The Unfolding of General Petri Nets

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Abstract. The unfolding of (1)-safe Petri nets to occurrence nets is well understood. There is a universal characterization of the unfolding of a safe net which is part and parcel of a coreflection from the category of occurrence nets to the category of safe nets. The unfolding of general Petri nets, nets with multiplicities on arcs whose markings are multisets of places, does not possess a directly analogous universal characterization, essentially because there is an implicit symmetry in the multiplicities of general nets, and that symmetry is not expressed in their traditional occurrence net unfoldings. In the present paper, we show how to recover a universal characterization by placing a symmetry on the occurrence net unfoldings of general Petri nets. We show that this is part of a coreflection between enriched categories of general Petri nets with symmetry and occurrence nets with symmetry.

1 Introduction

There is a vast array of models for concurrency, including a variety of kinds of Petri net. One of the disadvantages of having so many different models is that definitions of constructions are repeated on different models, and there is significant reinvention of concepts such as bisimilarity in the different models. In [1], it was shown how category theory could be applied to describe the relationships between several different models by giving corefections\(^1\) between the categories representing such models.

Only partial results have been achieved in relating Petri nets to other models for concurrency through corefections. In particular, they are restricted to (1)-safe nets, and do not apply to apply to the more general forms of net where transitions can take or deposit more than one token in a place or the initial marking contains more than one token. Occurrence nets are special kinds of Petri net, the structure of which illuminates how the occurrences of transitions in their runs causally depend and conflict with each other. As such, they are useful in giving the semantics of other forms of net, and the category of occurrence nets lies at the heart of corefections between nets and other models for concurrency. For instance, it has been shown [2] that there is a coreflection between occurrence nets and event structures. There is also a coreflection between occurrence nets and (1)-safe Petri nets [2, 3]. The corefections compose to yield

\(^1\) Corefections are adjunctions in which the unit is an isomorphism
a coreflection between safe nets and event structures, and thereon to other models for concurrency. At the core of this coreflection is the operation of unfolding a safe net to form its occurrence net semantics.

Significantly, the the results cannot be extended beyond safe Petri nets to the more general form of Petri net where transitions can take more than one token from a place and deposit more than one token in a place to yield a coreflection. We cannot obtain a coreflection between the category of occurrence nets and the category of these more general nets. The reason for this is that the occurrence net unfolding necessarily contains symmetric occurrences given rise to by the markings of the general net not necessarily being sets.

In [4], a categorical formulation of symmetry on event structures was given along with a scheme for applying symmetry to any category satisfying appropriate constraints. Following that work, we define categories of nets with symmetry. Given a general net, we shall construct a symmetry on its unfolding. This extends to giving a coreflection up to symmetry between the categories of general nets with symmetry and occurrence nets with symmetry.

2 Varieties of Petri nets

It is unfortunately beyond the scope of the current paper to give anything but the essential definitions of Petri nets; we instead refer the reader to [5, 1]. We give some preliminary definitions relating to multisets in Appendix A, though a fuller treatment can be found in the appendix of [6].

A Petri net consists of a set of places and a set of transitions, and possesses an appealing graphical interpretation in which its places are drawn as circles and its transitions are drawn as rectangles. Some (possibly infinite) number of tokens, drawn as dots, may reside inside each place, forming an \( \infty \)-multirelation called a marking. Arcs labelled with a natural number may be drawn from a place into a transition, indicating the finite number of tokens in the place that are consumed by the occurrence of the transition. Arcs labelled with a natural number or \( \infty \) may be drawn from a transition to a place, indicating the number of tokens to be deposited in the place by the occurrence of an event. More formally:

**Definition 1 (General Petri net).** A general Petri net is a 5-tuple,

\[
G = (P, T, \text{Pre}, \text{Post}, \mathcal{M}),
\]

comprising a set \( P \) of places; a set \( T \) of transitions; a precondition multirelation, \( \text{Pre} \subseteq \mu T \times P \); a postcondition \( \infty \)-multirelation, \( \text{Post} \subseteq \mu \infty T \times P \); and a set \( \mathcal{M} \) of \( \infty \)-multisets of \( P \), forming the set of initial markings of \( G \). Every transition must consume at least one token:

\[
\forall t \in T \exists p \in P. \text{Pre}[t, p] > 0.
\]

This is a mild generalization of the standard definition of Petri net in that we allow there to be a set of initial markings rather than just one initial marking,
and will prove important later. In the case where a general net has precisely one initial marking, we say that the net is *singly-marked*.

As described above, the marking of the net changes through the occurrence of transitions. This is commonly referred to as the *token game* for nets, and gives rise to a relation between markings labelled by a finite multiset of transitions, \( M \xrightarrow{A} M' \) defined as

\[
M \xrightarrow{A} M' \text{ iff } Pre \cdot A \leq M \text{ and } M' = M - Pre \cdot A + Post \cdot A.
\]

Note that \( M - Pre \cdot A \) is always defined if \( Pre \cdot A \leq M \) since \( A \) is finite and \( Pre \) is a multirelation rather than an \( \infty \)-multirelation; the possibility of a transition removing an infinite number of tokens from a place containing an infinite number of tokens to yield an undefined number of tokens in that place is excluded.

The transition relation yields a notion of *reachable marking*, saying that a marking \( M' \) is reachable if there is some initial marking \( M \) from which, following some finite number of transitions, the marking \( M' \) is obtained.

A morphism between from a general net \( G \) to a net \( G' \) embeds the structure of \( G \) into \( G' \) in a way that also embeds the behaviour of \( G \) into \( G' \).

**Definition 2 (Net morphisms).** Let \( G = (P, T, Pre, Post, M) \) and \( G' = (P', T', Pre', Post', M') \) be general Petri nets. A morphism \((\eta, \beta) : G \rightarrow G'\) is a pair consisting of a partial function \( \eta : T \rightarrow T' \) and an \( \infty \)-multirelation \( \beta \subseteq \mu_\infty P \times P' \) which jointly satisfy:

- for all \( M \in M \): \( \beta \cdot M \in M' \)
- for all \( t \in T \): \( \beta \cdot (Pre \cdot t) = Pre' \cdot \eta(t) \) and \( \beta \cdot (Post \cdot t) = Post' \cdot \eta(t) \)

We write \( \eta(t) = * \) if \( \eta(t) \) is undefined and in the above requirement regard * as the empty multiset, so that if \( \eta(t) = * \) then \( \beta \cdot (Pre \cdot t) \) and \( \beta \cdot (Post \cdot t) \) are both empty (see [1]).

We denote the category of general Petri nets with multiple initial markings \( \text{Gen} \), and denote by \( \text{Gen}_s \) the category of singly-marked general nets (those general nets with precisely one initial marking).

One restriction on general nets is to require that multirelations \( Pre \) and \( Post \) are relations, and that every initial marking must be a set of places rather than an \( \infty \)-multiset. Transitions therefore take at most one token from any place and deposit at most one token in any place. We shall call such nets \( \text{PT nets} \), and define the category \( \text{PT} \) to have \( \text{PT nets} \) as objects with the morphisms described above. Accordingly, \( \text{PT} \) is a full subcategory of \( \text{Gen} \). We also define the category \( \text{PT}_s \) to consist of singly-marked \( \text{PT nets} \), which is a full subcategory of \( \text{Gen}_s \).

The relations \( Pre \) and \( Post \) of a \( \text{PT} \) net \( G = (P, T, Pre, Post, M) \) may equivalently be seen as a *flow relation* \( \triangleright_G \subseteq (P \times T) \cup (P \times T) \) describing how places and transitions are connected:

\[
p \triangleright_G t \iff Pre(p, t) \quad t \triangleright_G p \iff Post(t, p).
\]

We may equivalently define a \( \text{PT} \) net \( G = (P, T, \triangleright_G, M) \) by giving its flow relation. When writing the flow relation, we shall omit the subscript \( G \) where no
confusion arises. We shall use $x$ and $y$ to range over elements of $P \cup T$ and write $x \in G$ to mean that $x \in P \cup T$. As standard, we write $\triangleright^*$ for the reflexive, transitive closure of $\triangleright$ and $\triangleright^+$ for the irreflexive transitive closure of $\triangleright$. We say that $y$ is reachable from $x$ if $x \triangleright^* y$.

It need not be the case that every marking following from an initial marking of a PT net can itself be represented as a set of places. Consider, for example, the following (singly-marked) PT net:

![PT net diagram]

The initial marking of the net consists only of the place $a$. Following the occurrence of $t$, the marking has one token in $b$ and one token in $a$. The event $t$ can occur again, this time yielding one token in $a$ and two tokens in $b$.

**Definition 3 (Safe net).** A safe net is a PT net for which every reachable marking (from any initial marking) is a set.

We reserve the symbol $\mathcal{N}$ to range over safe nets. We will call the transitions of safe nets events and their places conditions, writing $(B, E, \triangleright, M)$ for a safe net. We call $b \in B$ a precondition of $e \in E$ if $b \triangleright e$ and call $b$ a postcondition of $e$ if $e \triangleright b$. The set of preconditions of $e$ is written $\cdot e$ and the set of postconditions of $e$ is written $e \cdot$. The sets $\cdot b$ and $b \cdot$, of pre-events and post-events respectively of a condition $b$, are defined similarly. We denote the category of safe nets $\mathbf{Safe}$ and singly-marked safe nets $\mathbf{Safe_s}$, with morphisms as described above.

Occurrence nets were introduced in [7] as a class of net suited to giving the semantics of more general kinds of net in a way that represents the causal dependencies of event occurrences and how they conflict with each other. They can be thought of as safe nets with acyclic flow relations such that every condition occurs as a postcondition of at most one event and for every condition there is a reachable marking containing that condition, and for every event there is a reachable marking in which the event can occur. We extend their original definition to account for the generalization to having a set of initial markings.

**Definition 4 (Occurrence net).** An occurrence net $O = (B, E, \triangleright, M)$ is a safe net satisfying the following restrictions.

1. $\forall M \in \mathcal{M} : \forall b \in M : (\cdot b = \emptyset)$
2. $\forall b' \in B : \exists M \in \mathcal{M} : \exists b \in M : (b \triangleright^* b')$
3. $\forall b \in B : (|\cdot b| \leq 1)$
4. $\triangleright^+$ is irreflexive and, for all $e \in E$, the set $\{e' \mid e' \triangleright^* e\}$ is finite.
5. $\# \text{ is irreflexive, where}$

\[
\begin{align*}
e#_{me'} & \iff e \in E \land e' \in E \land e \neq e' \land \cdot e \cap \cdot e' \neq \emptyset \\
b#_{mb'} & \iff \exists M, M' \in \mathcal{M} : (M \neq M' \land b \in M \land b' \in M') \\
x#x' & \iff \exists y, y' \in E \cup B : y#_{m} y' \land y \triangleright^* x \land y' \triangleright^* x'
\end{align*}
\]
Singly-marked occurrence nets can be seen to coincide with the original definition of occurrence net. We denote the category of occurrence nets $\text{Occ}$ and the category of singly-marked occurrence nets $\text{Occ}_s$, both with morphisms as described above.

The flow relation $\triangleright$ of an occurrence net $O$ indicates how occurrences of events and conditions causally depend on each other and the relation $\#$ indicates how they conflict with each other, with $\#_m$ representing immediate conflict. The concurrency relation $\text{co}_O \subseteq (B \cup E) \times (B \cup E)$ may be defined as follows:

$$x \text{ co}_O y \iff \neg(x \# y \text{ or } x \triangleright^+ y \text{ or } y \triangleright^+ x)$$

We often drop the subscript $O$ when we write $\text{co}_O$. The concurrency relation is extended to sets of conditions $A$ in the following manner:

$$\text{co} A \iff \forall b, b' \in A : b \text{ co } b' \text{ and } \{e \in E : \exists b \in A. e \triangleright^+ b\} \text{ is finite}$$

**Proposition 1.** Let $O = (B, E, \triangleright, M)$ be an occurrence net. Any subset $A \subseteq B$ satisfies $\text{co} A$ if there exists a reachable marking $M$ of $O$ such that $A \subseteq M$. $\square$

The final class of net that we shall make use of is causal nets. These are well-known representations of paths of general nets.

**Definition 5 (Causal net).** A causal net $C = (B, E, \triangleright, M)$ is an occurrence net with at most one initial marking for which the conflict relation $\#$ is empty.

It is easy to see that causal nets might be equivalently be formulated as occurrence nets for which $M$ contains at most one initial marking and there are no two distinct events $e, e' \in E$ such that $\bullet e \cap \bullet e' = \emptyset$. The category of causal nets is denoted $\text{Caus}$, which is readily seen to be a full subcategory of $\text{Gen}$. A path of a general net $G$ is a causal net $C$ together with a morphism $m : C \rightarrow G$ in $\text{Gen}$.

### 2.1 Classical results

Occurrence nets provide a convenient link between more general forms of net and other models for concurrency. In [2, 3], a coreflection between (singly-marked) occurrence nets and a category of event structures was given, and there is a coreflection between the category of occurrence nets and (coherent) asynchronous transition systems [1]. Of more relevance to our current work, there is a coreflection [2] between the categories of singly-marked safe nets and singly-marked occurrence nets:

$$\text{Occ}_s \perp \perp \text{Safe}_s$$

The left adjoint is the inclusion of occurrence nets into safe nets and the right adjoint unfolds a safe net to an occurrence net. We shall not give the formal definition of the unfolding here since it arises as an instance of a more general
construction later on. It can be thought of, however, as a generalization of the
process of creating a synchronization tree from a transition system. The counit
of the adjunction is a morphism in the category of safe nets from the unfolding
of the safe net back into the original safe net. An example unfolding is presented
below, with the safe net \( N \) unfolded to \( U(N) \) on the right. Conditions and events
of \( U(N) \) are labelled with their image under the counit.

The key result in obtaining the coreflection is that the unfolding satisfies the
following **cofreeness** property: Let \( N \) be a safe net and \( U(N) \) be its occurrence net
unfolding with counit \( \varepsilon_N : U(N) \rightarrow N \). For any occurrence net \( O \) and morphism
\( (\eta, \beta) : O \rightarrow N \) in \( \text{Safe}_s \), there is a unique morphism \( (\theta, \alpha) : O \rightarrow U(N) \) such
that the following diagram commutes:

\[
\begin{array}{ccc}
U(N) & \xrightarrow{\varepsilon_N} & N \\
\downarrow{(\theta, \alpha)} & & \downarrow{(\eta, \beta)} \\
O & & \end{array}
\]

Though we are unable to find the result in the literature, it can also be shown
that an unfolding operation on PT nets gives rise to a coreflection:

\[
\text{Occ}_s \perp \xrightarrow{\mathcal{U}} \text{PT}_s
\]

### 2.2 Results on general nets

A similar result to those above, giving a coreflection between (singly-marked)
ocurrence nets and (singly-marked) general nets does not exist. The definition
of the unfolding of a general net \( G \) to an occurrence net \( U(G) \) can, however, be
obtained by straightforwardly extending the definition of unfolding of a safe net.
The way in which the unfolding is constructed is slightly technical, involving an
inductive definition, but it can be neatly uniquely characterized as follows:
Proposition 2. The unfolding $U(G) = (B, E, \triangleright, M_0)$ of $G = (P, T, \text{Pre}, \text{Post}, M)$ is the unique occurrence net to satisfy

\[
\begin{align*}
B &= \{(M, p, i) \mid M \in M \& p \in P \& 0 \leq i < M[p]\} \\
E &= \{(A, t) \mid A \subseteq B \& t \in T \& \text{co} A \& \beta \cdot A = \text{Pre} \cdot t\} \\
M_0 &= \{((M, p, i) \mid (M, p, i) \in B) \mid M \in M\},
\end{align*}
\]

where $\text{co}$ and $\#$ are the concurrency and conflict relations arising from $\triangleright$ on $B$ and $E$. Furthermore, $\eta$ and $\beta$ defined as

\[
\eta(A, t) = t \quad \beta(X, p, i) = p
\]

form a morphism $\varepsilon_G = (\eta, \beta) : U(G) \rightarrow G$ in $\text{Gen}_t$.

The reason why we fail to obtain a coreflection between $\text{Occ}_s$ and $\text{Gen}_s$ is that the uniqueness property required for cofreeness fails. That is, for any occurrence net $O$ and morphism $(\pi, \gamma) : O \rightarrow G$ there is a map $(\theta, \alpha) : O \rightarrow U(G)$ such that $\varepsilon_G \circ (\theta, \alpha) = (\pi, \gamma)$, but the map $(\theta, \alpha)$ may not be the unique such map.

To see this, consider the net $G$ in Figure 1(a) with its unfolding $U(G)$ in Figure 1(b). The embedding $\varepsilon_G = (\eta, \beta) : U(G) \rightarrow G$ is as defined in Prop. 2, relating both conditions in the unfolding to the place $p$ and sending both events in the unfolding to $t$. The morphism $\text{swap} : U(G) \rightarrow U(G)$ which swaps the conditions $b_1$ and $b_2$ and swaps the events $e_1$ and $e_2$ is drawn in Figure 1(c).

We have $\varepsilon_G \circ \text{id}_G = \varepsilon_G$ and $\varepsilon_G \circ \text{swap} = \varepsilon_G$. Considering the requirements for cofreeness, there is therefore no unique morphism $h : U(G) \rightarrow U(G)$ associated with the morphism $\varepsilon_G : U(G) \rightarrow G$ such that $\varepsilon_G \circ h = \varepsilon_G$, so the cofreeness property for the unfolding fails.

The two conditions in $U(G)$ are symmetric; they arise from unfolding the two tokens residing in the same place in the initial marking. Similarly, the two events in the unfolding are symmetric. There is no unique morphism in the above example because we are able to choose symmetric elements in the unfolding.

Our goal shall be to show that there is a unique morphism up to symmetry. We first summarize the part of the cofreeness property that does hold.

Theorem 1. Let $G$ be a general Petri net, $O$ be an occurrence net and $(\theta, \alpha) : O \rightarrow G$ be a morphism in $\text{Gen}$. There is a morphism $(\pi, \gamma) : O \rightarrow U(G)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
U(G) & \xrightarrow{(\eta, \beta) = \varepsilon_G} & G \\
\downarrow{(\pi, \gamma)} & & \downarrow{(\theta, \alpha)} \\
O & & \end{array}
$$

Furthermore, if the net $G$ is a PT net then $(\pi, \gamma)$ is the unique such morphism.

\[\square\]
\[(a) \text{ General net } G^b_2 = (M, p, 2) \quad e_2 = (b_2, t) \]

\[(b) \text{ Unfolding } U(G) \]

\[(c) \text{ Morphism } \text{swap} : U(G) \rightarrow U(G) \]

Fig. 1. Example unfolding of a general net

It will be of use later to note that if the multirelation \(\alpha\) above is a partial function then so is \(\gamma\).

To define symmetry on nets, we will need to work in a category with pullbacks. Whereas it was shown in [8] that the category of singly-marked safe nets has pullbacks, the category of singly-marked general nets does not. This failure led to the earlier relaxation of the definition of nets to permit them to have a set of initial markings rather the precisely one initial marking. However, even the category \(\text{Gen}\) does not have pullbacks. This can be seen straightforwardly by considering, for example, the requirements on any candidate pullback against itself of the morphism embedding a net with two places each initially containing one token into the net with just one place containing two tokens. To obtain a category of general nets with pullbacks, we restrict attention to folding morphisms between general nets:

\textbf{Definition 6.} A morphism \((\eta, \beta) : G \rightarrow G'\) is a folding morphism if the multirelation \(\beta\) can be represented as a partial function.

Denote the category of general nets with folding morphisms \(\text{Gen}_f\), with full subcategory \(\text{Occ}_f\) of occurrence nets with folding morphisms. We use symbols \(f, g, h\) for folding maps and write \(f(p) = p'\) and \(f(t) = t'\) with the obvious interpretations.

\textbf{Proposition 3.} The category \(\text{Gen}_f\) has pullbacks and binary products.

\textit{Proof.} Let \(G_0, G_1\) and \(G'\) be general nets. Denote the places of \(G'\) by \(P'\), the places of \(G_0\) by \(P_0\), etc. Suppose that there are folding morphisms \(f_0 = (\eta_0, \beta_0) : G_0 \rightarrow G'\) and \(f_1 = (\eta_1, \beta_1) : G_1 \rightarrow G'\). For brevity of the definition, assume that \(f_0\) and \(f_1\) are total on both transitions and conditions; the extension to the more general case is straightforward. Define the net \(Q\) with places \(P_Q\), transitions \(T_Q\),
etc. simultaneously with the definition of folding morphisms $q_0 = (\theta_0, \gamma_0) : Q \to G_0$ and $q_1 = (\theta_1, \gamma_1) : Q \to G_1$ as:

$$P_Q = \{(p_0, p_1) \mid p_0 \in P_0 \text{ and } p_1 \in P_1 \text{ and } f_0(p_0) = f_1(p_1)\}$$

$$T_Q = \{(C, t_0, t_1, D) \mid C \subseteq \mu P_Q \text{ and } D \subseteq \mu_\infty P_Q \text{ and } \eta_0(t_0) = \eta_1(t_1) \text{ and } \gamma_0 \cdot C = \text{Pre}_0 \cdot t_0 \text{ and } \gamma_1 \cdot C = \text{Pre}_1 \cdot t_1 \text{ and } \gamma_0 \cdot D = \text{Post}_0 \cdot t_0 \text{ and } \gamma_1 \cdot D = \text{Post}_1 \cdot t_1\}$$

$$\text{Pre}_Q \cdot (C, t_0, t_1, D) \triangleq C \quad \text{Post}_Q \cdot (C, t_0, t_1, D) \triangleq D$$

$$\mathbb{M}_Q = \{M \mid M \subseteq \mu_\infty P_Q \text{ and } \gamma_0 \cdot M \in \mathbb{M}_0 \text{ and } \gamma_1 \cdot M \in \mathbb{M}_1\}$$

with the folding morphisms acting on places as

$$\gamma_0(p_0, p_1) \triangleq p_0 \quad \gamma_0(p_0, p_1) \triangleq p_1$$

and on transitions as

$$\theta_0(C, t_0, t_1, D) \triangleq t_0 \quad \theta_1(C, t_0, t_1, D) \triangleq t_1.$$

It can be shown that the net $Q$ is a pullback of $f_0$ against $f_1$ in the category $\text{Gen}_f$, though we omit the proof here. The category $\text{Gen}_f$ has a terminal object (the net with no places, no transitions and the single empty initial marking) and pullbacks, and so by a general result of category theory (see [9]) has binary products. \hfill \square

### 3 Categories with symmetry

It is shown in [4] how symmetry can be defined on event structures, and more generally on any category of models satisfying certain properties. We briefly present the definitions here before introducing symmetry to nets.

The definition of symmetry makes use of open maps [10]. Let $\mathcal{C}$ be a category with a distinguished subcategory $\mathcal{P}$ of path morphisms. A morphism $f : X \to Y$ in $\mathcal{C}$ is $\mathcal{P}$-open if, for any map $s : P \to Q$ in $\mathcal{P}$ and morphisms $p : P \to X$ and $q : Q \to Y$, if the diagram

\[
P \xrightarrow{p} X \\
\downarrow{s} \quad \downarrow{f} \\
Q \xrightarrow{q} Y
\]

commutes then there is a morphism $h : Q \to X$ such that the diagram

\[
P \xrightarrow{p} X \\
\downarrow{s} \quad \downarrow{f} \\
\downarrow{h} \\
Q \xrightarrow{q} Y
\]
commutes, i.e. \( p = h \circ s \) and \( q = f \circ h \). It can be shown purely diagrammatically that open maps compose, and therefore form a subcategory, are preserved under pullbacks and the product of open maps is open.

Let \( \mathcal{C} \) be a category with a subcategory of path objects \( \mathbb{P} \). If \( \mathcal{C} \) has binary products and pullbacks, it can be extended with symmetry to yield a category \( \mathcal{SC} \). The objects of \( \mathcal{SC} \) are tuples \((X, S, l, r)\) consisting of an object \( X \) of \( \mathcal{C} \) and two \( \mathbb{P} \)-open maps \( l, r : S \to X \) which make \((l, r)\) an equivalence [11] (see Appendix B). The requirements on \( l \) and \( r \) are slightly weaker than those in [4] in that we do not require that the maps \( l \) and \( r \) are jointly monic. The morphisms of \( \mathcal{SC} \) are morphisms of \( \mathcal{C} \) that preserve symmetry. Let \( f : X \to Y \) be a morphism in \( \mathcal{C} \) and \((X, S, l, r)\) and \((X', S', l', r')\) be objects of \( \mathcal{SC} \). The morphism \( f : X \to Y \) preserves symmetry if there is a morphism \( h : S \to S' \) such that the following diagram commutes:

\[
\begin{array}{c}
X \\
\downarrow f \\
X'
\end{array}
\begin{array}{cccccc}
S & \rightarrow & X \\
\downarrow l & & \downarrow f \\
S' & \rightarrow & X'
\end{array}
\begin{array}{c}
S \\
\downarrow h \\
S'
\end{array}
\begin{array}{c}
X \\
\downarrow g \\
X'
\end{array}
\]

With the definition of symmetry on objects, the equivalence relation of symmetry \( \sim \) on morphisms can be defined.

**Definition 7.** Let \( f, g : (X, S, l, r) \to (X', S', l', r') \) be morphisms in \( \mathcal{SC} \). Define \( f \sim g \) iff there is a morphism \( h : X \to X' \) in \( \mathcal{C} \) such that following diagram commutes in \( \mathcal{C} \):

\[
\begin{array}{c}
X \\
\downarrow f \\
X'
\end{array}
\begin{array}{cccc}
S & \rightarrow & X \\
\downarrow l & & \downarrow g \\
S' & \rightarrow & X'
\end{array}
\begin{array}{c}
S \\
\downarrow h \\
S'
\end{array}
\begin{array}{c}
X \\
\downarrow g \\
X'
\end{array}
\]

Composition of morphisms in \( \mathcal{SC} \) coincides with composition in \( \mathcal{C} \) and the two categories share the same identity maps. The category \( \mathcal{C} \) is more fully described as a category enriched in equivalence relations.

### 3.1 Occurrence nets with symmetry

We can apply the abstract definition to obtain a category of occurrence nets with symmetry. The category of occurrence nets with folding morphisms \( \text{Occ}_t \) has pullbacks. The pullback of \( f_0 : O_0 \to O' \) against \( f_1 : O_1 \to O' \) is obtained by taking the the pullback \( Q, q_0, q_1 \) of \( f_0 \) against \( f_1 \) in the category \( \text{Gen}_t \) and then unfolding the result to obtain the pullback \( \mathcal{U}(Q) \) in \( \text{Occ}_t \) with pullback morphisms \( q_0 \circ \varepsilon_Q \) and \( q_1 \circ \varepsilon_Q \). The construction of \( Q \) ensures that it is a PT net, so Theorem 1 allows us to show that the result is indeed the pullback in the category \( \text{Occ}_t \). Products in the category \( \text{Occ}_t \) are constructed similarly. Using the prior construction, we obtain the category \( \text{SOcc}_t \) of occurrence nets with symmetry, interpreting open maps to be with respect to the category \( \text{Caus}_t \).
4 Symmetry in unfolding

Given a general net $G$, we wish to impose a symmetry on its unfolding $U(G)$. With respect to this symmetry, we will be able to show uniqueness up to symmetry of the morphism obtained in Theorem 1 when unfolding an arbitrary general net.

The symmetry on the unfolding is obtained by unfolding the kernel pair of the inclusion morphism $\varepsilon_G : U(G) \rightarrow G$, which is the pullback of $\varepsilon_G$ against itself in $\text{Gen}_f$:

$$
\begin{array}{c}
S \\
\downarrow \quad t \\
U(G) \downarrow \varepsilon_G \\
\varepsilon_G \rightarrow G
\end{array}
$$

To see that $(U(G), U(S), l \circ \varepsilon_S, r \circ \varepsilon_S)$ is a symmetry, we must show that the morphisms $l \circ \varepsilon_S$ and $r \circ \varepsilon_S$ are $\text{Caus}_f$-open and form an equivalence. A purely diagrammatic argument shows that $l \circ \varepsilon_S$ and $r \circ \varepsilon_S$ are an equivalence. Open morphisms from occurrence nets into general nets can be characterized in the following way:

**Proposition 4.** Let $O$ be an occurrence net and $G$ be a general net. A morphism $f : O \rightarrow G$ is $\text{Caus}_f$-open in $\text{Gen}_f$ if, and only if, it is total on conditions and events, reflects any initial marking of $G$ to an initial marking of $O$ and satisfies the following property:

for any subset $A$ of conditions of $O$ such that $\text{co} A$ for which there exists a transition $t$ of $G$ such that $f \cdot A = \text{Pre}_G \cdot t$, there exists an event $e$ of $O$ such that $A = \text{Pre}_O \cdot e$ and $f(e) = t$.

The morphism $\varepsilon_G : U(G) \rightarrow G$ of Proposition 2 is readily seen to satisfy this property, and is therefore $\text{Caus}_f$-open. The pullback of open morphisms is open [10], and so the morphisms $l$ and $r$ are $\text{Caus}_f$-open. Note that a morphism between occurrence nets is $\text{Caus}_f$-open in $\text{Occ}_f$ iff it is $\text{Caus}_f$-open in $\text{Gen}_f$.

**Proposition 5.** The tuple $(U(G), U(S), l \circ \varepsilon_S, r \circ \varepsilon_S)$ is an occurrence net with symmetry.

We can now show that we obtain cofreeness ‘up to symmetry’:

**Theorem 2.** Let $G$ be a general Petri net and $O$ be an occurrence net. For any morphism $g : O \rightarrow G$ in $\text{Gen}_f$, there is a morphism $h : O \rightarrow U(G)$ such that

$$
\begin{array}{c}
U(G) \quad \varepsilon_G \\
\downarrow \quad \quad \downarrow g \\
\uparrow h \\
O
\end{array}
$$

commutes, i.e. $\varepsilon_G \circ h = g$. Furthermore, any morphism $h' : W \rightarrow U(G)$ such that $\varepsilon_G \circ h' = g$ satisfies $h \sim h'$ with respect to the symmetry on $U(G)$ described above.
5 A coreflection up to symmetry

We show how the results of the last section are part of a more general coreflection from occurrence nets with symmetry to general nets with symmetry. In the last section, we showed how to unfold a general net to an occurrence net with symmetry. For the coreflection, we need to extend this construction to unfold general nets themselves with symmetry.

Fortunately we can adopt the method of Sect. 3.1 as the category meets the conditions required there. The category $\text{Gen}_f$ has pullbacks and products, with open maps determined by the subcategory $\text{Caus}_f$. Morphisms of general nets with symmetry and the 'up to symmetry' equivalence relation on morphisms follow from the general definitions in Sect. 3.1. This provides us with the definition of the category $S\text{Gen}_f$. Again, we obtain a category enriched in equivalence relations.

To show that the 'inclusion' $I : S\text{Occ}_f \rightarrow S\text{Gen}_f$ taking an occurrence net with symmetry $(O, S, l, r)$ to a general net with symmetry is a functor, it is necessary to show that the transitivity property holds of the symmetry in $S\text{Gen}_f$. This relies on the following lemma.

**Lemma 1.** Consider the following pullback in $\text{Gen}_f$:

\[
\begin{array}{ccc}
P & \xrightarrow{p} & N_1 \\
\downarrow q & & \downarrow f \\
N_2 & \xrightarrow{g} & N
\end{array}
\]

If $N_1$, $N$ and $N_2$ are occurrence nets and $f$ and $g$ are total, then $P$ is also an occurrence net, and both $p$ and $q$ are total.

We now have a functor $I : S\text{Occ}_f \rightarrow S\text{Gen}_f$ respecting $\sim$ regarding an occurrence net with symmetry $(O, S, l, r)$ itself directly as a general net with symmetry.

We define the unfolding operation on objects of the category of general nets with symmetry. Its extension to a pseudofunctor will follow from the biadjunction. Let $(G, S_G, l, r)$ be a general net with symmetry. It is ‘unfolded’ to the occurrence net with symmetry $U(G, S_G, l, r) = (U(G), S_0, l', r')$ defined in the following commuting diagram:

\[
\begin{array}{ccc}
S_0 & \cong U(S_1) & \xrightarrow{\varepsilon_{S_1}} S_1 \\
\downarrow & & \downarrow \langle l_1, r_1 \rangle \\
U(G) \times \text{Gen}_f U(G) & \xrightarrow{\varepsilon_G \times \varepsilon_G} G \times \text{Gen}_f G
\end{array}
\]
where $\varepsilon_G : U(G) \to G$ is the folding map given earlier — it will be a morphism of $SGen_f$ by Proposition 6 below.

It is useful to have an alternative way to construct the morphisms $l_1$ and $r_1$ above; it will help in showing that $U$ is well-defined in that it gives a general net with symmetry.

**Proposition 6.** The pullback above can be constructed as:

For example, it now becomes clear why $l_1$ and $r_1$, and hence $l'$ and $r'$, are open, being the composition of open maps themselves obtained as pullbacks of open maps. Theorem 1 generalizes to give a cofreeness result:

**Theorem 3.** Let $G = (G, S_G, l_G, r_G)$ be a general net with symmetry and $O = (O, S_O, l_O, r_O)$ be an occurrence net with symmetry. For any $g : O \to G$ in $SGen_f$, there is a morphism $h : O \to U(G)$ such that the following diagram commutes:

Furthermore, $h$ is unique up to symmetry: any $h' : O \to U(G)$ such that $\varepsilon_G \circ h' \sim g$ satisfies $h \sim h'$.

Technically, we have a biadjunction from $SOcc_f$ to $SGen_f$ with $I$ left biadjoint to $U$ (which extends to a pseudofunctor). Its counit is $\varepsilon$ and its unit is a natural isomorphism $O \cong U(O)$. In this sense, we have established a coreflection from $SOcc_f$ to $SGen_f$ up to symmetry.

**6 Related work**

Occurrence nets were first introduced in [7] together with the operation of unfolding singly-marked safe nets. The coreflection between occurrence nets and safe nets was first shown in [2].

Engelfriet defines the unfolding of singly-marked PT nets in [12]. Rather than giving a coreflection between the categories, the unfolding is characterized as the
greatest element of a complete lattice of occurrence nets embedding into the PT
net.

A coreflection between a subcategory of (singly-marked) general nets and
and a category of embellished forms of transition system was given in [13]. There,
the restriction to particular kinds of net morphism is of critical importance;
taking the more general morphisms of general Petri nets presented here would
have resulted in the cofreeness property failing for an analogous reason to the
failure of cofreeness of the unfolding of general nets to occurrence nets without
symmetry.

An adjunction is given between a subcategory of singly-marked general nets
and the category of occurrence nets in [14]. The restriction imposed on the
morphisms of general nets there, however, precludes in general there being a
morphism from \( U(G) \) to \( G \) in their category of general nets, if \( U(G) \) the oc-
currence net unfolding of \( G \) is regarded directly as a general net. To obtain an
adjunction, the functor from category of occurrence nets into the category of
general nets is not regarded as the direct inclusion, but instead occurs through
a rather detailed construction and does not yield a coreflection.

7 Conclusion

We have shown that symmetry plays a vital rô le in characterizing the occurrence
net unfolding of general Petri nets and therefore in relating the categories of
general nets and occurrence nets.

We would like to extend our results to the category of general nets with
all morphisms, not just folding morphisms. It is becoming clear from this and
other work [15] that sometimes, in adjoining symmetry, models do not fit the
simple scheme outlined in [4] appropriate to event structures and stable families.
For example, the category of general nets with all morphisms does not have
pullbacks. Examples like that here and in [15] should suggest more liberal axioms
on categories of models which enable their extension with symmetry.

The generalization of occurrence nets to allow them to have a set of initial
markings leads to an interesting generalization of the existing coreflection be-
tween event structures and occurrence nets. We can show that no changes need
to be made to category of event structures due to this generalization to obtain
a coreflection between event structures and occurrence nets. One would then
expect this adjunction, with the extension beyond folding morphisms mentioned
above, to straightforwardly yield a coreflection between the category of event
structures with symmetry and the category of occurrence nets with symmetry.
We also anticipate that the coreflection between occurrence nets with symmetry
and general nets with symmetry could be factored through the category of PT
nets with symmetry.

References


A Multisets

Let the set of natural numbers \( \{0, 1, 2, \ldots\} \) be denoted \( \mathbb{N} \). A multiset over a set \( A \) is a vector of elements of \( \mathbb{N} \) indexed by elements of \( A \). For instance, let \( A = \{a_0, a_1\} \) and suppose that the multiset \( X \) contains two occurrences of \( a_0 \) and one of \( a_1 \); the corresponding multiset is:

\[
\begin{bmatrix}
a_1 \\ a_2
\end{bmatrix}
\]

Let the set of multisets over a set \( A \) be denoted \( \mu(A) \) and write \( X \subseteq \mu(A) \) if \( X \) is a multiset over \( A \). Let \( X[a] \) denote the value of the vector at \( a \). Write \( \emptyset_A \) for the empty multiset with basis \( A \). Denote multiplication of the multiset \( X \) by a scalar \( n \in \mathbb{N} \) by \( n \cdot X \). A multiset \( X \) with basis \( A \) is said to be finite if \( \sum_{a \in A} X[a] \) is finite.

Define the set \( \mathbb{N}^\infty = \mathbb{N} \cup \{\infty\} \). An \( \infty \)-multiset over the set \( A \) is a vector of elements of \( \mathbb{N}^\infty \) indexed by elements of \( A \). The set of all \( \infty \)-multisets over \( A \) is denoted \( \mu^\infty(A) \), and we write \( X \subseteq \mu^\infty(A) \) if \( X \) is an \( \infty \)-multiset over \( A \). Addition and multiplication on integers is extended to the element \( \infty \) by defining \( \infty + n = n + \infty = \infty \) (\( \forall n \in \mathbb{N}^\infty \)), \( \infty \cdot n = n \cdot \infty = \infty \) (\( \forall n \in \mathbb{N}^\infty \setminus \{0\} \)), \( \infty \cdot 0 = 0 \cdot \infty = 0 \).

Subtraction \( m - n \) of two elements \( m, n \in \mathbb{N}^\infty \) is a partial operation, defined iff \( n \leq m \) and \( n \neq \infty \). The result of \( \infty - \infty \) is undefined.

A multirelation \( R \) between sets \( A \) and \( B \) is a matrix of elements of \( \mathbb{N} \), and similarly an \( \infty \)-multirelation is a matrix of elements in \( \mathbb{N}^\infty \). The number of times that the element \( a \) is related to \( b \) is given by \( R[a, b] \), which is the natural number (or \( \infty \)) occurring in the \( a \)-indexed row and \( b \)-indexed column of \( R \). We write \( R \subseteq \mu(A \times B) \) if \( R \) is a multirelation between \( A \) and \( B \) and \( R \subseteq \mu^\infty(A \times B) \) if \( R \) is an \( \infty \)-multirelation between \( A \) and \( B \), noting the equivalent formulation of a multirelation as a multiset over the basis \( A \times B \).

Application of a multirelation \( R \subseteq \mu(A \times B) \) to a multiset \( X \subseteq \mu(A) \) is obtained as their inner product \( R \cdot X \). In particular, \( R \cdot X \subseteq \mu(B) \) and for any \( b \in B \)

\[
(R \cdot X)[b] = \sum_{a \in A} R[a, b] \cdot X[a].
\]

Care has to be taken since \( R \subseteq \mu(A \times B) \) may fail to yield a multiset if the above sum is \( \infty \) at any \( b \in B \); an \( \infty \)-multiset would be obtained. Application of an \( \infty \)-multirelation to an \( \infty \)-multiset is always defined, yielding an \( \infty \)-multiset.

A.1 Sets, relations and (partial) functions

We say that a multiset \( X \subseteq \mu(A) \) is a set if \( X[a] \leq 1 \) for all \( a \in A \). All the usual notation for sets is adopted in this situation, for example \( a \in X \) for \( X[a] = 1 \). We write \( f : X \to Y \) if \( f \) is a function from \( X \) to \( Y \).
A relation $R$ on sets $A$ and $B$, written $R \subseteq A \times B$ is a multirelation $R \subseteq \mu A \times B$ such that $R(a,b) \leq 1$ for all $a \in A$ and $b \in B$. We now write $R(a,b)$ or $aRb$ if, as a multirelation, $R[a,b] = 1$. We write $R^*$ for the reflexive, transitive closure of a relation $R$, and write $R^+$ for the irreflexive, transitive closure of $R$.

If $f$ is a partial function from set $X$ to set $Y$, written $f : X \rightarrow_* Y$, that is undefined on $x \in X$, we write $f(x) = *$. Equality includes undefinedness, so if $f : X \rightarrow_* Y$ and $f' : X' \rightarrow_* Y$ then $f(x) = f'(x')$ iff either both $f(x) = *$ and $f'(x') = *$ or there exists $y \in Y$ such that both $f(x) = y$ and $f'(x') = y$.

### B Equivalences

Assume a category $\mathcal{C}$ with pullbacks. Let $l, r : S \to G$ be a pair of morphisms in $\mathcal{C}$. They form an equivalence iff they satisfy:

**Reflexivity** there is a map $\rho$ such that

$$
\begin{array}{c}
\text{id}_G \\
\downarrow \\
G
\end{array}
\begin{array}{c}
\downarrow \\
S
\end{array}
\begin{array}{c}
r \\
\downarrow \\
G
\end{array}
\begin{array}{c}
\downarrow \\
l \\
\downarrow \\
G
\end{array}
\begin{array}{c}
\text{id}_G \\
\downarrow \\
G
\end{array}
$$

Commutes;  

**Symmetry** there is a map $\sigma$ such that

$$
\begin{array}{c}
\downarrow r \\
S
\end{array}
\begin{array}{c}
\downarrow l \\
S
\end{array}
\begin{array}{c}
\downarrow \sigma \\
G
\end{array}
\begin{array}{c}
\downarrow r \\
S
\end{array}
\begin{array}{c}
\downarrow l \\
G
\end{array}
$$

Commutes; and  

**Transitivity** there is a map $\tau$ such that

$$
\begin{array}{c}
\downarrow f \\
\downarrow g \\
Q
\end{array}
\begin{array}{c}
\downarrow f \\
\downarrow g \\
S
\end{array}
\begin{array}{c}
\downarrow l \\
\downarrow r \\
S
\end{array}
\begin{array}{c}
\downarrow f \\
\downarrow g \\
S
\end{array}
\begin{array}{c}
\downarrow l \\
\downarrow r \\
S
\end{array}
\begin{array}{c}
\downarrow l \\
\downarrow r \\
G
\end{array}
\begin{array}{c}
\downarrow l \\
\downarrow r \\
G
\end{array}
\begin{array}{c}
\downarrow l \\
\downarrow r \\
G
\end{array}
$$

Commutes, where $Q,f,g$ is a pullback of $r,l$. 