Interactive Theorem Proving in Industry

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I wrote an automatic theorem prover in Swansea for myself and became shattered with the difficulty of doing anything interesting in that direction and I still am. I greatly admired Robinson’s resolution principle, a wonderful breakthrough; but in fact the amount of stuff you can prove with fully automatic theorem proving is still very small. So I was always more interested in amplifying human intelligence than I am in artificial intelligence.
Automated theorem proving

The 1970s and 1980s saw intense interest in purely automated theorem proving techniques:

- Robinson's resolution method and other techniques for first-order logic
- Knuth-Bendix completion for equational logic
- Boyer-Moore style automation of inductive proof
- Shostak and Nelson-Oppen work on cooperating decision procedures, congruence closure

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This led to a renaissance of formalization of all kinds, in pure mathematics and verification.
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We are actively trying to combine the power of automated techniques with the generality and reliability of interactive ones to produce the smoothest and most effective synthesis.
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Intel’s diverse activities

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If the Intel® Software and Services Group (SSG) were split off as a separate company, it would be in the top 10 software companies worldwide.
Intel’s diverse verification problems

This gives rise to a corresponding diversity of verification problems, and of verification solutions.

- Propositional tautology/equivalence checking (FEV)
- Symbolic simulation
- Symbolic trajectory evaluation (STE)
- Temporal logic model checking
- Combined decision procedures (SMT)
- First order automated theorem proving
- Interactive theorem proving

Integrating all these is a challenge!
The Flyspeck project

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This presents a similar integration challenge, since ultimately we would like a unified and completely formal proof.
Sharing results or sharing proofs?

A key dichotomy is whether we want to simply:

- Transfer results, effectively assuming the soundness of tools
- Transfer proofs or other 'certificates' and actually check them in a systematic way.

The first is generally easier and still useful. The latter is more ultimately satisfying and allows us to retain 'LCF-quality' results.
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Interfaces between interactive provers

Transferring results:

- hol90 $\rightarrow$ Nuprl: Howe and Felty 1997
- ACL2 $\rightarrow$ HOL4: Gordon, Hunt, Kaufmann & Reynolds 2006

Transferring proofs:

- HOL4 $\rightarrow$ Isabelle/HOL: Skalberg 2006
- HOL Light $\rightarrow$ Isabelle/HOL: Obua 2006
- Isabelle/HOL $\rightarrow$ HOL Light: McLaughlin 2006
- HOL Light $\rightarrow$ Coq: Keller 2009

More comprehensive solutions for exchange between HOL-like provers include work by Hurd et al. (OpenTheory) and Adams (importing into HOL Zero).
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Several reasonably fast solutions, e.g. Weber and Amjad, *Efficiently Checking Propositional Refutations in HOL Theorem Provers*
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Such integrations are currently an active theme, e.g. Isabelle’s “Sledgehammer”.
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While these work, the process of checking incurs a sometimes dramatic slowdown, and are sensitive to implementation details of the target prover.
Arithmetical theories: linear arithmetic

Generally works quite well for universal formulas over $\mathbb{R}$ or $\mathbb{Q}$. 

Farkas’s Lemma implies that any unsatisfiable set of inequalities has a linear combination that’s ‘obviously false’ like $1 < 0$.

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Thus we can reduce equation-solving to ideal membership, solvable using Gröbner bases.
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The similar but more intricate Positivstellensatz generalizes this to inequalities of all kinds.
Arithmetical theories: universal theory of reals (2)

The appropriate certificates can be found in practice via semidefinite programming (SDP). For example

\[ 23x^2 + 6xy + 3y^2 - 20x + 5 = 5(2x - 1)^2 + 3(x + y)^2 \geq 0 \]

\[ ax^2 + bx + c = 0 \Rightarrow b^2 - 4ac \geq 0 \]

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Other examples

There has been some research on at least the following:

▶ SMT: seems feasible to combine and generalize methods for SAT and theories.
▶ Explicit-state or BDD-based symbolic model checking: seems hard to separately certify and emulation is slow.
▶ Computer algebra: some easy case like factorization, indefinite integrals. Others like definite integrals are much harder.

Major research challenge: which algorithms lend themselves to this kind of efficient checking? Which ones seem essentially not to?

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Fully integrated automation?

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Suppose we have many efficient decision procedures implemented by external tools. How can we put them together? Effectively combination methods like Nelson-Oppen and Shostak solve this problem for quantifier-free theories. Even mild extensions with quantifiers rapidly become undecidable, such as linear integer arithmetic with one function symbol, when we can characterize squaring:

\[(\forall n. f(-n) = f(n)) \land f(0) = 0 \land (\forall n. 0 \leq n \Rightarrow f(n+1) = f(n) + n + n + 1)\]

and then multiplication by \(m = n \cdot p \iff (n + p)^2 = n^2 + p^2 + 2m\)
Quantifiers + theories

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▶ SMT solvers are improving their ability to instantiate quantifiers

Can sometimes exploit types to instantiate quantifiers systematically, and other heuristics often seem to work well in practice.
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- Effective exchange and checking of proofs between tools seems to be the best way of maintaining the ‘LCF advantage’.
- Several significant problems still seem hard to treat effectively via a certification, including model checking state enumeration and full quantifier elimination or general nonlinear optimization.
- The final challenge will probably lie in the effective combination of a variety of certified techniques, which broadly involves the combination of quantifier and theory reasoning.