Formal verification of floating-point arithmetic at Intel

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6 June 2012
Summary

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- Formal verification, testing and models
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- Perspectives and future prospects
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- As a result, the conversion failed.
- The rocket veered off its flight path and exploded, just 40 seconds into the flight sequence.
Patriot missile failure

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- A Scud missile travels 500 m in that time
Intel’s FDIV bug

Intel has also had at least one major floating-point issue:

- Error in the floating-point division (FDIV) instruction on some early Intel® Pentium® processors

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For example:

- $2^{160}$ possible pairs of floating point numbers (possible inputs to an adder).
- Vastly higher number of possible states of a complex microarchitecture.
Formal verification

Formal verification: mathematically prove the correctness of a design with respect to a mathematical formal specification.
Analogy with mathematics

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- $li(n) = \int_0^n du/\ln(u)$
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No change of sign at all had ever been found despite testing up to $n = 10^{10}$ (in the days before computers).
Similarly, extensive testing of hardware or software may still miss errors that would be revealed by a formal proof.
Verification vs. testing

Verification has some advantages over testing:

▶ Exhaustive.
▶ Improves our intellectual grasp of the system.

However:

▶ Difficult and time-consuming.
▶ Only as reliable as the formal models used.

▶ How can we be sure the proof is right?
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Formal verification is hard

Writing out a completely formal proof of correctness for real-world hardware and software is difficult.

- Must specify intended behaviour formally
- Need to make many hidden assumptions explicit
- Requires long detailed proofs, difficult to review

The state of the art is quite limited.
Software verification has been around since the 60s, but there have been few major successes.
Models versus the real world

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However, these are rare and apparently well controlled by existing engineering best practice.
Faulty hand proofs

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- Lemmas 1, 2, and 3 were all false.
- The proof of the main induction in the final theorem was wrong.
- The main result, however, was correct!
A more promising approach is to have the proof checked (or even generated) by a computer program.

- It can reduce the risk of mistakes.
- The computer can automate some parts of the proofs.

There are limits on the power of automation, so detailed human guidance is often necessary.
A variety of verification methods

There is a diverse world of formal verification methods, trading automation for generality / efficiency, most of which are in active use at Intel.

- Propositional tautology/equivalence checking (FEV)
- Symbolic simulation
- Symbolic trajectory evaluation (STE)
- Temporal logic model checking
- Combined decision procedures (SMT)
- First order automated theorem proving
- Interactive theorem proving
A spectrum of formal techniques

Traditionally, formal verification has been focused on complete proofs of functional correctness. But recently there have been notable successes elsewhere for ‘semi-formal’ methods involving abstraction or more limited property checking.

- Airbus A380 avionics
- Microsoft SLAM/SDV

One can also consider applying theorem proving technology to support testing or other traditional validation methods like path coverage. These are all areas of interest at Intel.
Our work

We have formally verified correctness of various floating-point algorithms.

- Division and square root (Marstein-style, using fused multiply-add to do Newton-Raphson or power series approximation with delicate final rounding).
- Transcendental functions like \( \log \) and \( \sin \) (table-driven algorithms using range reduction and a core polynomial approximations).

Proofs use the HOL Light prover

- [http://www.cl.cam.ac.uk/users/jrh/hol-light](http://www.cl.cam.ac.uk/users/jrh/hol-light)
Our HOL Light proofs

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- Elementary number theory and real analysis
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▶ Proving bounds on rational approximations
▶ Verifying errors in polynomial approximations
Example: tangent algorithm

- The input number $X$ is first reduced to $r$ with approximately $|r| \leq \pi/4$ such that $X = r + N\pi/2$ for some integer $N$. We now need to calculate $\pm \tan(r)$ or $\pm \cot(r)$ depending on $N$ modulo 4.
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- If the reduced argument $r$ is still not small enough, it is separated into its leading few bits $B$ and the trailing part $x = r - B$, and the overall result computed from $\tan(x)$ and pre-stored functions of $B$, e.g.

$$\tan(B + x) = \tan(B) + \frac{1}{\sin(B)\cos(B)} \frac{\tan(x)}{\cot(B) - \tan(x)}$$
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\tan(B + x) = \tan(B) + \frac{1}{\sin(B)\cos(B)}\tan(x) - \frac{\tan(x)}{\cot(B) - \tan(x)}
\]

- Now a power series approximation is used for \( \tan(r) \), \( \cot(r) \) or \( \tan(x) \) as appropriate.
Overview of the verification

To verify this algorithm, we need to prove:

▶ The range reduction to obtain $r$ is done accurately.
▶ The mathematical facts used to reconstruct the result from components are applicable.
▶ Stored constants such as $\tan(B)$ are sufficiently accurate.
▶ The power series approximation does not introduce too much error in approximation.
▶ The rounding errors involved in computing with floating point arithmetic are within bounds.

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Why mathematics?

Controlling the error in range reduction becomes difficult when the reduced argument \( X - N\pi/2 \) is small.

To check that the computation is accurate enough, we need to know:

*How close can a floating point number be to an integer multiple of \( \pi/2 \)?*

Even deriving the power series (for \( 0 < |x| < \pi \)):

\[
\cot(x) = \frac{1}{x} - \frac{1}{3}x - \frac{1}{45}x^3 - \frac{2}{945}x^5 - \ldots
\]

is much harder than you might expect.
Why HOL Light?

We need a general theorem proving system with:

- High standard of logical rigor and reliability
- Ability to mix interactive and automated proof
- Programmability for domain-specific proof tasks
- A substantial library of pre-proved mathematics

Other theorem provers such as ACL2, Coq and PVS have also been used for verification in this area.
The value of formal verification

Formal verification has contributed in many ways, and not only the obvious ones:

- Uncovered bugs, including subtle and sometimes very serious ones
- Revealed ways that algorithms could be made more efficient
- Improved our confidence in the (original or final) product
- Led to deeper theoretical understanding

This experience seems quite common.
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Perspectives and future prospects

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- We need more research on making formal verification more efficient and automatic so it can be applied more widely, and applied by relative non-experts.
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- We need more research on making formal verification more efficient and automatic so it can be applied more widely, and applied by relative non-experts.

- We need computer science curricula at universities to provide more rigorous treatment of mathematical rigor, logic and formal proof so that more programmers and engineers are able to deploy formal techniques.