An overview of automated reasoning

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Talk overview

- What is automated reasoning?
- Early history and taxonomy
- Automation, its scope and limits
- Interactive theorem proving

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Nowadays, by 'automatic and algorithmic' we mean 'using a computer program'.

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- Express *statement* of theorems in a formal language. (Leibniz's *characteristica universalis*.)
- Use automated algorithmic manipulations on those formal expressions. (Leibniz's *calculus ratiocinator*).

Theoretical and practical limitations

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  - Attempts to mimic human intelligence
  - An interactive 'proof assistant' guided by a human

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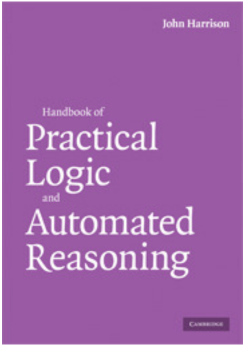
- To check the correctness of proofs in mathematics, supplementing or even replacing the existing 'social process' of peer review etc. with a more objective criterion.
- To extend rigorous proof from pure mathematics to the verification of computer systems (programs, hardware systems, protocols etc.), supplementing or replacing the usual testing process.

Just to fix notation

English	Our notation	Other common notations
false	$\bot$	0, F
true	$\top$	1, T
not p	$\neg p$	$\overline{p}$, $-p$, $\sim p$
p and q	$p \wedge q$	pq, $p \& q$, $p \cdot q$
p or q	$p \vee q$	$p + q$, $p \mid q$, p *or* q
p implies q	$p \Rightarrow q$	$p \leq q$, $p \rightarrow q$, $p \supset q$
p iff q	$p \Leftrightarrow q$	$p = q$, $p \equiv q$, $p \sim q$
for all x, p	$\forall x.\ p$	$(\forall x)p$, $(x)p$
there exists x such that p	$\exists x.\ p$	$(\exists x)p$, $(Ex)p$

For more details

An introductory survey of many central results in automated reasoning, together with actual OCaml model implementations
`http://www.cl.cam.ac.uk/~jrh13/atp/index.html`



Early history and taxonomy

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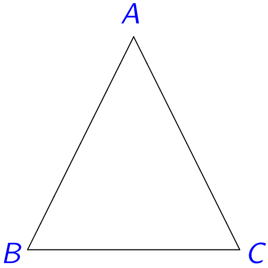
Modern work is dominated by machine-oriented approach but there have been some successes for the AI approach.

A theorem in geometry (1)

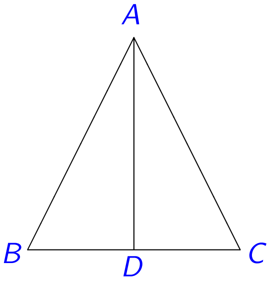
Example of AI approach in action:



If the sides AB and AC are equal (i.e. the triangle is isosceles), then the angles ABC and ACB are equal.

A theorem in geometry (2)

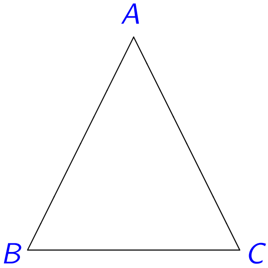
Pick bisector D of the line BC:



and then use the fact that the triangles ABD and ACD are congruent.

A theorem in geometry (3)

Originally found by Pappus but not in many books:



Simply, the triangles ABC and ACB are congruent.

The Robbins Conjecture (1)

Huntington (1933) presented the following axioms for a Boolean algebra:

$$\begin{aligned} x + y &= y + x \\ (x + y) + z &= x + (y + z) \\ n(n(x) + y) + n(n(x) + n(y)) &= x \end{aligned}$$

Herbert Robbins conjectured that the Huntington equation can be replaced by a simpler one:

$$n(n(x + y) + n(x + n(y))) = x$$

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In October 1996, a (key lemma leading to) a proof was found by McCune's program EQP.

The successful search took about 8 days on an RS/6000 processor and used about 30 megabytes of memory.

The scope of automation

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- Validity/satisfiability in many temporal logics is decidable.
- Validity in first-order logic is *semidecidable*, i.e. there are complete proof procedures that may run forever on invalid formulas
- Validity in higher-order logic is not even *semidecidable* (or anywhere in the arithmetical hierarchy).

Some specific theories

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Many of these naturally generalize known algorithms like linear/integer programming and Sturm's theorem.

Quantifier elimination

Many decision methods based on quantifier elimination, e.g.

- $\mathbb{C} \models (\exists x.\, x^2 + 1 = 0) \Leftrightarrow \top$
- $\mathbb{R} \models (\exists x.\, ax^2 + bx + c = 0) \Leftrightarrow a \neq 0 \wedge b^2 \geq 4ac \vee a = 0 \wedge (b \neq 0 \vee c = 0)$

If we can decide variable-free formulas, quantifier elimination implies completeness.

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There are also many more specialized symbolic algorithms, some of which can produce proofs or other certificates.

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However, modern fast algorithms (distantly descended from the Davis-Putnam algorithm from the 60s) are surprisingly effective on big problems.

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Many first-order algorithms and their implementations blend the key ideas from both threads (e.g. superposition).

FOL example

This is Łoś's well-known 'non-obvious' fact:

$$
\begin{aligned}
&(\forall x\ y\ z.\ P(x,y) \land P(y,z) \Rightarrow P(x,z)) \land \\
&(\forall x\ y\ z.\ Q(x,y) \land Q(y,z) \Rightarrow Q(x,z)) \land \\
&(\forall x\ y.\ Q(x,y) \Rightarrow Q(y,x)) \land \\
&(\forall x\ y.\ P(x,y) \lor Q(x,y)) \\
&\Rightarrow (\forall x\ y.\ P(x,y)) \lor (\forall x\ y.\ Q(x,y))
\end{aligned}
$$

Most people take more time to solve this than automated first-order provers.

Equational logic examples

A simple group theory exercise,

$$
\begin{aligned}
&(\forall x\ y\ z.\ x \cdot (y \cdot z) = (x \cdot y) \cdot z) \wedge \\
&(\forall x.\ 1 \cdot x = x) \wedge \\
&(\forall x.\ \mathsf{inv}(x) \cdot x = 1) \\
&\Rightarrow \mathsf{inv}(\mathsf{inv}(x)) = x
\end{aligned}
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$$

The conceptual core of the Eckmann-Hilton argument that certain homotopy groups are abelian:

$$
\begin{array}{l}
(\forall x.\ 1 \cdot x = x)\wedge \\
(\forall x.\ x \cdot 1 = x)\wedge \\
(\forall x.\ 1 + x = x)\wedge \\
(\forall x.\ x + 1 = x)\wedge \\
(\forall w\ x\ y\ z.\ (w \cdot x) + (y \cdot z) = (w + y) \cdot (x + z)) \\
\Rightarrow \forall x\ y.\ x \cdot y = x + y
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$$

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Both were used in some early theorem provers and program verification systems and some of the key ideas are still used today. Modern systems are usually based on SAT as the core, hence 'satisfiability modulo theories' (SMT).

SMT examples

SMT systems can handle purely equational reasoning (without embedded quantifiers)

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or with arithmetic theories too:

$$f(v - 1) - 1 = v + 1 \wedge f(u) + 1 = u - 1 \wedge u + 1 = v \Rightarrow \bot$$

Quantifiers and theories?

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For example, using linear integer arithmetic with one function symbol, when we can characterize squaring:

$$(\forall n. f(-n) = f(n)) \wedge f(0) = 0 \wedge (\forall n. 0 \leq n \Rightarrow f(n+1) = f(n) + n + n + 1)$$

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There are special cases that work (e.g. based on type structure), and many SMT systems have incomplete heuristics for quantifiers.

Interactive theorem proving

Interactive theorem proving (1)

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In practice, we need an interactive arrangement, where the user and machine work together.
The user can delegate simple subtasks to pure first order proof search or one of the decidable subsets.
However, at the high level, the user must guide the prover.

Interactive theorem proving (2)

The idea of a more 'interactive' approach was already anticipated by pioneers, e.g. Wang (1960):

> *[...] the writer believes that perhaps machines may more quickly become of practical use in mathematical research, not by proving new theorems, but by formalizing and checking outlines of proofs, say, from textbooks to detailed formalizations more rigorous that* Principia *[Mathematica], from technical papers to textbooks, or from abstracts to technical papers.*

However, constructing an effective and programmable combination is not so easy.

SAM

First successful family of interactive provers were the SAM systems:

> *Semi-automated mathematics is an approach to theorem-proving which seeks to combine automatic logic routines with ordinary proof procedures in such a manner that the resulting procedure is both efficient and subject to human intervention in the form of control and guidance. Because it makes the mathematician an essential factor in the quest to establish theorems, this approach is a departure from the usual theorem-proving attempts in which the computer* unaided *seeks to establish proofs.*

SAM V was used to settle an open problem in lattice theory.

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Ideas from all these systems are used in present-day systems. (Corbineau's declarative proof mode for Coq . . .)

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- What if the inference rules of the logic are specified incorrectly? It's easy and common to make mistakes connected with variable capture.
- What if the proof checker has a bug? They are often large and complex pieces of software not developed to high standards of rigour

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- de Bruijn approach — generate proofs that can be certified by a simple, separate checker.
- LCF approach — reduce all rules to sequences of primitive inferences implemented by a small logical kernel.

The checker or kernel can be much simpler than the prover as a whole.
Nothing is ever certain, but we can potentially achieve very high levels of reliability in this way.

Key ideas behind LCF

- Implement in a strongly-typed functional programming language (usually a variant of ML)
- Make `thm` ('theorem') an abstract data type with only simple primitive inference rules
- Make the implementation language available for arbitrary extensions.

Proof styles

Directly invoking the primitive or derived rules tends to give proofs that are *procedural*.

A *declarative* style (*what* is to be proved, not *how*) can be nicer:

- Easier to write and understand independent of the prover
- Easier to modify
- Less tied to the details of the prover, hence more portable

Mizar pioneered the declarative style of proof.

Recently, several other declarative proof languages have been developed, as well as declarative shells round existing systems like HOL and Isabelle.

Finding the right style is an interesting research topic.

Procedural proof example

```
let NSQRT_2 = prove
 (`!p q. p * p = 2 * q * q ==> q = 0`,
  MATCH_MP_TAC num_WF THEN REWRITE_TAC[RIGHT_IMP_FORALL_THM] THEN
  REPEAT STRIP_TAC THEN FIRST_ASSUM(MP_TAC o AP_TERM `EVEN`) THEN
  REWRITE_TAC[EVEN_MULT; ARITH] THEN REWRITE_TAC[EVEN_EXISTS] THEN
  DISCH_THEN(X_CHOOSE_THEN `m:num` SUBST_ALL_TAC) THEN
  FIRST_X_ASSUM(MP_TAC o SPECL [`q:num`; `m:num`]) THEN
  ASM_REWRITE_TAC[ARITH_RULE
   `q < 2 * m ==> q * q = 2 * m * m ==> m = 0 <=>
    (2 * m) * 2 * m = 2 * q * q ==> 2 * m <= q`] THEN
  ASM_MESON_TAC[LE_MULT2; MULT_EQ_0; ARITH_RULE `2 * x <= x <=> x = 0`]);;
```

Declarative proof example

```
let NSQRT_2 = prove
 (`!p q. p * p = 2 * q * q ==> q = 0`,
  suffices_to_prove
   `!p. (!m. m < p ==> (!q. m * m = 2 * q * q ==> q = 0))
         ==> (!q. p * p = 2 * q * q ==> q = 0)`
   (wellfounded_induction) THEN
  fix [`p:num`] THEN
  assume("A") `!m. m < p ==> !q. m * m = 2 * q * q ==> q = 0` THEN
  fix [`q:num`] THEN
  assume("B") `p * p = 2 * q * q` THEN
  so have `EVEN(p * p) <=> EVEN(2 * q * q)` (trivial) THEN
  so have `EVEN(p)` (using [ARITH; EVEN_MULT] trivial) THEN
  so consider (`m:num`,"C",`p = 2 * m`) (using [EVEN_EXISTS] trivial) THEN
  cases ("D",`q < p \/ p <= q`) (arithmetic) THENL
   [so have `q * q = 2 * m * m ==> m = 0` (by ["A"] trivial) THEN
    so we're finished (by ["B"; "C"] algebra);
    so have `p * p <= q * q` (using [LE_MULT2] trivial) THEN
    so have `q * q = 0` (by ["B"] arithmetic) THEN
    so we're finished (algebra)]);;
```

The Seventeen Provers of the World (1)

- ACL2 — Highly automated prover for first-order number theory without explicit quantifiers, able to do induction proofs itself.
- Alfa/Agda — Prover for constructive type theory integrated with dependently typed programming language.
- B prover — Prover for first-order set theory designed to support verification and refinement of programs.
- Coq — LCF-like prover for constructive Calculus of Constructions with reflective programming language.
- HOL (HOL Light, HOL4, ProofPower) — Seminal LCF-style prover for classical simply typed higher-order logic.
- IMPS — Interactive prover for an expressive logic supporting partially defined functions.

The Seventeen Provers of the World (2)

- Isabelle/Isar — Generic prover in LCF style with a newer declarative proof style influenced by Mizar.
- Lego — Well-established framework for proof in constructive type theory, with a similar logic to Coq.
- Metamath — Fast proof checker for an exceptionally simple axiomatization of standard ZF set theory.
- Minlog — Prover for minimal logic supporting practical extraction of programs from proofs.
- Mizar — Pioneering system for formalizing mathematics, originating the declarative style of proof.
- Nuprl/MetaPRL — LCF-style prover with powerful graphical interface for Martin-Löf type theory with new constructs.

The Seventeen Provers of the World (3)

- Omega — Unified combination in modular style of several theorem-proving techniques including proof planning.
- Otter/IVY — Powerful automated theorem prover for pure first-order logic plus a proof checker.
- PVS — Prover designed for applications with an expressive classical type theory and powerful automation.
- PhoX — prover for higher-order logic designed to be relatively simple to use in comparison with Coq, HOL etc.
- Theorema — Ambitious integrated framework for theorem proving and computer algebra built inside Mathematica.

For more, see Freek Wiedijk, *The Seventeen Provers of the World*, Springer Lecture Notes in Computer Science vol. 3600, 2006.

Conclusions

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- Traditional automated methods have their limitations, but many of these tools are remarkably powerful.
- There is a rich and diverse group of interactive proof assistants, which are integrating many automated tools in a sound way, and being used for a variety of applications.