sigma^n_k(a)$ for some $a \in \omega^\omega$, and similarly for $\Pi^n_1$. 

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3 Measurability entails large cardinal assumptions

Most of the following is from [Bek91]

In this section, we give an exposition of Shelah’s result [She84] that if every projective set of reals is Lebesgue measurable, then $\omega_1^V$ is inaccessible in $L$. While Shelah’s original proof used complex forcing ideas, we will instead follow [Bek91], who adapted his exposition from lectures given by S. Todorcevic at UC Boulder in 1987 (the first combinatorial proof was given by Raisonnier in [Rai84])

We will produce a definable (in fact, a $\Sigma^1_3$) non-measurable set of reals, assuming that $\omega_1$ is not inaccessible in $L$. A corollary of this is that, if all projective sets are measurable, then $\omega_1$ is inaccessible in $L$. For the following section, we will take the reals to be the Cantor space $2^\omega$ instead of $\omega^\omega$, and we will often conflate $2^\omega$ with $\mathcal{P}(\omega)$, the power set of $\omega$.

We begin with a theorem (the proof can be found in any good textbook on measure theory). $E \subseteq 2^\omega$ is a tail set if, whenever $x \in E$ and $x|_{\omega \setminus n} = y|_{\omega \setminus n}$ for some $n$ (where we identify $n$ with the set of its predecessors), then $y \in E$ (i.e. membership of a tail set is unaffected by finite changes). We have:

**Theorem 3.1** (Kolmogorov’s 0-1 Law). If $E$ is a tail set, and $E$ is measurable, then $\mu(E) = 0$ or $1$.

3.1 Some non-measurable sets of $2^\omega$ which we will need later

**Proposition 3.2.** Let $U$ be a free ultrafilter on $\omega$. Then $U$ regarded as a subset of $2^\omega$ is not measurable.

**Proof.** As $U$ is free, it is not principal, so $U$ is a tail set in $2^\omega$. So, if $U$ is measurable, $\mu(U) = 0$ or $\mu(U) = 1$. Now, let $T : 2^\omega \to 2^\omega$ be the ‘bit-flipping’ operator: $T(x) = \omega \setminus x$. $T$ preserves measure, and, since $U$ is an ultrafilter, $x \in U \iff T(x) \not\in U \iff T(x) \in T(U)$. So $U \cap T(U) = \emptyset$, $2^\omega = U \cup T(U)$ so $\mu(U) = \mu(T(U)) = \frac{1}{2}$, contradiction. 

**Lemma 3.3.** Let $F$ be a filter on $\omega$ such that $\emptyset \not\in F$ and $\omega \setminus n \in F$ for each $n \in \omega$. Then, if $F$ is measurable, $\mu(F) = 0$.

**Proof.** Note that $F$ is a tail set. Thus, if $F$ is measurable, then its measure is either 0 or 1. It cannot be 1 as $F \cap T(F) = \emptyset$, so $\mu(F) + \mu(T(F)) > 1$ contradicting that $\mu$ is a probability measure on $2^\omega$.

The following lemma is useful for our next result:
Lemma 3.4. Suppose $X \subseteq 2^\omega$ meets every compact subset of $2^\omega$ of positive measure. Then $\mu^*(X)$, the outer measure of $X$, is non-zero.

Proof. Recall that the outer measure of a set $X$ is given by

$$\mu^*(X) = \inf\{\mu(A) \mid X \subset A, \ A \text{ Lebesgue measurable}\}$$

Also, as $\mu$ is a regular measure, we know that for all $\varepsilon > 0$ and for all $X$ measurable there exists a $U$ open such that $X \subseteq U$ and $\mu(X) > \mu(U) - \varepsilon$. Thus, in the definition of outer measure, it suffices to only consider the infimum over open sets.

We will prove the contrapositive: if $\mu^*(X) = 0$, then there exists a compact $B$ such that $\mu(B) > 0$ and $X \cap B = \emptyset$. As $2^\omega$ is compact, the compact subsets of $2^\omega$ are precisely the closed subsets, and so we will find some $t \in 2^{<\omega}$ such that $O(t) \cap X = \emptyset$.

As $\mu^*(X) = 0$, for all $\varepsilon > 0$ there is some $U_\varepsilon$ open such that $X \subseteq U_\varepsilon$ and $\mu(U_\varepsilon) < \varepsilon$. Pick $\varepsilon < 2^{-n}$ for some large $n$. Then we know $U_{2^{-n}} = \bigcup O(s_i)$ for some $s_i \in 2^{<\omega}$ with $|s_i| > n$, and this union is disjoint, so that none of the $s_i$'s are comparable. Hence, the $s_i$'s form an antichain in $2^{<\omega}$. However, this antichain is not maximal, because $\bigcup O(s_i) \neq 2^{<\omega}$, so there is some $t$ which is incomparable to the $s_i$'s, and so $O(t) \cap U_{2^{-n}} = \emptyset$ and so $O(t)$ is compact, of positive measure and disjoint from $X$.

We also require the following theorem (given without proof):

Theorem 3.5 (Lebesgue Density Theorem). Let $A$ be a Lebesgue measurable set of $2^\omega$, let $x \in A$. Let

$$d_n(x) = \frac{\mu(A \cap O(x \upharpoonright n))}{\mu(O(x \upharpoonright n))}$$

Then $d_n(x) \to 1$ as $n \to \infty$ for almost all $x \in A$.

Proof. See, e.g., [Rud87].

Proposition 3.6. Let $\mathcal{F}$ be a filter on $\omega$ with the property that, for any increasing sequence $\langle n_i \mid i \in \omega \rangle$ of natural numbers, there is $a \in \mathcal{F}$ such that for all $i$, $|a \cap n_i| \leq i$ (where we identify $n$ with the set of its predecessors). Then $\mathcal{F}$ is not measurable.

Proof. Note that $\mathcal{F}$ contains Frechet’s filter, as otherwise it would be principal, contradicting the hypotheses (if $\mathcal{F}$ principal, then there is some $X \subseteq \omega$ such that $\bigcap a = X$).

If $X$ finite, pick some $\langle n_i \mid i \in \omega \rangle$ such that $n_0 > \max(X)$, contradiction. Otherwise enumerate the elements of $X$ in increasing order as $x_1, x_2, \ldots$ and pick some $n_i > x_{i+1}$ for each $i$, contradiction). Hence, if $\mathcal{F}$ is measurable, $\mu(\mathcal{F}) = \mu^*(\mathcal{F}) = 0$.

Now, if we show that $\mathcal{F}$ meets every compact subset of $2^\omega$ of positive measure, then we can conclude that $\mathcal{F}$ is not measurable, as then $\mu^*(\mathcal{F}) > 0$, contradiction.
First, fix some notation. For \( X \subseteq 2^\omega \), let \( T_X \) be the subtree of \( 2^{<\omega} \) consisting of all the \( f \in X \) ordered by extension, and let \( T_X(n) \) be the restriction of the above tree to the \( n^{th} \) level. Thus, \( T_X(n) = \{ f \mid n \in \omega, f \in X \} \), ordered by extension.

Let \( A \) be a compact subset of \( 2^\omega \), \( \mu(A) > 0 \). We aim to show \( \mathcal{F} \) meets \( A \). By the Lebesgue Density Theorem, we can refine \( A \) to a compact subset \( B \), still of positive measure, such that there is a sequence \( \langle T(i) \mid i \in \omega \rangle \) of finite maximal antichains of \( T_B \) such that, if \( i < j \), then \( \max\{ |s| \mid s \in T_B(i) \} < \min\{ |t| \mid t \in T_B(j) \} \) and, for all \( i \), for all \( s \in T_B(i) \),

\[
\mu(O(s) \cap B) > \left( 1 - \frac{1}{2^i} \right) \mu(O(s)) \tag{3}
\]

Let \( n_i \) be the minimal integer larger than \( \max\{|s| \mid s \in T(i)\} \). Pick \( a \in \mathcal{F} \) such that \( |a \cap n_{i+2}| \leq i \) for all \( i \in \omega \). We will inductively construct an increasing sequence \( \langle s_i \mid i \in \omega \rangle \) satisfying:

1. \( s_i \in T_B(i) \)
2. \( O(s_i) \cap B \) is non-empty
3. \( m \in a \cap n_i \) implies \( s_i(m) = 1 \).

Suppose we have \( s_i \). Let \( H = \{ x \in 2^\omega \mid \forall n \in [ |s_i|, n_{i+1} ] , \ x(n) = 1 \} \). By our choice of \( a \), \( \mu(H) \geq \frac{1}{2^{i-1}} \) (essentially because we specify values for at most \( i - 1 \) elements). Therefore, by (3), \( (O(s_i) \cap B) \cap H \) is non-empty: notice that \( O(s_i) \) and \( H \) are independent, so \( \mu(O(s_i) \cap H) = \mu(O(s_i)) \mu(H) \geq \frac{1}{2^{i-1}} \mu(O(s_i)) \). Thus, if \( O(s_i) \cap (B \cap H) \) were empty, then \( \mu(O(s_i) \cap (B \cup H)) = \mu(O(s_i) \cap B) + \mu(O(s_i) \cap H) \geq (1 - \frac{1}{2^i}) \mu(O(s_i)) + \frac{1}{2^i} \mu(O(s_i)) > \mu(O(s_i)) \), contradiction. Fix some \( x \in H \) in this intersection. As \( T_B(i+1) \) is a maximal antichain, \( \exists s_{i+1} \in T_B(i+1) \) such that \( s_{i+1} \subset x \). This is our inductive step.

Finally, let \( b = \bigcup s_i \). Note first \( a \subseteq b \) (considered as subsets of \( \omega \)), so \( b \in \mathcal{F} \). To show \( b \in B \), argue by contradiction. If \( b \notin B \), \( \exists t \in 2^{<\omega} \) such that \( b \supseteq t \) and \( O(t) \cap B \) is empty (\( B \) is compact which implies \( B \) is closed). Thus, \( \exists i \in \omega \) such that \( O(s_i) \cap B = \emptyset \), contradiction. \( \square \)

### 3.2 Null sets in \( L[x] \)

First of all, another theorem from measure theory:

**Theorem 3.7** (Fubini’s Theorem). Let \( X \) be a measure space, suppose \( A \subseteq X \times X \) is measurable. Then \( \mu(A) = 0 \) iff for almost all \( x \), \( A_x = \{ y \in X \mid \langle x, y \rangle \in A \} \) is of measure 0.

(cf. [Jec02]).
Proposition 3.8. To every real $x$, we can associate a $\Sigma^1_2(x)$ set $A(x) \subseteq 2^\omega \times 2^\omega$ such that $A(x)$ is measurable iff the union of all Borel sets of measure 0 coded in $L[x]$ is of measure 0. Moreover, $A(x)$ is defined by:

$\langle u, v \rangle \in A(x)$ iff there is a countable model $M$ of enough ZFC and $V = L[x]$ (so that $M = L_\alpha[x]$ by reflection) such that there is a real $z$ in $M$ which codes a null $G_\delta$ set containing $u$ but no $y <_{L[x]} z$ codes a null $G_\delta$ set containing $v$ (where $<_{L[x]}$ is a well-ordering of $L[x]$).

Remark.

1. The above union of Borel sets may be uncountable

2. Again, it suffices to only consider $G_\delta$ Borel sets, rather than arbitrary ones (by the regularity of the Lebesgue measure)

3. To better motivate the definition of $A(x)$, let $\langle z_\xi \mid \xi < \delta \rangle$ be a sequence of reals enumerating all the $G_\delta$ codes for null sets in $L[x]$. For a limit $\lambda$, set $\check{G}_\lambda = G_\lambda \setminus (\bigcup_{\eta < \lambda} G_\eta)$; finally, $G = \bigcup_{\nu < \delta} G_\nu$. Then, $A(x)$ can be viewed as a subset of $G \times G$:

$\langle u, v \rangle \in A(x)$ iff there exist $\xi, \eta$ such that $u \in \check{G}_\xi$ and $v \in \check{G}_\eta$ with $\xi < \eta$.

Proof. Suppose $\mu^*(G) = c > 0$ (so the union of all Borel null sets has positive measure).

Define a measure $m$ on $\mathcal{B}(G)$, the Borel subsets of $G$, by $m(B) = \mu^*(G \cap B)$ where $B$ is in $\mathcal{B}(2^\omega)$ such that $B \cap G = B$. Then $A(x) \subseteq G \times G$ cannot be $m$-measurable by Fubini’s theorem (similarly to how we show any well-order of $2^\omega$, regarded as a subset of $(2^\omega)^2$ is non-measurable).

Conversely, suppose the union of all $G_\delta$ sets of measure 0 coded in $L[x]$ is of measure 0. We will show all $\Sigma^1_2(x)$ sets of reals are measurable. Let $A$ be a $\Sigma^1_2(x)$ set, let $A = \{ z \in 2^\omega \mid \varphi(x, z) \}$. $z \in A$ iff $L[x, z] \models \varphi(x, z)$ (by Shoenfield’s Absoluteness Theorem, cf. [Jec02]). Let $B = \mathcal{B}^L[x]$, where $\mathcal{B}$ is the measure algebra on $2^\omega$. Forcing with $B$ adds a random real $r$; let $\check{r}$ be its canonical name. Let $C = ||\varphi(\check{x}, \check{r})||_B$ (the truth value of $\varphi(\check{x}, \check{r})$ in the Boolean-valued model of ZFC taking truth values in $\mathcal{B}$). Note that $C$ is a Borel set.

In $V$, we claim that $A \Delta C$ is of measure 0, and so $A$ is Lebesgue measurable. Let $S = 2^{\omega} \setminus \bigcup_{\xi} G_\xi$ where the union runs over all $G_\delta$ null sets coded in $L[x]$. Then we claim that $(A \Delta C) \cap S = \emptyset$: suppose not. Pick $r \in (A \Delta C) \cap S$.

Case 1. $r$ is in $C \setminus A$. Then $r$ is generic over $L[x]$, and so $L[x, r] \models \varphi(x, r)$, so $x \in A$, contradiction.

Case 2. $r$ is in $A \setminus C$. Then $r \notin C$, so $L[x, r] \models \neg \varphi(x, r)$, and, since $V \models \varphi(x, r)$ iff $L[x, r] \models \varphi(x, r)$, if follows that $V \models \neg \varphi(x, r)$, so $x \notin A$, contradiction. \qed
3.3 A $\Sigma_3^1$ filter on $2^\omega$

Let $a \subseteq \omega$ (so $a \in 2^\omega$) be a real, and partition $[2^\omega]^2$ into $K^a \coprod (K^a)^c$, where $\{x, y\} \in K^a$ iff $\Delta(x, y) \in a$, where $\Delta(x, y) = \min\{j \mid x(j) \neq y(j)\}$.

We say a set $X \subseteq 2^\omega$ is $a$-homogeneous if $[X]^2 \subseteq K^a$. Note we can identify $[2^\omega]^2$ as a subset of $2^\omega \times 2^\omega$: $[2^\omega]^2 = \{(x, y) \mid x < y\}$ (in the lex. order). With this identification, the closure of any $a$-homogenous set is $a$-homogeneous (as being $a$-homogeneous is a head property).

For a parameter $x \in 2^\omega$, we associate a subset $\mathcal{F}(x)$ of $2^\omega$ defined by: $b \in \mathcal{F}(x)$ iff there exists $\langle F_i \mid i \in \omega \rangle$, a sequence of closed $b$-homogeneous sets such that $\bigcup_{i \in \omega} F_i \supseteq L(x) \cap 2^\omega$

(i.e. this union contains all the reals in $L(x)$). Note that $\mathcal{F}(x)$ is a $\Sigma_3^1(x)$ set.

**Proposition 3.9.** If $2^\omega \cap L[x]$ is uncountable, then $\mathcal{F}(x)$ is a filter that contains the Frechet filter, and so if $\mathcal{F}(x)$ is measurable then $\mu(\mathcal{F}(x)) = 0$.

**Proof.**

1. $\emptyset = (0,0,0,\ldots) \notin \mathcal{F}(x)$, as $K^\emptyset$ is empty and so there are no $\emptyset$-homogeneous sets.
2. Let $a = \omega \setminus n$, let $F_s = \{f \mid s \subset f, s \in 2^n\}$. Then:
   
   (a) For every $x, y \in F_s$, $\Delta(x, y) = n$ and so is in $a$. Hence, $F_s$ is $a$-homogeneous, and so is $\overline{F_s}$, the closure of $F_s$.

   Hence $a \in \mathcal{F}(x)$ for each $n$.
3. $\bigcup F_s = 2^\omega \supseteq L(x) \cap 2^\omega$
4. Let $a, b \in \mathcal{F}(x)$ and $F_i^a, F_j^b$ the corresponding family of $a$ (resp. $b$)-homogeneous sets. Let $F_{ij} = F_i \cap F_j$. Then $\bigcup_{i,j \in \omega} F_{ij}$ covers $L(x) \cap 2^\omega$, and for $x, y \in F_{ij}$, $\Delta(x, y) \in a \cap b$. Hence $a \cap b \in \mathcal{F}(x)$.
5. Suppose $a \subseteq b$ (so $b(i) = 1$ whenever $a(i) = 1$), let $\langle F_i \mid i \in \omega \rangle$ be the corresponding family of closed $a$-homogeneous sets. Then $F_i$ are $b$-homogeneous too, so $b \in \mathcal{F}(x)$. \hfill $\square$

Now, we would like to show that $\mathcal{F}(x)$ meets every compact subset of $2^\omega$ (and hence that $\mathcal{F}(x)$ is non-measurable). However, we need the following lemma first, which gives us a correspondence between elements of $\omega^\omega$ (i.e. functions $\omega \to \omega$) and null sets in $2^\omega$.

**Lemma 3.10.** To any function $f$ in $\omega^\omega$, we can associate a measure 0 $G_\delta$ set $N_f \subseteq 2^\omega$ and to any open $U$ in $2^\omega$ of measure less than 1 we associate a function $\phi_U : \omega \to [\omega]^<\omega$ with $|\phi_U(n)| \leq n$ for each $n$ such that, if $N_f$ is a subset of $U$, then $f$ is ‘almost captured’ by $\phi_U$: there is an $m$ such that, for all $n > m$, $f(n) \in \phi_U(n)$.

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Proof. For \( n, m \in \omega \), define \( B_{n,m} = \{ x \in 2^\omega \mid x(2^m \cdot 3^n \cdot 5^k) = 0 \ \forall k \leq n \} \). Note that the \( B_{n,m} \)'s are independent, clopen, and \( \mu(B_{n,m}) = \frac{1}{2^{n+1}} \) (as we specify values at \( n+1 \) indices).

Thus, \( N_m = \bigcup_{n \geq m} B_{n,m} \) is open, and

\[
\mu(N_m) \leq \sum_{n \geq m} \frac{1}{2^n} = \frac{1}{2^m}
\]

and so \( N_f = \bigcap_m N_m \) is a \( G_\delta \) set of measure 0.

Suppose \( U \) open in \( 2^\omega \), and \( \mu(U) < 1 \). Let \( K = 2^\omega \setminus U \). We can assume, without loss of generality, that

\[
\forall s \in 2^{<\omega}, \ (O(s) \cap K \neq \emptyset \rightarrow \mu(O(s) \cap K) > 0)
\]

Let \( T_K \) be the subtree of \( 2^{<\omega} \) such that for all \( s \in T_K, O(s) \cap K \neq \emptyset \).

Let \( K_s = O(s) \cap K \). For a fixed \( n \in \omega \), \( s \in 2^{<\omega} \), we will bound the size of \( A_s(n) = \{ m \mid K_s \cap B_{n,m} = \emptyset \} \). Note that \( |A_s(m)| \) is finite (because \( s \) is a finite string). Moreover,

\[
0 < \mu(O(s) \cap K) = \mu(K_s) \leq \mu \left( \bigcap_{m \in A_s(n)} 2^\omega \setminus B_{n,m} \right) = \prod_{n \in \omega} \left( 1 - \frac{1}{2^{n+1}} \right)^{|A_s(n)|}
\]

But

\[
\prod_{n \in \omega} \left( 1 - \frac{1}{2^{n+1}} \right)^{|A_s(n)|} \leq \prod_{n \in \omega} e^{-\frac{|A_s(n)|}{2^{n+1}}} = e^{-\sum_{n \in \omega} \frac{|A_s(n)|}{2^{n+1}}}
\]

and so \( \sum_{n \in \omega} \frac{|A_s(n)|}{2^{n+1}} \) converges (if not, \( e^{-\sum_{n \in \omega} \frac{|A_s(n)|}{2^{n+1}}} \) would be 0). In particular, \( |A_s(n)| \) gets very small very quickly as \( n \to \infty \).

Now, define \( \phi_U \). Fix a bijection \( i: 2^{<\omega} \to \omega \). For each \( s \in 2^{<\omega} \), there is some \( n(s) \) such that for all \( m > n(s) \), \( |A_s(m)| \leq \frac{1}{2^{i(s)+1}} \). Let \( \phi_U = \bigcup \{ A_s(m) \mid m \geq n(s), s \in T_K \} \); let \( \phi_U(n) \) for \( n > n(s) \) be \( \bigcup \{ A_s(n) \mid s \in T_K \} \).

Claim. \( |\phi_U(n)| < 2^{n+1} \) for all \( n \).

\[
|\phi_U(n)| \leq \sum_{n \geq n(s)} \frac{|A_s(n)|}{2^{n+1}} \leq \sum_{s \in T_K} \frac{1}{2^{i(s)+1}}
\]

but as \( i \) is a bijection,

\[
\sum_{s \in T_K} \frac{1}{2^{i(s)+1}} \leq \sum_{\ell \in \omega} 2^{-(\ell+1)} \leq 1
\]

Now, if \( N_f \subseteq U \) (i.e. \( N_f \cap K = \emptyset \); i.e. \( K \cap (\bigcap_{m \geq 0} B_m) = \emptyset \)), we claim that there are \( s, m \) such that \( K_s \cap B_m = \emptyset \). Otherwise, for all \( s, m, K_s \cap B_m \neq \emptyset \) and so \( K \cap B_m \) is
dense in $K$ (for each $m$) and so by the Baire Category Theorem so is $\bigcap_{m>0}(K \cap B_m)$, but $K \cap N_f = \emptyset$, contradiction. Hence, fix $s'$, $m'$ minimal such that $K_{s'} \cap B_{m'} = \emptyset$. For $m \geq m'$, $n(s')$, we have $B_{m,f(m)} \subseteq B_m \subseteq B_{m'}$. So $K \cap B_{m,f(m)} = \emptyset$, i.e. $f(m) \in A_{s'}(m)$. Therefore $f(m) \in \phi_U(m)$.

Now, we shrink $\phi$ and set $\tilde{\phi}$. Therefore $\bar{m}$. Every compact subset $B$ is dense in $K$.

**Proposition 3.11.** Every compact subset $B$ of $2^{\omega}$ of positive measure such that $A(x;B)$ (the analogue of $A(x)$ constructed in $L[x,B]$) is measurable meets $F(x)$.

**Remark.** Note that we can write $A(x;B)$ depending on the parameter $B$ as a function of a real single $A(y)$ where $y$ codes both $x$ and $B$.

**Proof.** As $A(x;B)$ is measurable, we can assume that the union of all Borel null sets coded in $L[x;B]$ has measure 0. By considering the proof of Proposition 3.6, it suffices to show that there exists $a \in F(x)$ such that, for the sequence $\langle n_i \mid i \in \omega \rangle$ which is determined by $B$ (as in the proof of 3.6), $a \cap n_i \mid i < j$.

For $f \in 2^{\omega} \cap L[x;B]$, let $\tilde{f}(i) = f \upharpoonright n_i$ (and we code finite strings of 0’s and 1’s as a single natural number). Then $\tilde{f} \in L[x;B]$ and hence is $N_f$ and, in particular, $N_{\tilde{f}}$ still has measure 0 in $L[x;B]$. Thus, $\bigcup_{f \in 2^{\omega} \cap L[x;B]} N_f$ is of measure 0. Pick $U$ open of measure < 1 which covers this union; i.e. $N_f \subseteq U$ for all $f \in 2^{\omega} \cap L[x;B]$. By Lemma 3.10, for all $f \in M \cap 2^{\omega}$, there is some $j$ and $\phi_U$ as above such that $\forall i > j$. Let $a$ be such that $n \in a$ (i.e. $a(n) = 1$) if, for $i$ minimal such that $n_i \geq n$, there exist $s,t \in \phi_U(n_i)$, $s \neq t$ such that $\Delta(s,t) = n$. Then $|a \cap (n_{i-1}, n_i)| \leq i$ and so $|a \cap n_i| \leq i^2$. Passing to a suitable subsequence of $\langle n_i \rangle$ (say $\langle n_{i+1} \rangle$), we get what we want.

Now, we just have to show that $a \in F(x)$; i.e. there is some sequence $\langle F_i \mid i \in \omega \rangle$ of closed $a$-homogeneous sets such that $\bigcup_{i \in \omega} F_i \supseteq 2^{\omega} \cap L[x]$. For each $s$ in $\bigcup_{i \in \omega} 2^{n_i}$, let $F_s$ be the set of all $f \in 2^{\omega} \cap L[x;B]$ such that $s \subseteq f$ and for all $i \geq |s|$ $f \upharpoonright n_i \in \phi_U(i)$. It is clear that $\bigcup_{s \in 2^{n_i}} \bigcup_{i \in \omega} 2^{n_i} \subseteq 2^{\omega} \cap L[x]$. Now, we show each $F_s$ is $a$-homogeneous. For $f,g \in F_s$, consider $\Delta(f,g)$. Let $i$ be minimal such that $f \upharpoonright n_i \neq g \upharpoonright n_i$; say $t = f \upharpoonright n_i$, $u = g \upharpoonright n_i$. Then $t,u \in \phi_U(n_i)$ and $\Delta(f,g) = \Delta(t,u) = n < n_i$, and so, by the definition of $a$, $n \in a$ and we are done. \qed
3.4 Can you take Solovay’s inaccessible away?

We now come to the main theorem of this section:

**Theorem 3.12.**

1. If every \( \Sigma_1^2 \) set is measurable, then, for all \( x \in 2^{\omega} \) such that \( 2^{\omega} \cap L[x] \) is uncountable, \( \mathcal{F}(x) \) is not measurable.

2. If every \( \Sigma_1^3 \) set is measurable, then \( L \models ‘\omega_1 \text{ is inaccessible}’ \).

**Proof.** Recall that all \( \Sigma_1^3 \) sets are \( \Sigma_1^3(x) \) in some \( x \).

1. This follows from Propositions 3.6, 3.8 and 3.11: if every \( \Sigma_1^2 \) set is measurable, then this means that \( A(x) \) is measurable for each \( x \), and so, for all \( x \), the union of all Borel measure 0 sets coded in \( L[x] \) is of measure 0. This implies that \( \mathcal{F}(x) \) meets every compact subset of \( 2^{\omega} \), and hence \( \mathcal{F}(x) \) is not measurable.

2. Suppose \( L \models ‘\omega_1 \text{ is not inaccessible}’ \). Then, as GCH holds in \( L \), we have that \( L \models ‘\omega_1 \text{ is a successor}’ \) (the property of being regular is downwards absolute, as any \( f \) which witnesses the singularity of \( \alpha \) in \( L \) would witness the singularity of \( \alpha \) in \( V \)), say \( L \models ‘\omega_1 = \alpha^+’ \) for \( \alpha \) countable (in \( V \)). Let \( x \) be a code for \( \alpha \) (say a well-ordering of \( \mathbb{N} \) of order-type \( \alpha \)). Then \( L[x] \models ‘\alpha \text{ is countable}’ \), \( L[x] \models ‘\omega_1 \text{ is the first uncountable cardinal}’ \); i.e. \( \omega_1^{L[x]} = \omega_1 \). Hence \( 2^{\omega} \cap L[x] \) is uncountable, and, by 1) \( \mathcal{F}(x) \) is not measurable; but \( \mathcal{F}(x) \) is \( \Sigma_3^1(x) \), contradiction.

\[ \Box \]

4 Larger large cardinals decide the measurability of more sets

A primary reference for this is [Bek91]

The above two sections have showed that large cardinal notions (such as inaccessibility) decide the measurability of all \( \text{On}^\omega \)-definable sets of reals in a forcing extension of \( V \); by passing to a suitable inner model of \( V[G] \), we get a model of set theory in which every set of reals is measurable. However, this is quite a small subclass of sets, and so the natural question is whether stronger large cardinal assumptions settle the decidability of more sets of reals. In this section, we will give a brief overview of the argument given by Shelah and Woodin in [SW90] showing that, assuming the existence of a supercompact cardinal, every set of reals in \( L(\mathbb{R}) \) (in \( V^\mathbb{R} \)) is measurable.
4.1 Supercompact cardinals and forcing absoluteness

A cardinal $\kappa$ is $\lambda$-supercompact if there exists an elementary embedding $j : V \rightarrow M$ of the universe into a transitive class $M$ such that $\kappa$ is the critical point of $j$ (i.e. $j(\alpha) = \alpha$ for all $\alpha < \kappa$), $j(\kappa) > \lambda$ and $M^\lambda \subseteq M$ (i.e. $M$ contains all its $\lambda$-sequences).

A cardinal $\kappa$ is supercompact if $\kappa$ is $\lambda$-supercompact for all $\lambda$. From now on, let $\kappa$ be supercompact.

We say that an ordinal is large enough (or simply large) if $\lambda$ is strong limit and $\text{cf}(\lambda) > \kappa$.

Lemma 4.1. In $L(\mathbb{R})$ every set is definable using a real and a finite sequence of large enough ordinals.

Proof. Let $X$ be the class of sets definable using a real and a finite sequence of large ordinals. Then $X$ is a submodel of $L(\mathbb{R})$ (as $L(\mathbb{R})$ is the class of all sets definable using parameters from the reals and every finite sequence of large ordinals is eventually captured at some stage of the hierarchy in $L(\mathbb{R})$). Also, $\mathbb{R} \subseteq X$. So take the transitive collapse of $X$, $\tilde{X}$. Then $\tilde{X} = L(\mathbb{R})$ (as $L(\mathbb{R})$ is the least transitive class model of ZFC with the above properties)

We will begin with some lemmas about forcing with weakly compact cardinals. Note that every supercompact cardinal is weakly compact.

Lemma 4.2. If $\kappa$ is weakly compact and $Q$ is a $\kappa$-c.c. poset of cardinality $\kappa$, then for every $A \subseteq Q$ of size less than $\kappa$ there exists a $\tilde{Q}$ also of size less than $\kappa$ such that $A \subseteq \tilde{Q} \subseteq Q$ and $\tilde{Q}$ is a regular subposet of $Q$ (i.e. every maximal antichain in $\tilde{Q}$ is a maximal antichain in $Q$).

Proof. By a $\Pi^1_1$-indescribability argument (see [Jec02])

Applying this lemma to $\text{Col}(\omega_1, \kappa) \ast \dot{\mathcal{I}}^+$, we see that for every real $x$ in $V^{\text{Col}(\omega_1, \kappa) \ast \dot{\mathcal{I}}}$, there is some $Q_x$ in $V$, $|Q_x| < \kappa$ and $V$-generic $G_x$ over $Q_x$ such that $x \in V[G_x]$. Without loss of generality (see Proposition 0.3) we can assume that $Q_x$ is $\text{Col}(\omega, \alpha)$ for some $\alpha < \kappa$.

We shall also need the notion of a saturated ideal on $\omega_1$. Let $\mathcal{I}$ be an ideal on $\omega_1$, let $\mathcal{I}^+$ be $\mathcal{P}(\omega_1) \setminus \mathcal{I}$ (so if $\mathcal{I} = \text{NS}_{\omega_1}$, the non-stationary subsets of $\omega_1$, then $\mathcal{I}^+$ is the set of all stationary subsets of $\omega_1$). We say an ideal on $\omega_1$ is saturated if, for any $\{X_\alpha \mid \alpha < \omega_2\}$, there are $\beta < \gamma < \omega_2$ such that $X_\beta \cap X_\gamma \in \mathcal{I}^+$. Equivalently, $\mathcal{I}$ is saturated if the algebra $\mathcal{P}(\omega_1) / \mathcal{I}$ has the $\aleph_2$-c.c.

Now, if $\mathcal{I}$ is a saturated ideal, then forcing with $\mathcal{I}^+$ gives rise (via a generic ultrapower embedding) to an elementary embedding $j : V \rightarrow M$ into a transitive class which contains all the reals.
Importantly, in $\mathcal{V}^{\text{Col}(\omega_1, \kappa)}$ for $\kappa$ supercompact, we can define a saturated ideal $\mathcal{I}$ on $\omega_1$ (cf. [Bek91, Chapter 2]), and so, forcing with $\mathcal{I}^+$ inside $\mathcal{V}^{\text{Col}(\omega_1, \kappa)}$, we get an embedding of $\mathcal{V}^{\text{Col}(\omega_1, \kappa)}$ into some transitive class $M$ which contains all the reals of $\mathcal{V}^{\text{Col}(\omega_1, \kappa)}$. Let $V_1 = \mathcal{V}^{\text{Col}(\omega_1, \kappa)}$ and $V_2 = \mathcal{V}^{\text{Col}(\omega_1, \kappa) \ast \mathcal{I}}$.

Now the following lemma assures the existence of a generic in $V_2$ for the Levy collapse of $\kappa$ to $\omega$:

**Lemma 4.3.** In $V_2$, there is a poset $\mathcal{Q}$ such that:

1. $\check{\mathcal{Q}}$ adds no reals

2. In $V_3 = \mathcal{V}^{\text{Col}(\omega_1, \kappa) \ast \mathcal{I} \ast \mathcal{Q}}$, there is a $\mathcal{V}$-generic $H$ over $\text{Col}(\omega, \kappa)$ such that $\mathcal{R}^{V_2} = \mathcal{R}^{V[H]}$.

**Proof.** In $V_2$, define $\mathcal{Q} = \{(\alpha, H_\alpha) \mid \alpha < \kappa\}$ and $H_\alpha$ is $\mathcal{V}$-generic over $\text{Col}(\omega, \alpha)$. $(\alpha, H_\alpha) \leq (\beta, H_\beta)$ iff $\alpha \geq \beta$ and $H_\alpha \upharpoonright \text{Col}(\omega, \beta) = H_\beta$.

Then forcing with $\mathcal{Q}$ over $V_2$ does what we want (cf. [Bek91])

The following absoluteness theorem is the key ingredient for our result:

**Theorem 4.4.** Let $\varphi$ be a formula, $r \in \mathcal{R}$, $\vec{\alpha}$ a finite collection of large enough ordinals. Then $L(\mathcal{R}) \models \varphi[r, \vec{\alpha}]$ iff $\mathcal{V}^{\text{Col}(\omega, \kappa)} \models \check{L(\mathcal{R})} \models \check{\varphi[r, \vec{\alpha}]}$.

**Proof.** By Lemma 4.3, $L(\mathcal{R})^{V_2} = L(\mathcal{R})^{V[H]}$ (as $V_2$ and $V[H]$ have the same reals and the same ordinals: forcing extensions add no ordinals). Hence, to prove the theorem, it suffices to prove that $L(\mathcal{R}) \models \varphi[r, \vec{\alpha}]$ iff $V_2 \models \check{L(\mathcal{R})} \models \check{\varphi[r, \vec{\alpha}]}$ (as $H$ is $\text{Col}(\omega, \kappa)$-generic over $V$.

Note that, in $V_2$, we have the elementary generic ultrapower embedding $j : V_1 \rightarrow M$ of $V_1 = \mathcal{V}^{\text{Col}(\omega_1, \kappa)}$ into a transitive class $M$ which has all the reals of $V_1$. Then, $L(\mathcal{R}) \models \varphi[r, \vec{\alpha}]$ iff $V_1 \models \check{L(\mathcal{R})} \models \check{\varphi[r, \vec{\alpha}]}$ (as $\text{Col}(\omega_1, \kappa)$ adds no functions with domain $\omega$ and hence adds no reals, so $L(\mathcal{R})^{V_2} = L(\mathcal{R})^{V_1}$).

As $j$ elementary and $j$ fixes all the ordinals of $\vec{\alpha}$ and also all reals, $V_1 \models \check{L(\mathcal{R})} \models \check{\varphi[r, \vec{\alpha}]}$ iff $M \models \check{L(\mathcal{R})} \models \varphi[r, \vec{\alpha}]$. However, as $\mathcal{R}^{V_2} = \mathcal{R}^{V_2}$, $M \models \check{L(\mathcal{R})} \models \varphi[r, \vec{\alpha}]$ iff $V_2 \models \check{L(\mathcal{R})} \models \varphi[r, \vec{\alpha}]$.

**Corollary 4.5.** If $\varphi$, $r$, $\vec{\alpha}$ as above, and $Q$ is a poset of size $< \kappa$, then $L(\mathcal{R}) \models \varphi[r, \vec{\alpha}]$ iff $\mathcal{V}^Q \models \check{L(\mathcal{R})} \models \check{\varphi[r, \vec{\alpha}]}$.

**Proof.** First, a small claim.

**Claim.** $Q \times \text{Col}(\omega, \kappa) \cong \text{Col}(\omega, \kappa)$
Let $\lambda = |Q|$, let $\tilde{Q} = Q \times \text{Col}(\omega; \{\lambda\})$. Then (by Proposition 0.3 again) we see that $\tilde{Q} \cong \text{Col}(\omega; \{\lambda\})$. Also, $V^Q \models \lambda = \omega$ and $|\tilde{Q}| = \lambda$. Thus,

$$Q \times \text{Col}(\omega, \kappa) = \prod_{\alpha < \lambda} \text{Col}(\omega, \{\alpha\}) \times (Q \times \text{Col}(\omega, \{\lambda\})) \times \prod_{\lambda < \alpha < \kappa} \text{Col}(\omega, \{\alpha\})$$

$$\cong \prod_{\alpha < \lambda} \text{Col}(\omega, \{\alpha\}) \times \text{Col}(\omega, \{\lambda\}) \times \prod_{\lambda < \alpha < \kappa} \text{Col}(\omega, \{\alpha\})$$

$$\cong \text{Col}(\omega, \kappa)$$

Now, by Proposition 0.1, we know that $\kappa$ is still supercompact in $V^Q$. By Theorem 4.4 applied to $V^Q$, we have

$$V^Q \models L(\mathbb{R}) \models \varphi[r, \vec{\alpha}] \iff (V^Q)^{\text{Col}(\omega, \kappa)} \models L(\mathbb{R}) \models \varphi[r, \vec{\alpha}]$$

But $\text{Col}(\omega, \kappa)$ is definable in $V$, so the iterated forcing $Q \ast \text{Col}(\omega, \kappa)$ can be recast as a one step product forcing $Q \times \text{Col}(\omega, \kappa)$ in $V$, and from the above $Q \times \text{Col}(\omega, \kappa) \cong \text{Col}(\omega, \kappa)$. Hence

$$V^Q \models L(\mathbb{R}) \models \varphi[r, \vec{\alpha}] \iff V^{\text{Col}(\omega, \kappa)} \models L(\mathbb{R}) \models \varphi[r, \vec{\alpha}] \iff L(\mathbb{R}) \models \varphi[r, \vec{\alpha}] \quad \square$$

4.2 Large cardinals imply that every reasonably definable set of reals is Lebesgue measurable

We now come to the main theorem we wish to prove:

**Theorem 4.6.** Assume the existence of a supercompact cardinal. Then every set of reals in $L(\mathbb{R})$ is Lebesgue measurable.

**Proof.** Let $A = \{x \mid L(\mathbb{R}) \models \varphi(x, r, \alpha)\}$ be a set defined by a real and a finite sequence of large ordinals. We wish to prove that $A$ is Lebesgue measurable.

Let $B$ be the measure algebra of $\mathbb{R}$ in $V$, let $B = |L(\mathbb{R}) \models \varphi(\hat{c}, r, \vec{\alpha})|_B$ where $\hat{c}$ is a canonical name for the random real added by forcing over $B$. $B$ is a Borel set, and we claim $B \Delta A$ is of measure 0. By Corollary 4.5, it suffices to show that $B \Delta A$ is of measure 0 in $V^Q$, where $Q = \text{Col}(\omega, \{\lambda\})$ and $\lambda = (2^{\aleph_0})^V$. $|Q| = 2^{\aleph_0} < \kappa$ (as $\kappa$ inaccessible).

Note that the set of $V$-random reals in $V^Q$ is of measure 1 (as $V^Q \models |\mathbb{R}^V| = \aleph_0$), so

$$V^Q \models C = \mathbb{R} \setminus \{\text{measure 0 sets coded in } V\}$$

Thus, it is sufficient to show that $V^Q \models C \cap (B \Delta A^Q) = \emptyset$. 

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Note that for any random real $c$ in $V^Q$, $V[c] \models \kappa$ is supercompact’ (because $|B| = 2^{\aleph_0} < \kappa$). Hence, as $V^Q$ is a forcing extension of $V[c]$, Corollary 4.5 applies again, this time to $V[c]$ rather than $V$, i.e.

$$V[c] \models \varphi(c, r, \vec{\alpha}) \iff V^Q \models \varphi(c, r, \vec{\alpha})$$

Let $c$ be random in $V^Q$ such that $c \in B \triangle A$.

**Case 1.** $c \in B \setminus A$. Then as $c \in B$, $V[c] \models \varphi(c, r, \vec{\alpha})$, and so $V^Q \models \varphi(c, r, \vec{\alpha})$, so $V^Q \models c \in A$, contradiction.

**Case 2.** $c \in A \setminus B$. Then $V[c] \not\models \varphi(c, r, \vec{\alpha})$, and so $V^Q \not\models \varphi(c, r, \vec{\alpha})$. Hence $V^Q \models c \not\in A$, contradiction.

Hence, $V^Q \models \mu(B \triangle A) = 0$, and so by Corollary 4.5 we have $V \models \mu(B \triangle A) = 0$, and so $A$ is Lebesgue measurable.

5 Concluding remarks

### 5.1 Optimal assumptions for measurability in $L(\mathbb{R})$

The assumption that a supercompact cardinal exists is a very strong statement. It is natural to ask whether weaker large cardinal assumptions suffice to prove the measurability of all sets of reals in $L(\mathbb{R})$.

In [SW90], Shelah and Woodin prove that, in the end, a sufficient condition to prove that every set of reals in $L(\mathbb{R})$ is Lebesgue measurable is that $\omega$ many Woodin cardinals exist, with a measurable cardinal above them all. A Woodin cardinal is a cardinal $\kappa$ such that, for all $f: \kappa \to \kappa$, there exists a $\lambda < \kappa$ with $\{f(\alpha) \mid \beta < \lambda\} \subseteq \lambda$ and an elementary embedding $j: V \to M$ into a transitive class $M$ with critical point $\lambda$ and $V_{j(f)(\lambda)} \subseteq M$.

Woodin cardinals were later found to be very important for the investigation of determinacy hypotheses extending ZFC - see the next subsection.

### 5.2 Determinacy

Fix a subset $X$ of $\omega^\omega$. A game on $\omega$ with payoff set $X$ is where we have players, I and II, who each take it in turns to pick an element of $\omega$. After $\omega$ many steps, I wins if the resulting $f \in \omega^\omega$ is an element of $X$; otherwise II wins. A strategy for I (resp. II) is a function which, at each stage when I (resp. II) plays, looks at the finite string obtained so far and deterministically tells I (or II) what number to play next.

A game is determined if a winning strategy exists for either player. It is almost immediately obvious that every finite set is determined, and that countable sets are also
determined by a diagonalisation argument. The first step on the ladder of Borel determinacy was taken by Gale and Stewart [GS53] who proved that every open and closed game (i.e. those games with open or closed payoff sets) are determined. Through strenuous effort, the determinacy of more and more sets in the Borel hierarchy was proven, until Martin proved that every Borel set is determined (see [Mar75]). This is the limit to what is provable in ZFC - the determinacy of $\Pi_1^1$ sets is equiconsistent with the existence of a measurable cardinal.\footnote{AD implies that $\omega_1$ is measurable (see [Jec02, pp. 633-636], and the existence of a measurable cardinal implies all $\Pi_1^1$ sets are determined (see [Mar70])}

The Axiom of Determinacy (AD) states that every game is determined. AD implies that every set of reals is Lebesgue measurable (see [MS64]). In fact, determinacy for a pointclass implies most other regularity properties, such as the property of Baire and the perfect set property. Hence, we can immediately see that AD and AC cannot coexist (we can also hands-on construct a subset of $\omega^\omega$, using AC, which cannot be determined). From the above remarks, we know that AD is independent of ZF, and it was speculated where, or indeed whether, determinacy fitted in with the hierarchy of large cardinal assumptions which are the natural extensions of ZF.

Woodin settled this question definitively: he proved that the existence of infinitely many Woodin cardinals is equiconsistent with AD ([Kan09, p. 466]). In fact, if we assume that there exist infinitely many Woodin cardinals with a measurable cardinal above them all, then we have the very strong result that AD holds in $L(\mathbb{R})$ (see [Nee10] for an exposition of this proof).

This interplay between determinacy, large cardinals and the descriptive set theory of the reals has been one of the driving forces of contemporary set theory, and has highlighted the fascinating connection in the set-theoretic universe between questions of ‘width’ of the universe (questions such show many sets of reals there are and how well behaved they are) and the ‘height’ of the universe (in other words, large cardinal assumptions).

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\footnote{AD implies that $\omega_1$ is measurable (see [Jec02, pp. 633-636], and the existence of a measurable cardinal implies all $\Pi_1^1$ sets are determined (see [Mar70])}
References


