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A NEW DEFINITION OF MORPHISM ON PETRI NETS

(Extended Abstract)

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Introduction

Petri nets are a fundamental model of concurrent processes and have a wide range of applications (see [Br], [Pe]). In this paper we address the problem of how to structure nets, define constructions on them and understand the behaviour of a compound process, represented as a net, from the behaviour of its components. The constructions follow from a new definition of morphism on Petri nets--it is not the same as Petri's original. The morphisms respect the "token game"--the dynamic behaviour of nets--unlike Petri's. The category of nets with the new morphisms has a product which is closely related to various parallel compositions which have been defined on labelled Petri nets for synchronising processes (see e.g. the compositions on nets defined in [LSC] and section 3). It has a coproduct which is a generalised form of the "sum" operation as used for example in [M]. There are pleasing relations with other categories too.

One can use Petri nets to give semantics to programming languages. But, what is the semantics of nets? In themselves nets are complicated objects whose behaviour is rather intricate. When, for instance, do Petri nets have the same behaviour? Attempting to answer these questions leads naturally to occurrence nets first introduced in [NWP1,2]. Occurrence nets form a subcategory which bears a pleasant relation to the larger category of nets; the inclusion functor has a right adjoint which is an operation taking a net to its unfolding to a net of condition and event occurrences. (This construction was introduced in [NWP1,2, W] but without this abstract characterisation.) It is argued that the meaning, or semantics, of a net is its occurrence net unfolding so that two nets are regarded as having essentially the same behaviour if they have isomorphic unfoldings. In a similar way there is an adjunction between the category of occurrence nets and the category of (prime) event structures. Thus allied with the work of [W1,2] there are functors which serve as a bridge between Petri net models and the interleaving models used in e.g. [M] and [HBR].

These successes give force to the new definition of morphism on nets. They counter a criticism frequently levelled at Petri nets, that their mathematics is unwieldy.

Unfortunately for lack of space all proofs have been omitted. They will be included in a report of the Computer Laboratory, University of Cambridge.

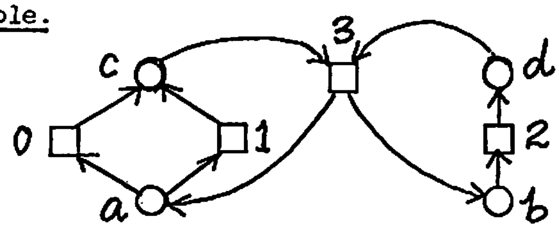
1. Petri nets

Petri nets model processes in terms of how the occurrences of events incur changes in local states, called conditions. This is expressed by a causal dependency (or flow) relation between sets of events and conditions, and it is this structure which determines the dynamic behaviour of nets once the causal dependency relation is given a natural interpretation.

1.1 Definition. A Petri net is a 3-tuple (B,E,F) where
B is a non-null set of conditions
E is a set of events, and
$F \subseteq (B \times E) \cup (E \times B)$ is the causal dependency relation which satisfies the restrictions $\{b \in B \mid bFe\}$ are non-null for all events $e \in E$.
$\{b \in B \mid eFb\}$

Nets are often drawn as graphs in which events are represented as boxes and conditions as circles with directed arcs between them to represent the flow relation. Here is an example.

1.2 Example.



1.3 Notation. Let $N = (B,E,F)$ be a net. Let $A \subseteq B \cup E$.
Define $^\bullet A = \{y \in B \cup E \mid \exists a \in A.\ yFa\}$ and
$A^\bullet = \{y \in B \cup E \mid \exists a \in A.\ aFy\}$.
When A is a singleton $\{a\}$ we abbreviate $\{a\}^\bullet$ to $a^\bullet$ and $^\bullet\{a\}$ to $^\bullet a$. When e is an event we call $^\bullet e$ its set of preconditions and $e^\bullet$ its postconditions.

The dynamic behaviour of nets is based on these principles which specify how the occurrence of events affect the holding of conditions--a condition is said to hold when it is true. They express the intended meaning of the causal dependency relation.

(i) An occurrence of an event e ends the holding of its preconditions $^\bullet e$ and begins the holding of its postconditions $e^\bullet$.

(ii) (a) The holding of a condition b, when it ends, ends because of the occurrence of a unique event in $b^\bullet$.
(b) The holding of a condition b, when it begins, begins because of the occurrence of a unique event in $^\bullet b$.

Of course we need a way to specify what conditions hold. We introduce an idea of global state which just specifies what subset of conditions hold.

1.4 Definition. A marking of a net is a non-null subset of conditions.
The marking of a net changes over time according to rules, commonly called "the token game" because a marking is often specified by laying tokens on those conditions in the marking; as events occur tokens are picked-up and put-down in accord with the principles above. From the principles it follows, only informally, of course, that an event can occur only once all its preconditions hold and none of its postconditions which are not preconditions hold. Then the event is said to have concession. Nets allow more than one event to occur together but there are situations where the occurrence of one event excludes the occurrence of another and vice versa - a phenomenon called conflict. Consider two events which both have concession but which have a precondition in common. From the principle (ii)(a) it follows that only one of them can occur; otherwise they would both end the holding of the condition b. They are in forwards conflict. Now consider two events which both have concession but which have a postcondition in common. By (ii)(b) only one of them can occur. They are in backwards conflict. We formally define the token game which specifies how the marking changes as events occur.

1.5 Definition. (The token game) Let $N = (B,E,F)$ be a Petri net. Let M be a marking. Say an event $e \in E$ has concession at M iff $^\bullet e \subseteq M$ & $(e^\bullet \setminus {}^\bullet e) \cap M = \emptyset$. Let e,e' be events with concession at M. Say e and e' are in forwards conflict at M iff $e \neq e'$ & $^\bullet e \cap {}^\bullet e' \neq \emptyset$.
Say they are in backwards conflict at M iff $e \neq e'$ & $e^\bullet \cap e'^\bullet \neq \emptyset$.
Let M and M' be markings. Let A be a finite subset of E. Define $M \xrightarrow{A} M'$ iff

(i) $\forall e \in A$.e has concession at M and
(ii) $\forall e,e' \in A$.e,e' are not in conflict, and
(iii) $M' = (M \setminus {}^\bullet A) \cup A^\bullet$.

In this situation the events A are said to occur concurrently.

A marking M' is said to be reachable from a marking M iff $M = M_0 \xrightarrow{A_0} M_1 \xrightarrow{A_1} \cdots \xrightarrow{A_{n-1}} M_n = M'$ for subsets of events $A_0, A_1, \ldots, A_{n-1}$ and markings $M_0, M_1, \ldots M_n$.

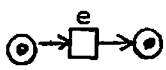
Remark. There are other versions of the token game in which more than one token is allowed on a condition; conditions are allowed a certain multiplicity so that they can model, for example, the availability of a number of resources. We shall not allow more than one token on a condition, partly for simplicity and partly because it is intended that more complicated nets should ultimately be abbreviations for the simpler nets we consider (see e.g. [GR]). The nets we consider are almost, but not quite, those nets called condition-event systems in [Br].

1.6 Example. Consider the net of example 1.2. Initially the net is marked $\{a,b\}$. The events 0, 1 are in both forwards and backwards conflict so either 0 or 1, but not both can occur. Certainly the event 2 can occur. It is not in conflict with either 0 or 1 so 2 can occur concurrently with 0 or 1, but not both. For example, taking $M' = \{c,d\}$ and $A = \{0,2\}$ we have $M \xrightarrow{A} M'$. Of course from the marking M' the event 3 can occur giving rise to the marking M again, and we can start all over again, perhaps letting event 1 occur this time.

Generally a process is modelled by a Petri net with an initial marking from which it reaches other markings as events occur.

1.7 Definition. A Petri net with initial marking is a structure (B,E,F,M) where (B,E,F) is a Petri net and M is a marking called the initial marking. Markings reachable from the initial markings are called reachable markings. An event e is said to be in contact at a marking M' if ${}^{\bullet}e \subseteq M' \,\&\, (e^{\bullet} \setminus {}^{\bullet}e) \cap M' \neq \emptyset$.

A net with initial marking is contact-free iff there is not contact at any reachable marking.

The event e in the net  is in contact and the net  is not contact-free. The net of example 1.2 with initial marking $\{a,b\}$ is contact-free however.

Contact-free nets have the pleasant property that an event can occur at a reachable marking iff its preconditions are included in the marking. If one accepts the earlier principles, the behaviour of nets with contact is weird; it seems an event is prevented from occurring by the knowledge of what would happen in the future if it did--see the above examples. For this reason it is difficult to understand their behaviour. Later when we come to associate an occurrence net unfolding with the behaviour of a net--thus giving nets a formal semantics in terms of more basic nets--we shall be able to do this only for nets which are contact-free. One view of nets with contact is that they are improper descriptions. As has been remarked, there are other token games in which conditions can have multiple holdings. For such nets the above principles are invalid. The understanding of such nets is less settled; for example the question of how to unfold such a net to an occurrence net (as in §5) is unsettled, though a start has been made in [GR].

2. The new definition of morphism on nets

Our definition of morphism on nets involves binary relations, sometimes specialised to being partial or total functions. Here are the elementary notations, properties and operations on relations we shall use:

2.1 Notation. A relation from a set X to a set Y is a subset $R \subseteq X \times Y$. When $(x,y) \in R$ we write xRy. A relation R has an opposite or (converse) relation, R^{op}, given by $R^{op} = \{(y,x) \mid xRy\}$. Clearly $xRy \Leftrightarrow yR^{op}x$.

When the relation R satisfies the property $\forall y,y' \in Y\ \forall x \in X. xRy \,\&\, xRy' \Rightarrow y = y'$ the relation R is said to be a partial function. A partial function R is said to be total when it satisfies the additional property $\forall x \in X \exists y \in Y. xRy$.

The composition of relations is defined as follows: Let R be a relation from a set X to a set Y and S a relation from the set Y to a set Z. The composition of R with S is the relation $S \circ R$ from X to Z given by $S \circ R = \{(x,z) \in X \times Z \mid \exists y \in Y. xRy \,\&\, ySz\}$. Note the order of the composition which follows that generally used for functions. We shall frequently miss-out the composition symbol $\circ$ and write $S \circ R$ as just SR.

When a relation R is a partial function, and we are thinking of it as taking an argument x and giving a value R(x), it is useful to have a symbol to invoke when the value R(x) does not exist. We use * to represent undefined and so write $R(x) = * \Leftrightarrow \nexists y. xRy$ when R is a partial function from X to Y.

If R is a relation from X to Y and $A \subseteq X$ we define the image of A under R to be the set RA given by $RA = \{y \in Y \mid \exists x \in A. xRy\}$. Note the clash with abbreviated relation composition; any ambiguities can be resolved from the context.

A morphism from a net $N_0 = (B_0, E_0, F_0, M_0)$ to a net $N_1 = (B_1, E_1, F_1, M_1)$ specifies how the dynamic behaviour of N_0 induces the dynamic behaviour of N_1. It consists of two parts.

One is a partial function $\eta \subseteq E_0 \times E_1$ on events where $e_0\ \eta\ e_1$ means the occurrence of e_0 implies the simultaneous occurence of e_1. Think of the event e_1 as being a component of the event e_0. We assume an event e_1 only occurs in N_1 if some e_0 occurs in N_0 with $e_0 \eta e_1$.

The other part of the morphism is a relation $\beta \subseteq B_0 \times B_1$ between conditions. A relation $b_0\ \beta\ b_1$ means the holding of b_0 implies the coincident holding of b_1 i.e. when b_0 begins or ends holding then so does b_1 --they have the same extent. We assume a condition b_1 holds in N_1 only if there is a unique condition b_0 which holds in N_0 with $b_0\ \beta\ b_1$.

This understanding implies several properties of a morphism $(\eta,\beta): N_0 \rightarrow N_1$ which we take as our formal definition below. Firstly every condition which holds initially in N_1 should be the image under β of a unique condition which holds initially in N_0 property (i). Secondly if $b_0\ \beta\ b_1$ the occurrence of an event beginning the holding of b_0 should imply the simultaneous occurrence of an event beginning the holding of b_1, and similarly the occurrence of an event ending b_0 should imply the simultaneous occurrence of an event ending b_1, property (ii). Thirdly if $e_0\ \eta\ e_1$ the occurrence of e_0 should imply the conditions ${}^\bullet e_1$ end holding and the conditions $e_1^\bullet$ begin holding which gives property (iii).

2.2 <u>Definition</u>. Let $N_i = (B_i, E_i, F_i, M_i)$ be nets for i=0,1. Define a <u>morphism</u> of nets from N_0 to N_1 to be a pair of relations (η,β) with $\eta \subseteq E_0 \times E_1$, a <u>partial function</u>, and $\beta \subseteq B_0 \times B_1$ such that:

(i) $M_1 = \beta M_0$ and $\forall b_1 \in M_1 \exists! b_0 \in M_0.\ b_0 \beta b_1$,

(ii) If $b_0 \beta b_1$ then $\eta \cap ({}^\bullet b_0 \times {}^\bullet b_1)$ is a total function ${}^\bullet b_0 \rightarrow {}^\bullet b_1$
and $\eta \cap (b_0^\bullet \times b_1^\bullet)$ is a total function $b_0^\bullet \rightarrow b_1^\bullet$,

(iii) If $e_0 \eta e_1$ then $\beta^{op} \cap ({}^\bullet e_1 \times {}^\bullet e_0)$ is a total function ${}^\bullet e_1 \rightarrow {}^\bullet e_0$
and $\beta^{op} \cap (e_1^\bullet \times e_0^\bullet)$ is a total function $e_1^\bullet \rightarrow e_0^\bullet$.

If further η is total we say the morphism (η,β) is <u>synchronous</u>. When η and β are total functions we say the morphism (η,β) is a <u>folding</u>. When η and β are the inclusion relations $\eta: E_0 \subseteq E_1$ and $\beta: B_0 \subseteq B_1$ we say N_0 is a <u>subnet</u> of N_1.

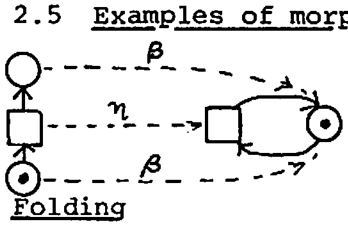
Subnets provide the simplest example of morphisms on nets. They have a simple characterisation and arise naturally by restricting a net to a subset of events.

2.3 <u>Proposition</u>. Let $N_0 = (B_0, E_0, F_0, M_0)$ and $N_1 = (B_1, E_1, F_1, M_1)$ be nets. Then N_0 is a subnet of N_1 iff $B_0 \subseteq B_1, E_0 \subseteq E_1, M_0 = M_1$ and $\forall e_0 \in E_0 \forall b \in B_1 .\ e_0 F_1 b \Leftrightarrow e_0 F_0 b$, & $\forall e_0 \in E_0 \forall b \in B_1\ b F_1 e_0 \Leftrightarrow b F_0 e_0$.

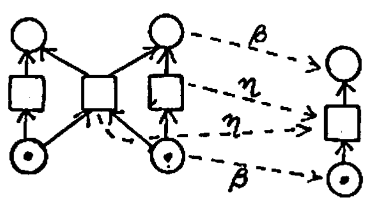
2.4 <u>Proposition</u>. Let $N = (B,E,F,M)$ be a net. Let $E' \subseteq E$. Define the <u>restriction</u> of N to E', written $N \restriction E'$, to be (B,E',F',M) where $F' = F \cap ((B \times E') \cup (E' \times B))$. The restriction $N \restriction E'$ is a subnet of N.

Of course morphisms can be more complicated as the following examples show.

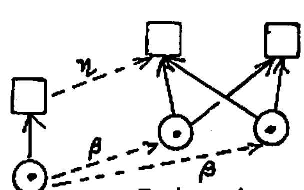
2.5 <u>Examples of morphisms</u>:



<u>Folding</u>



<u>Projection</u> (3.3)



<u>Injection</u> (3.9)

When $(\eta,\beta): N_0 \rightarrow N_1$ is a morphism β preserves the pre and postconditions of a set of events. Far more, when N_1 is contact-free the morphism respects the dynamic behaviour of nets; a play of the token game in N_0 induces a play of the token game in N_1. This further justifies our definition of morphism.

2.6 Lemma. Let $(\eta,\beta):N_0\to N_1$ be a morphism between nets N_0 and N_1. Let A be a subset of the events N_0. Then $\beta({}^\bullet A) = {}^\bullet(\eta A)$ and $\beta(A^\bullet) = (\eta A)^\bullet$.

2.7 Theorem. Let $N = (B_i,E_i,F_i,M_i)$ be nets for $i = 0,1$. Let N_1 be contact-free. Let $(\eta,\beta): N_0\to N_1$ be a morphism of nets. Let C be a reachable marking of N_0 and suppose $C\xrightarrow{A}C'$ in N_0. Then βC is a reachable marking of N_1 and $\beta C\xrightarrow{\eta A}\beta C'$ in N_1.
Further, for all reachable markings C of N_0, $\forall b_1 \in \beta C\exists! b_0 \in C. b_0\, \beta b_1$.

From now on we shall insist all nets are contact-free. Contact-free nets with morphisms of nets form a category.

2.8 Definition. Define Net to be the category of contact-free nets with morphisms as above and composition given by $(\eta_0,\beta_0) \circ (\eta_1,\beta_1) = (\eta_0\circ\eta_1,\beta_0\circ\beta_1)$. Define Net_{syn} to be the subcategory with synchronous morphisms on nets.

3. Categorical constructions

The categorical constructions in Net and Net_{syn} which we introduce will depend on the properties of two more basic categories. One is well-known; it is the category of sets with partial functions. It corresponds to that part of morphisms on nets which act between sets of events. The other is the category of marked sets and corresponds to that part of morphisms on nets which act between sets of conditions while respecting the initial marking.

3.1 Lemma. Let Set_* be the category of sets and partial functions given in definition 2.1. Set_* has products and coproducts of the following form where E_0 and E_1 are sets:

Their product, to within isomorphism, is $E_0 \times_* E_1$ with projections π_0, π_1 where

$$E_0 \times_* E_1 = \{(e_0,*)\mid e_0 \in E_0\} \cup \{(*,e_1)\mid e_1 \in E_1\} \cup \{(e_0,e_1)\mid e_0 \in E_0 \,\&\, e_1 \in E_1\},$$

and $\pi_0(x,y) = x, \pi_1(x,y) = y$.

Their coproduct, to within isomorphism, is $E_0 + E_1 =_{def} \{0\} \times E_0 \cup \{1\} \times E_1$ with injections $in_0(e_0) = (0,e_0)$ and $in_1(e_1)$ $(=(1,e_1))$ for $e_0 \in E_0$ and $e_1 \in E_1$.

3.2 Lemma. Define a marked set to be a pair of sets (B,M) where $M \subseteq B$. Define a morphism of marked sets from (B_0,M_0) to (B_1,M_1) to be a relation $R \subseteq B_0 \times B_1$ such that $RM_0 = M_1$ and

$$\forall b_0,b'_0 \in M_0 \forall b_1 \in M_1. b_0 R b_1 \;\&\; b'_0 R b_1 \Rightarrow b_0 = b'_0.$$

Define composition to be the usual composition of relations given in 2.1. Then marked sets with the morphisms above form a category with identity morphisms the identity relations. It has products and coproducts of the following form, where (B_0,M_0) and (B_1,M_1) are marked sets:

Their product, to within isomorphism, is $(B_0 + B_1, M_0 + M_1)$ with projections the relations ρ_0 and ρ_1 given by $(b,0)\rho_0 b$ for $b \in B_0$ and $(b,1)\rho_1 b$ for $b \in B_1$.

Their coproduct, to within isomorphism, is (B,M) with injections ι_0 and ι_1 where

$$B = \{(b_0,*)\mid b_0 \in B_0\setminus M_0\} \cup \{(*,b_1)\mid b_1 \in B_1\setminus M_1\} \cup \{(b_0,b_1)\mid b_0\in B_0 \;\&\; b_1\in B_1\},$$

$$M = M_0 \times M_1,$$

$$b_0\iota_0 b \Leftrightarrow \exists b_1 \in B_1\cup\{*\}.\; b = (b_0, b_1),$$
$$b_1\iota_1 b \Leftrightarrow \exists b_0 \in B_0\cup\{*\}.\; b = (b_0, b_1).$$

We shall use the above facts and notation in defining the constructions as nets.

3.3 Definition. (The products of nets) Let $N_0 = (B_0,E_0,F_0,M_0)$ and $N_1 = (B_1,E_1,F_1,M_1)$ be contact free nets. Let $\pi_0 : E_0 \times_* E_1\to E_0$ and $\pi_1 : E_0 \times_* E_1 \to E_1$ be the projections from the product of sets in Set_* given in 3.1. Let ρ_0: $(B_0 + B_1,M_0 + M_1) \to (B_0,M_0)$ and ρ_1: $(B_0 + B_1, M_0 + M_1) \to (B_1, M_1)$ be the projections from the product of marked sets given in 3.2.

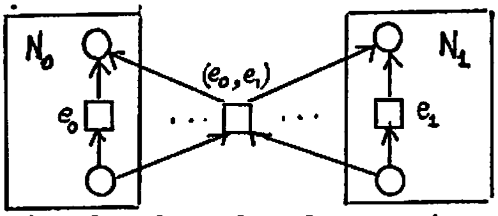
Define the product of the nets, $N_0 \times N_1$, to be the net (B,E,F,M) where $B = B_0 + B_1$, $M = M_0 + M_1$, $E = E_0 \times_* E_1$ and

$$eFb \Leftrightarrow (\exists e_0 \in E,\ b_0 \in B_0 .\ e\pi_0 e_0 \ \&\ b\rho_0 b_0 \ \&\ e_0 F_0 b_0)$$
$$\text{or}(\exists e_1 \in E,\ b_1 \in B_1 .\ e\pi_1 e_1 \ \&\ b\rho_1 b_1 \ \&\ e_1 F_1 b_1),$$

$$bFe \Leftrightarrow (\exists e_0 \in E,\ b_0 \in B_0 .\ e\pi_0 e_0 \ \&\ b\rho_0 b_0 \ \&\ b_0 F_0 e_0)$$
$$\text{or}(\exists e_1 \in E,\ b_1 \in B_1 .\ e\pi_1 e_1 \ \&\ b\rho_1 b_1 \ \&\ b_1 F_1 e_1).$$

Define <u>projection</u> morphisms of nets: $\Pi_0 = (\pi_0, \rho_0) : N_0 \times N_1 \rightarrow N_0$
$\Pi_1 = (\pi_1, \rho_1) : N_0 \times N_1 \rightarrow N_1$.

The product construction can be summarised in a simple picture. Disjoint copies of the two nets N_0 and N_1 are juxtaposed and extra events of synchronisation of the form (e_0, e_1) are adjoined, for e_0 an event of N_0 and e_1 an event of N_1; an extra event (e_0, e_1) has as preconditions those of its components ${}^{\bullet}e_0 \cup {}^{\bullet}e_1$ and similarly postconditions $e_0^{\bullet} \cup e_1^{\bullet}$.



The product on nets is closely related to various forms of parallel composition which have been defined on nets to model synchronised communication--see [LSC]. Imagine that the events of nets are labelled in order to specify how they can or cannot synchronise with events in the environment--the synchronisation algebras of [W1,W2] are a way of formalising this idea. Then the parallel composition of two labelled nets will be modelled as a restriction of the product of those synchronised events--of the from (e_0, e_1)--and those unsynchronised events--of the form $(e_0, *)$ and $(*, e_1)$--allowed by the discipline of synchronisation.

3.4 <u>Theorem</u>. The above construction $N_0 \times N_1$, Π_0, Π_1 is a product in <u>Net</u>, the category of nets.

Of course the token game tells us how we can view a net as giving rise to a transition system in which the arrows between states are associated with sets of events imagined to occur concurrently. Let us see how the product construction looks from this point of view.

3.5 <u>Theorem</u>. Let $N_0 \times N_1$, $\Pi_0 = (\pi_0, \rho_0)$ and $\Pi_1 = (\pi_1, \rho_1)$ be a product of nets. Then M is a reachable marking of $N_0 \times N_1$ and $M \xrightarrow{A} M'$ iff

$\rho_0 M$ is a reachable marking of N_0 and $\rho_0 M \xrightarrow{\pi_0 A} \rho_0 M'$ and
$\forall e, e' \in A\ \forall e_0 \in E_0 .\ e\pi_0 e_0 \ \&\ e'\pi_0 e_0 \Rightarrow e = e'$ and

$\rho_1 M$ is a reachable marking of N_1 and $\rho_1 M \xrightarrow{\pi_1 A} \rho_1 M'$ and
$\forall e, e' \in A\ \forall e_1 \in E_1 .\ e\pi_1 e_1 \ \&\ e'\pi_1 e_1 \Rightarrow e = e'$.

A similar story can be told for $\underline{\underline{\text{Net}}}_{\text{syn}}$.

3.6 <u>Definition</u>. (Synchronous product) Let $N_0 = (B_0, E_0, F_0, M_0)$ and $N_1 = (B_1, E_1, F_1, M_1)$ be contact-free nets. Define their <u>synchronous product</u> $N_0 \otimes N_1$ to be the restriction $N_0 \times N_1 \restriction (E_0 \times E_1)$ with synchronous projections $\Pi_0' = (\pi_0', \rho_0)$ and $\Pi_1' = (\pi_1', \rho_1)$ where $\pi_0'(e_0, e_1) = e_0$ and $\pi_1'(e_0, e_1) = e_1$.

3.7 <u>Theorem</u>. The above construction $N_0 \otimes N_1$, Π_0', Π_1' is a product of $\underline{\underline{\text{Net}}}_{\text{syn}}$, the category of nets with synchronous morphisms.

3.8 <u>Example</u>. One can repeat a ticking clock as the simple net $\Omega =$ .
Given an arbitrary contact-free net N it is a simple matter to serialise, or interleave, its event occurrences; just synchronise them one at a time with the ticks of the clock. This amounts to forming the synchronous product $N \otimes \Omega$ of N with Ω.

Now we give the form of coproducts in $\underline{\underline{\text{Net}}}$ and $\underline{\underline{\text{Net}}}_{\text{syn}}$.

3.9 Definition. (The coproduct of nets)

Let $N_0 = (B_0, E_0, F_0, M_0)$ and $N_1 = (B_1, E_1, F_1, M_1)$ be contact-free nets.

Let $in_0 : E_0 \to E_0 + E_1$ and $in_1 : E_1 \to E_0 + E_1$ be the injections into the coproduct of sets in $\underline{Set}_*$ given in 3.2. Let $\iota_0 : (B_0, M_0) \to (B,M)$ and $\iota_1 : (B_1, M_1) \to (B,M)$ be the injections into the coproduct of marked sets given in 3.3.

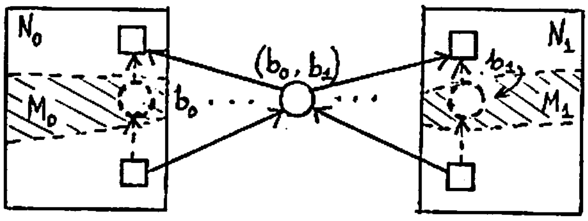
Define the coproduct of the nets, $N_0 + N_1$, to be the net (B,E,F,M) where

$$
\begin{aligned}
&(B,M) \text{ is the coproduct of marked sets} \\
&E = E_0 + E_1, \\
&eFb \Leftrightarrow (\exists e_0 \in E_0, b_0 \in B_0 . e_0 in_0 e \;\&\; b_0 \iota_0 b \;\&\; e_0 F_0 b_0) \\
&\quad \text{or } (\exists e_1 \in E_1, b_1 \in B_1 . e_1 in_1 e \;\&\; b_1 \iota_1 b \;\&\; e_1 F_1 b_1), \\
&bFe \Leftrightarrow (\exists e_0 \in E_0, b_0 \in B_0 . e_0 in_0 e_0 \;\&\; b_0 \iota_0 b \;\&\; b_0 F_0 e_0) \\
&\quad \text{or } (\exists e_1 \in E_1, b_1 \in B_1 . e_1 in_1 e \;\&\; b_1 \iota_1 b \;\&\; b_1 F_1 e_1).
\end{aligned}
$$

Define injection morphisms of nets:

$$
\begin{aligned}
I_0 &= (in_0, \iota_0) : N_0 \to N_0 + N_1 \\
I_1 &= (in_1, \iota_1) : N_1 \to N_0 + N_1 .
\end{aligned}
$$

The coproduct construction can be summarised in a simple picture. The two nets N_0 and N_1 are laid side by side and then a little surgery is performed on their initial markings. For each pair of conditions b_0 in the initial marking of N_0 and b_1 in the initial marking of N_1 a new condition (b_0, b_1) is created and made to have the same pre and post events as b_0 and b_1 together--think of it as exclusive or of b_0 and b_1. The conditions in the original initial markings are removed and replaced by a new initial marking consisting of these newly created conditions. Here is the picture:



3.10 Theorem. The above construction $N_0 + N_1$, I_0, I_1 is a coproduct in the categories $\underline{Net}$ and $\underline{Net}_{syn}$.

Again the construction translates over to a natural construction on transition systems.

3.11 Theorem. Let $N_0 + N_1, I_0 = (in_0, \iota_0)$ and $I_1 = (in_1, \iota_1)$ be the coproduct of nets. Then M is a reachable marking of $N_0 + N_1$ and $M \to M'$ iff

$$
\begin{aligned}
&\exists M_0, A_0, M_0' .\; M_0 \xrightarrow{A_0} M_0' \;\&\; A = in_0 A_0 \;\&\; M = \iota_0 M_0 \;\&\; M' = \iota_0 M_0' \quad \text{or} \\
&\exists M_1, A_1, M_1' .\; M_1 \xrightarrow{A_1} M_1' \;\&\; A = in_1 A_1 \;\&\; M = \iota_1 M_1 \;\&\; M' = \iota_1 M_1' .
\end{aligned}
$$

4. The semantics of Petri nets

Here we show how an occurrence net, in which conditions and events stand for occurrences, can be associated with a contact-free net. The occurrence net we associate with a contact-free net will be built up essentially by unfolding the net to its occurrences. This unfolding is a canonical representative of the behaviour of the original net. Occurrence nets and the operation of unfolding a net to an occurrence net were first introduced in [NPW1,2 and W] and the reader should look there for more motivation. (Note causal nets were rechristened "occurrence nets" in [Br]--such nets are not as general as the ones here.)

In general because of the presence of backwards conflict that part of a net causing an event or condition need not be unique. We wish events and conditions of an occurrence net to correspond to occurrences (as in the case for Petri's causal nets). From this point of view backwards conflict is undesirable as it allows a holding of a condition to occur in more than one way, so we impose (i). Following this view we ban loops in the F^+ relation and ensure any occurrence depends on only a finite number of event occurrences —axiom (iii) — and insist no event is in conflict with itself —axiom (iv). For occurrence nets there is an especially simple definition of a concurrency relation and conflict relation which was previously only defined with respect to a marking. We take the initial marking to consist of those conditions b such that ${}^{\bullet}b = \emptyset$.

4.1 Definition. An occurrence net is a net (B,E,F,M) for which the following restrictions are satisfied:

(i) $\forall b \in B. |{}^{\bullet}b| \leqslant 1$,

(ii) $b \in M \Leftrightarrow {}^{\bullet}b = \emptyset$,

(iii) F^+ is irreflexive and $\forall e \in E.\ \{e' \in E \mid e'F^*e\}$ is finite.

(iv) # is irreflexive where $e \#_1 e' \Leftrightarrow_{def} e \in E \ \&\ e' \in E \ \&\ {}^{\bullet}e \cap {}^{\bullet}e' \neq \emptyset$ and

$$x \# x' \Leftrightarrow_{def} \exists e, e' \in E. e \#_1 e' \ \&\ eF^*x \ \&\ e'F^*x'.$$

Suppose N = (B,E,F,M) is an occurrence net. We call the relation $\#_1$ defined above the immediate conflict relation and # the conflict relation. We define the concurrency relation, co, between pairs $x, y \in B \cup E$ by: $x \text{ co } y \Leftrightarrow_{def} \neg(xF^+y \text{ or } yF^+x \text{ or } x\#y)$.

4.2 Proposition. Let N = (B,E,F,M) be an occurrence net. Then N is contact-free and every event has concession at some reachable marking and every condition holds at some reachable marking.

Let e,e' be two events of N. Let b,b' be two conditions of N.

The relations $\#_1 \subseteq E^2$ and $\# \subseteq (B \cup E)^2$ are binary, symmetric, irreflexive relations. The relation of immediate conflict $e \#_1 e'$ holds iff there is a reachable marking of N at which the events e and e' are in conflict.

The relation co is a binary, symmetric, reflexive relation between conditions and events of N. We have b co b' iff there is a reachable marking of N at which b and b' both hold. We have e co e' iff there is a reachable marking at which e and e' can occur concurrently.

4.3 Definition. Write Occ for the category of occurrence nets with net morphisms.

We can define the unfolding a contact-free net inductively to obtain an occurrence net satisfying the following theorem.

4.4 Theorem. Let N = (B,E,F,M) be a contact-free net. There is a unique occurrence net $\mathcal{N} = (\mathcal{B}, \mathcal{E}, \mathcal{F}, \mathcal{M})$ and folding $f = (\eta, \beta)$ which satisfy:

$\mathcal{B} = \{(0,b) \mid b \in M\} \cup \{(\{e'\},b) \mid e' \in \mathcal{E} \ \&\ b \in B \ \&\ \eta(e')F b\}$,

$\mathcal{E} = \{(S,e) \mid S \subseteq \mathcal{B} \ \&\ e \in E \ \&\ \beta S = {}^{\bullet}e \ \&\ \forall b', b'' \in S.\ b' \text{ co } b''\}$,

$x \mathcal{F} y \Leftrightarrow \exists w,z.\ y = (w,z) \ \&\ x \in W$,

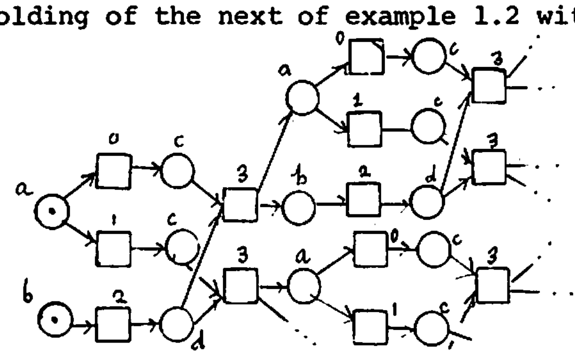
$\mathcal{M} = \{(\emptyset,b) \mid b \in M\}$

and $e' \eta e \Leftrightarrow \exists S \subseteq \mathcal{B}.\ e' = (S,e)$.

$b' \beta b \Leftrightarrow (b \in M \ \&\ b' = (\emptyset,b))$ or $\exists e' \in \mathcal{E}.\ b' = (\{e'\},b)$.

4.5 Definition. Write $\mathcal{U}$N for the occurrence net defined above. Call it the unfolding of N.

4.6 Example. The unfolding of the next of example 1.2 with initial marking {a,b} looks like this:



Although the unfolding construction is quite natural it is, by itself, quite unwieldy. Imagine proving for example that unfolding preserves products. Fortunately the unfolding construction has an abstract characterisation which implies such facts immediately. Unfolding is cofree. It is a right adjoint to the inclusion functor

$\underline{\underline{Occ}} \to \underline{\underline{Net}}$, and right adjoints preserve limits and in particular products (see [Mac]). The unfolding of an occurrence net is naturally isomorphic to the original net which makes this adjunction a coreflection.

4.7 Theorem. Let N be a contact-free Petri net. Then the occurrence net unfolding $\mathcal{U}N$ and folding f are cofree over N i.e. for any morphism $g : N_1 \to N$ with N_1 an occurrence net there is a unique morphism $h : N_1 \to \mathcal{U}N$ such that $f \circ h = g$. In fact $\underline{\underline{Occ}}$ is a coreflective subcategory of $\underline{\underline{Net}}$.

Thus, from the coreflection we know the product of two occurrence nets N_0, N_1 in $\underline{\underline{Occ}}$ is $N_0 \times_{Occ} N_1 \cong \mathcal{U}N \times_{Occ} \mathcal{U}N \cong \mathcal{U}(N_0 \times_{Net} N_1)$, the unfolding of their product in $\underline{\underline{Net}}$. Although we cannot make the full case here, the coreflection relates parallel compositions using contact-free nets to parallel compositions using occurrence nets and vice versa. The idea is to label events by elements of a synchronisation algebra, specifying how events synchronise, and to obtain parallel compositions by restricting events of the product in accord with the algebra--see [W1,2]. The coproduct of occurrence nets in $\underline{\underline{Occ}}$ is their coproduct in $\underline{\underline{Net}}$--this follows from the coreflection. Coproducts are not always preserved by $\mathcal{U}$ however (right adjoints only preserve limits not necessarily colimits). Still they are preserved on a full subcategory of $\underline{\underline{Net}}$ with objects those nets whose initial markings consist solely of conditions with no pre-events.

5. A coreflection between nets and event structures

We show there is a coreflection between the category of occurrence nets and a category of event structures. The functors provide a translation between the Petri net model of computation and that based on event structures. The work [W1,2] provides another coreflection which bridges the gap between event structures and synchronisation trees, at the basis of the interleaving models of CCS and CSP (see e.g. [M] and [HBR]). Coreflections compose so relating Petri nets to other established work in concurrency.

The event structures are of the simple form introduced in [NPW1,2]. They are essentially occurrence nets with the conditions stripped away to leave the causal dependency and conflict relations. (They are called prime event structures in [W1].)

5.1 Definition. A (prime) event structure is a triple $(E, \leqslant, \#)$ consisting of

(i) E a set of events,

(ii) $\leqslant$ the causal dependency relation, a partial order on E and

(iii) # the conflict relation a binary symmetric relation on E

which satisfy $e \# e' \leqslant e'' \Rightarrow e \# e''$ and $\{e' \in E \mid e' \leqslant e\}$ is finite.

Event structures carry a natural idea of configuration (or state), the left-closed, conflict-free subsets of events. Intuitively a configuration is a set of events that occur in some history of a process; it should only be possible for an event to occur once the events on which it causally depends have occurred and it should be impossible for two events in conflict to occur in the same history.

5.2 Definition. Let $(E, \leqslant, \#)$ be an event structure. Let $x \subseteq E$. Say x is left-closed iff $\forall e, e' \in E.\, e \leqslant e' \in x \Rightarrow e \in x$. Say x is conflict-free iff $\forall e, e' \in x.\, \neg(e \# e')$. Write $\mathcal{L}(E, \leqslant, \#)$ for the set of left-closed conflict-free subsets.

Clearly an occurrence net determines an event structure.

5.3 Definition. Le $N = (B,E,F,M)$ be an occurrence net. Define $\mathcal{E}(N) = (E, F^* \upharpoonright E, \# \upharpoonright E)$ where # is defined in 4.1.

Event structures possess a definition of morphism which is respected by $\mathcal{E}$ making $\mathcal{E}$ into a functor.

5.4 Definition. Define $\underline{P}$, the category of event structures, to have prime event structures as objects and morphisms $\theta : (E_0, \leqslant_0, \#_0) \to (E_1, \leqslant_1, \#_1)$ those partial functions $\theta : E_0 \to_* E_1$ which satisfy

$$\forall x \in \mathcal{L}(E_0).\ (\theta x \in \mathcal{L}(E_1)\ \ \&\ (\forall e,e'\in x.\ \theta(e) = \theta(e) \neq * \Rightarrow e = e'))$$

The identity morphisms are the identity functions and composition is the usual composition of partial functions.

5.5 Theorem. Let $N_i = (B_i, E_i, F_i, M_i)$ be occurrence nets for $i = 0,1$. Let $f = (\eta,\beta) : N_0 \to N_1$ be a morphism of nets. Then $\mathcal{E}f = \eta : \mathcal{E}N_0 \to \mathcal{E}N_1$ is a morphism of event structures, making $\mathcal{E}$ a functor $\underline{Occ} \to \underline{P}$.

Conversely an event structure can be identified with a canonical occurrence net. The basic idea is to produce an occurrence net with as many conditions as are consistent with the causal dependency and conflict relations of the event structure. But we do not want more than one condition with the same beginning and ending events --we want an occurrence net which is "condition-extensional" in the terms of [Br]. Thus we can identify the conditions with pairs of the form (e,A) where e is an event and A is a subset of events causally dependant on e and with every distinct pair of events in A in conflict. But not quite, we also want initial conditions $(\emptyset,A)$ with no beginning events (see [NPW], though note a small but important change; we introduce the isolated condition $(\emptyset,\emptyset)$.)

5.6 Definition. Let $(E,\leq,\#)$ be an event structure. Define $\mathcal{N}(E,\leq,\#)$ to be (B,E,F,M) where

$$M = \{(\emptyset,A) \mid A \subseteq E\ \&\ \forall a,a'\in A.\ a(\# \cup 1)a'\},$$
$$B = M \cup \{(e,A) \mid e\in E\ \&\ A \subseteq E\ \&(\forall a\in A.\ e<a)\ \&\ \forall a,a'\in A.\ a(\#\cup 1)a'\},$$
$$F = \{((c,A),e) \mid (c,A)\in B\ \&\ e\in A\} \cup \{(e,(e,A)) \mid (e,A) \in B\}.$$

As promised there is a coreflection between event structures and occurrence nets; the construction $\mathcal{N}$ provides the free occurrence net over an event structure.

5.7 Theorem. Let E be an event structure. Then $\mathcal{N}E$ is an occurrence net. Moreover, $\mathcal{E}\mathcal{N}E = E$. In fact $\mathcal{N}E$, $1_E : E \to \mathcal{E}\mathcal{N}E$ is free over E with respect to $\mathcal{E}$ i.e. for any morphism $g : E \to \mathcal{E}N$ in $\underline{P}$ with N an occurrence net there is a unique morphism $h : \mathcal{N}E \to N$ in $\underline{\underline{Occ}}$ such that $\mathcal{E}h = \mathcal{E}h \circ 1_E = g$.

5.8 Example. Left adjoints preserve colimits, and so coproducts. Thus by 5.7 we can deduce $\mathcal{N}(E_0 + E_1) \cong \mathcal{N}E_0 + \mathcal{N}E_1$ so $E_0 + E_1 = \mathcal{E}\mathcal{N}(E_0 + E_1) \cong \mathcal{E}(\mathcal{N}E_0 + \mathcal{N}E_1)$, which expresses the coproduct of event structures in terms of the coproduct of nets. Right adjoints preserve limits, and so products. By 5.7 and 4.7 we deduce : $E_0 \times_P E_1 = \mathcal{E}\mathcal{N}E_0 \times_P \mathcal{E}\mathcal{N}E_1 \cong \mathcal{E}(\mathcal{N}E_0 \times_{Occ} \mathcal{N}E_1) \cong \mathcal{E}\mathcal{U}(\mathcal{N}E_0 \times_{Net} \mathcal{N}E_1)$ which expresses the product of event structures in terms of the product of nets.

In fact whole denotational semantics for a wide range of languages ($Proc_L$ of [W1,2]) can be translated back and forth between different models using these techniques. The demonstration of this and the use of "net-embeddings" to define nets recursively must await the complete version of this paper.

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