Formal proof: current progress and outstanding challenges

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Summary of talk

- A century of formal proof
  - Poincaré on formal proof
  - From Principia Mathematica to the computer age
  - Major milestones in formalization
  - Development of mathematical libraries

- Current perspectives
  - The provers of the world
  - Foundations
  - Software architecture
  - Proof languages
  - Automation
  - Libraries

- More about HOL Light
  - Foundations and architecture
  - Decision procedures and automation
  - A tour of the libraries

- The future
A century of formal proof
What would Poincaré have thought?
I see in logistic only shackles for the inventor. It is no aid to conciseness — far from it, and if twenty-seven equations were necessary to establish that 1 is a number, how many would be needed to prove a real theorem?
Poincaré’s had a distinct aversion to formal logic

I see in logistic only shackles for the inventor. It is no aid to conciseness — far from it, and if twenty-seven equations were necessary to establish that 1 is a number, how many would be needed to prove a real theorem? If we distinguish, with Whitehead, the individual $x$, the class of which the only member is $x$ and [...] the class of which the only member is the class of which the only member is $x$ [...] do you think these distinctions, useful as they may be, go far to quicken our pace?
However, Poincaré’s was no stranger to errors

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- As a result of probing questions by Phragmén, Poincaré discovered a fundamental error *after* the prize had been awarded and the journal issue printed and even delivered to some subscribers.
- This was a very productive mistake: the new realization led to a much deeper understanding of dynamical systems and laid the foundations of modern chaos theory.
- However it was embarrassing and expensive for all concerned — Poincaré spent more than the competition prize money paying for the journal issues to be recalled and reprinted.
100 years since *Principia Mathematica*

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- This practical formal mathematics was to forestall objections to Russell and Whitehead’s ‘logicist’ thesis, not a goal in itself.
- The development was difficult and painstaking, and has probably been studied in detail by very few.
- Subsequently, the idea of actually formalizing proofs has not been taken very seriously.
Even Russell did not enjoy doing formal proofs

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However, now we have computers to check and even automatically generate formal proofs. Our goal is now not so much philosophical, but to achieve a real, practical, useful increase in the precision and accuracy of mathematical proofs.
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Because of these dual connections, interest in formal proofs is strongest among computer scientists, but some ‘mainstream’ mathematicians are becoming interested too.
A formal proof from 1910

This is p379 of Whitehead and Russell’s *Principia Mathematica*.
*54.43. \( \vdash \alpha, \beta \in 1. \exists \alpha \cap \beta = \Lambda. \equiv \alpha \cup \beta \in 2 \)

Dem.

\( \vdash \star 54.26. \exists \alpha = t'x. \beta = t'y. \exists \alpha \cup \beta \in 2. \equiv x \neq y. \)

\[ \star 51.231 \]

\[ \star 13.12 \]

\( \vdash (1). \star 11.11.35. \exists \)

\( \vdash (\exists x, y). \alpha = t'x. \beta = t'y. \exists \alpha \cup \beta \in 2. \equiv \alpha \cap \beta = \Lambda \)

\( \vdash (2). \star 11.54. \star 52.1. \exists \vdash \text{Prop} \)

From this proposition it will follow, when arithmetical addition has been defined, that \( 1 + 1 = 2. \)
A formal proof from 2010

\[
\text{let PNT = prove}
\]
\[
\left(\left(\forall n. \text{CARD}\ \{p \mid \text{prime}\ p \land p \leq n\}\right) / (n / \log(n))\right)
\]
\[
\quad \rightarrow \ 1\ \text{sequentially},
\]
\[
\text{REWRITE_TAC[PNT\_PARTIAL\_SUMMATION]\ THEN}
\]
\[
\text{REWRITE_TAC[SUM\_PARTIAL\_PRE]\ THEN}
\]
\[
\text{REWRITE_TAC[GSYM REAL\_OF\_NUM\_ADD; SUB\_REFL; CONJUNCT1 LE]\ THEN}
\]
\[
\text{SUBGOAL\_THEN \\{p \mid \text{prime}\ p \land p = 0\} = \{\}\ \text{SUBST1\_TAC THENL}
\]
\[
[\text{REWRITE_TAC[EXTENSION; IN\_ELIM\_THM; NOT\_IN\_EMPTY]\ THEN}
\]
\[
\text{MESON\_TAC[PRIME\_IMP\_NZ];}
\]
\[
\text{ALL\_TAC\ THEN}
\]
\[
\text{REWRITE_TAC[SUM\_CLAUSES; REAL\_MUL\_RZERO; REAL\_SUB\_RZERO]\ THEN}
\]
\[
\text{MATCH\_MP\_TAC REALLIM\_TRANSFORM\_EVENTUALLY\ THEN}
\]
\[
\text{EXISTS\_TAC}
\]
\[
\left(\forall n. \left((n + 1) / \log(n + 1) \times \sum\ \{p \mid \text{prime}\ p \land p \leq n\}\ (p \cdot \log(p) / p) - \sum\ (1..n) (\left(k. \sum\ \{p \mid \text{prime}\ p \land p \leq k\}\ (p \cdot \log(p) / p) \times ((k + 1) / \log(k + 1) - k / \log(k)))\right) / (n / \log(n))\right)
\]
\[
\text{CONJ\_TAC THENL}
\]
\[
[\text{REWRITE\_TAC[EVENTUALLY\_SEQUENTIAL\_ALLY]\ THEN \text{EXISTS\_TAC '1' THEN SIMP\_TAC[]};
\]
\[
\text{ALL\_TAC\ THEN}
\]
\[
\text{MATCH\_MP\_TAC REALLIM\_TRANSFORM\ THEN}
\]
\[
\text{EXISTS\_TAC}
\]
\[
\left(\forall n. \left((n + 1) / \log(n + 1) \times \log(n) - \sum\ (1..n) (\left(k. \log(k) \times ((k + 1) / \log(k + 1) - k / \log(k)))\right) / (n / \log(n))\right)
\]
\[
\text{REWRITE\_TAC[]}\ \text{THEN \text{CONJ\_TAC THENL}
\]
\[
[\text{REWRITE\_TAC[REAL\_ARITH}
\]
\[
\left(a \times x - s\right) / b - \left(a \times x' - s'\right)/ b':real =
\]
\[
\left((s' - s) - (x' - x) * a\right) / b'
\]
\[
\text{THEN}
\]
\[
\text{REWRITE\_TAC[GSYM SUM\_SUB\_NUMSEG; GSYM REAL\_SUB\_RDISTRIBUT]\ THEN}
\]
\[
\text{REWRITE\_TAC[GSYM REAL\_OF\_NUM\_ADD]\ THEN}
\]
\[
\text{MATCH\_MP\_TAC SUM\_PARTIAL\_LIMIT\_ALT\ THEN}
\]
At least the theorems are more substantial:

```
let PNT = prove
  (\((n. \&(CARD \{p \mid \text{prime } p \land p \leq n\}) \div (\&n / \log(\&n)))
   \longrightarrow \&1\) sequentially',
   REWRITE_TAC[PNT_PARTIAL_SUMMATION] THEN
   REWRITE_TAC[SUM_PARTIAL_PRE] THEN
   REWRITE_TAC[GSYM REAL_OF_NUM_ADD; SUB_REFL; CONJUNCT1 LE] THEN
   SUBGOAL_THEN '{p \mid \text{prime } p \land p = 0} = {}' SUBST1_TAC THENL
```

Moreover, we can arrange to have more readable proofs — see for example Bill Richter's talk.
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let PNT = prove
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The major landmarks

These are arguably the three major landmarks in the formalization of mathematics

1. The four-colour theorem (every planar map is 4-colourable) — Gonthier et al.
2. The odd order theorem (every finite group of odd order is solvable) — Gonthier et al.
3. The Flyspeck project (the Kepler Conjecture that no sphere packing beats face-centred cubic) — Hales et al.
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These are demonstrations that the technology can handle long and difficult proofs, and even that some leading mathematicians like Hales are willing to use them.
Formalized theorems and libraries of mathematics

Also important is the progress made on more modest building-blocks for mathematics, still including quite substantial results, e.g.

- Jordan Curve Theorem — Tom Hales (HOL Light), Andrzej Trybulec et al. (Mizar)
- Prime Number Theorem — Jeremy Avigad et al (Isabelle/HOL), John Harrison (HOL Light)
- First and second Cartan Theorems — Marco Maggesi et al (HOL Light)

In the process, provers are building up ever-larger libraries of pre-proved theorems that can be deployed in future proofs.
Current perspectives
A few notable general-purpose theorem provers

There is a diverse (perhaps too diverse?) world of proof assistants, with these being just a few:

- ACL2
- Agda
- Coq
- HOL (HOL Light, HOL4, ProofPower, HOL Zero)
- IMPS
- Isabelle
- Metamath
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See Freek Wiedijk’s book *The Seventeen Provers of the World* (Springer-Verlag lecture notes in computer science volume 3600) for descriptions of many systems and proofs that $\sqrt{2}$ is irrational.
Foundations

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  - Metamath and Isabelle/ZF (standard ZF/ZFC)
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  - HOL family and Isabelle/HOL (simple type theory)
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  - Calculus of inductive constructions (Coq)
  - Other typed formalisms (IMPS, PVS)
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Software architecture

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- de Bruijn approach — generate proofs that can be certified by a simple, separate checker.
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- Have suitable ‘certificates’ produced by an external tool checked in the inference kernel.
- Extend kernel with verified implementation (reflection).
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Many theorem provers including Coq, HOL Light and Isabelle/HOL (including the ‘archive of formal proofs’) also have large and every-expanding mathematical libraries.
More about HOL Light
There are many HOL provers, of which HOL Light is just one, all descended from Mike Gordon’s original HOL system in the late 1980s.
HOL Light primitive rules (1)

\[
\Gamma \vdash t = t \quad \text{REFL}
\]

\[
\Gamma \vdash s = t \quad \Delta \vdash t = u \quad \frac{}{\Gamma \cup \Delta \vdash s = u} \quad \text{TRANS}
\]

\[
\Gamma \vdash s = t \quad \Delta \vdash u = v \quad \frac{}{\Gamma \cup \Delta \vdash s(u) = t(v)} \quad \text{MK}_\text{COMB}
\]

\[
\Gamma \vdash s = t \quad \frac{}{\Gamma \vdash (\lambda x. s) = (\lambda x. t)} \quad \text{ABS}
\]

\[
\frac{}{\Gamma \vdash (\lambda x. t)x = t} \quad \text{BETA}
\]
HOL Light primitive rules (2)

\[
\begin{align*}
\{p\} & \vdash p & \text{ASSUME} \\
\Gamma \vdash p = q & \quad \Delta \vdash p & \quad \text{EQ_MP} \\
\Gamma \cup \Delta & \vdash q \\
\end{align*}
\]

\[
\begin{align*}
\Gamma \vdash p & \quad \Delta \vdash q & \quad \text{DEDUCT_ANTISYM_RULE} \\
(\Gamma - \{q\}) \cup (\Delta - \{p\}) & \vdash p = q \\
\end{align*}
\]

\[
\begin{align*}
\Gamma[x_1, \ldots, x_n] & \vdash p[x_1, \ldots, x_n] & \text{INST} \\
\Gamma[t_1, \ldots, t_n] & \vdash p[t_1, \ldots, t_n] \\
\end{align*}
\]

\[
\begin{align*}
\Gamma[\alpha_1, \ldots, \alpha_n] & \vdash p[\alpha_1, \ldots, \alpha_n] & \text{INST_TYPE} \\
\Gamma[\gamma_1, \ldots, \gamma_n] & \vdash p[\gamma_1, \ldots, \gamma_n] \\
\end{align*}
\]
Pushing the LCF approach to its limits

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HOL Light may represent the most “extreme” application of this philosophy.
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Pushing the LCF approach to its limits

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HOL Light may represent the most “extreme” application of this philosophy.

- HOL Light’s primitive rules are very simple, and the trusted core is just a few hundred lines of code.
- There is an extensive suite of automated tools built on top that all reduce to this foundation.
Some of HOL Light’s basic automation

- Simplifier for (conditional, contextual) rewriting.
- Tactic mechanism for mixed forward and backward proofs.
- Tautology checker.
- Automated theorem provers for pure logic, based on tableaux and model elimination.
- Linear arithmetic decision procedures over $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$.
- Differentiator for real functions.
- Generic normalizers for rings and fields.
- General quantifier elimination over $\mathbb{C}$.
- Gröbner basis algorithm over fields.
Some unusual automation

HOL Light has also introduced several novel automated proof methods, all of which were developed to answer real problems in formalization:

▶ Heuristic decision procedure for divisibility properties in number theory via a reduction to ideal membership. (For example, can prove the Chinese Remainder Theorem automatically.)

▶ Decision procedures for general 'triangle law' reasoning in normed spaces and general decision procedure for Hilbert spaces, using decidability results developed in work with Solovay and Arthan.

▶ 'Without loss of generality' tactics for simplifying goals in geometry by use of special coordinate systems, which can greatly simplify some Flyspeck goals.
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Partly as a result of Flyspeck, HOL Light is particularly strong in the area of topology, analysis and geometry in Euclidean space $\mathbb{R}^n$. 

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<tr>
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<td>11827</td>
<td>Convex sets and functions</td>
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<td>paths.ml</td>
<td>17066</td>
<td>Paths, simple connectedness etc.</td>
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<td>polytope.ml</td>
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<td>Faces, polytopes, polyhedra etc.</td>
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From this foundation complex analysis is developed and used to derive convenient theorems for $\mathbb{R}$ as well as more topological results.

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A tour of the libraries (2)

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It would be desirable to generalize much of the material to general topological spaces, metric spaces, measure spaces etc. Some work already by Bill Richter on general topology.
Some examples from topology

The Brouwer fixed point theorem:

\[ \forall f : \mathbb{R}^N \to \mathbb{R}^N \ s. \]
  \[
  \text{compact } s \land \text{convex } s \land \neg (s = \{\}) \land \]
  \[
  f \text{ continuous_on } s \land \text{IMAGE } f \ s \subseteq s
  \]
\[ \Rightarrow \exists x. x \in s \land f x = x \]

The Borsuk homotopy extension theorem:

\[ \forall f : \mathbb{R}^M \to \mathbb{R}^N \ g \ s \ t \ u. \]
  \[
  \text{closed_in (subtopology euclidean } t) \ s \land \]
  \[
  (\text{ANR } s \land \text{ANR } t \lor \text{ANR } u) \land \]
  \[
  f \text{ continuous_on } t \land \text{IMAGE } f \ t \subseteq u \land \]
  \[
  \text{homotopic_with } (\forall x. T) \ (s, u) \ f \ g
  \]
\[ \Rightarrow \exists g'. \text{homotopic_with } (\forall x. T) \ (t, u) \ f \ g' \land \]
  \[
  g' \text{ continuous_on } t \land \]
  \[
  \text{IMAGE } g' \ t \subseteq u \land \]
  \[
  \forall x. x \in s \Rightarrow g'(x) = g(x) \]
Some examples from convexity

The Krein-Milman (Minkowski) theorem

\[ \forall s : \text{real}^N \rightarrow \text{bool}. \]
\[ \text{convex } s \land \text{compact } s \]
\[ \implies s = \text{convex hull } \{ x \mid x \text{ extreme}_\text{point}_\text{of } s \} \]

Approximation of convex sets by polytopes w.r.t. Hausdorff distance:

\[ \forall s : \text{real}^N \rightarrow \text{bool} \, e. \]
\[ \text{bounded } s \land \text{convex } s \land 0 < e \]
\[ \implies \exists p. \text{polytope } p \land s \subseteq p \land \text{hausdist}(p, s) < e \]
Some examples from measure theory

Steinhaus’s theorem:

|⁻| `s:real^N→bool.`
  `lebesgue_measurable s \negligible s`  
  `=> ?d. &0 < d \`
  `ball(vec 0,d) SUBSET`
  `{x - y | x IN s \ y IN s}`

Luzin’s theorem:

|⁻| `!f:real^M→real^N s e.`
  `measurable s \ f measurable_on s \&0 < e`  
  `=> ?k. compact k \ k SUBSET s \`
  `measure(s DIFF k) < e \`
  `f continuous_on k`
Some examples from complex analysis

The Little Picard theorem:

\[- \forall f \ a \ b. \]
\[f \text{ holomorphic_on } (:\text{complex}) \land\]
\[\neg (a = b) \land \text{IMAGE } f (:\text{complex}) \cap \{a,b\} = \emptyset\]
\[\implies \exists c. \ f = \lambda x. \ c\]

The Riemann mapping theorem:

\[\neg \forall s. \text{open } s \land \text{simply_connected } s \iff s = \emptyset \lor s = (:\text{complex}) \lor\]
\[\exists f \ g. \ f \text{ holomorphic_on } s \land\]
\[g \text{ holomorphic_on } \text{ball}(Cx(&0),&1) \land\]
\[(\forall z. \ z \in s) \implies f(z) \in \text{ball}(Cx(&0),&1) \land\]
\[g(f(z)) = z) \land\]
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The future
Future prospects

There is still lots of scope for improving automation, either with off-the-shelf methods adapted to be provably sound, or new ideas. The steady increase in the stock of theorems in the prover libraries will continue and eventually make tackling a 'typical' mathematical problem much more tractable. New research in foundations may result in fundamentally better approaches to formalization and even have increasing influence back on mathematics itself. Given the diversity of theorem proving systems, it seems there will be still more research into sharing and importing and exporting proofs between them. We can further increase the soundness guarantees by rigorous verification down to the lowest levels as well as proof checking and proof auditing.
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