On concurrent games with payoff

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Abstract. The paper considers an extension of concurrent games with a **payoff**, *i.e.* a numerical value resulting from the interaction of two players. We extend a recent determinacy result on concurrent games [5] to a *value theorem*, *i.e.* a value that both players can get arbitrarily close to, whatever the behaviour of their opponent. This value is not reached in general, *i.e.* there is not always an optimal strategy for one of the players (there is for finite games). However when they exist, we show that optimal strategies are closed under composition, which opens up the possibility of computing optimal strategies for complex games compositionally from optimal strategies for their component games.

1 Introduction

Games are a well-established tool in mathematics, economics, logic, and of course computer science: in the latter, two-player games in particular are very widely used to model situations where an agent (*e.g.* a program) interacts with its environment (*e.g.* the user, the operating system). For instance, researchers in *game semantics* [9] have managed to build very precise (*fully abstract* [1, 8]) models of higher-order programming languages with various computational effects. Another particularly rich line of work has been the application of game-theoretic tools for algorithmic and verification purposes: one expresses a desirable property of a system as a game, and reduces the satisfaction of this property to the existence of a "good" strategy for this game. Here, the meaning of "good" can be either qualitative (positions are winning or losing, with each player wanting to reach a winning position) or quantitative (positions have a given *payoff*, with both players trying to maximize their payoff). For these purposes, one generally wants the games considered to be *determined*: qualitatively, this means that one of the players necessarily has a winning strategy, and quantitatively that the game has a well-defined *value* that well-chosen strategies can reach or get arbitrarily close to. For this reason, the classes of games considered for these purposes generally enjoy a *determinacy*: the most well-known such result is Martin's famous theorem [12] stating that for sequential, tree-like games whose winning positions form a *Borel set*, one of the players must have a winning strategy. It is well-known that Martin's theorem generalizes to the quantitative setting if the game is *zero-sum*, *i.e.* in each position the payoffs of the two players sum to zero. In the last decade, there has been a growing interest in extensions of these games with *concurrency*. One very successful definition of (turn-based) concurrent games has been proposed by Henzinger, de Alfaro *et. al.* [3, 4]: their games are based on Blackwell games [13], where at any point, the next state is decided by a function of parallel choices of both players. In these games, the *pure strategy* determinacy result of sequential games is weakened into a *mixed strategy* determinacy, where strategies are allowed to make probabilistic choices.

However in semantics, models of concurrent processes generally allow a more liberal, non turn-based form of concurrency. Starting with the work of Petri, many have come to advocate a view of concurrency based on *partial orders*, specifying the causal dependency between events – see [16] for an early summary of Petri's work and its relation with domain theory. Following this approach, several notions of concurrent games have been proposed as a basis for denotational semantics: in terms of closure operators [2] or asynchronous transition systems [15]. Recently, Winskel and Rideau introduced a more general setting for concurrent games [17]. It is based on the notion of *event structure* [18], a partial order of causal dependency on events with a consistency relation expressing non-deterministic choice. In the present paper, it is this framework that we will refer to by *concurrent games*. We showed in [5] that in this setting qualitative determinacy is satisfied for well-founded games meeting a structural condition called *race-freedom* expressing that moves of one player do not directly interfere with moves of the other. Here, we consider an extension of concurrent games with *zero-sum payoff*, and show a generalization of the qualitative determinacy result of [5] to a quantitative one. As the reader will see this is not a trivial exercise and requires a much finer analysis than for the qualitative case.

Note that we obtain *pure strategy* determinacy – our strategies do not make probabilistic choices, although they can act non-deterministically. There is an apparent contradiction with the line of work based on Blackwell games mentioned above, since they only have mixed strategy determinacy. This is due to a crucial difference between the two settings: in our games, no *fairness* assumption is made and strategies can legitimately choose *not to play*, possibly resulting in a deadlock if both strategies choose to do so. We argue that this is a desirable property, since very often in computer science we have to deal with systems that might not terminate. However from the game theory perspective, this implies that Blackwell games are *not* instances of our zero-sum concurrent games. (They *do* fit into our general framework, since fairness can be expressed by non zero-sum payoff by setting both players to be losing at incomplete positions.)

We also investigate quantitative features with respect to the *compositional* structure of concurrent games. In sequential games, strategies can be composed using a form of parallel composition and a hiding operation to make internal play invisible. This fact (first remarked on by Conway and emphasised by Joyal [10] in his analysis of Conway's work [6]) is seldom used in economics and algorithmics. However, it is at the very heart of game semantics, the compositional analysis of programs and programming languages in terms of games and strategies. Our concurrent games are compositional; in fact, the main result of [17] was to define and characterise strategies for which composition behaves well (*i.e.* is associative, and has identities). Not only is compositionality a prerequisite for building denotational models of programming languages (as they organize naturally as *categories*, see *e.g.* [11]), but it is also a very successful general approach for proving properties of complex programs. Adapting the earlier work on concurrent strategies, we show here that optimal strategies are stable under composition, thus building a bicategory of optimal strategies. This is a significant step towards a compositional analysis of optimal strategies: instead of modeling complex behaviours as payoff functions and then computing values and optimal strategies, construct complex optimal strategies by composition from elementary ones. Extensions with payoff should also prove useful for purely semantic purposes: pay-off is a powerful notion that allows us to express familiar *winning strategies* – as strategies of positive value – as well as more arcane game-theoretic notions, such as *well-bracketing* [14].

Outline. In Section 2, we recall the framework of concurrent games originally presented in [17]. In Section 3, we show how to enrich these concurrent games with payoff and introduce the notion of value of games and strategies. In Section 4, we prove the main result of our paper, the value theorem. Finally in Section 5, we investigate the compositional aspects of payoff games; in particular we show that optimal strategies are stable under composition and form a bicategory.

2 Preliminaries

2.1 Event structures

An *event structure* comprises $(E, \leq, \mathrm{Con})$, consisting of a set E, of *events* which are partially ordered by $\leq$, the *causal dependency relation*, and a nonempty *consistency relation* Con consisting of finite subsets of E, which satisfy

$$
\begin{aligned}
&\{e' \mid e' \leq e\} \text{ is finite for all } e \in E,\\
&\{e\} \in \mathrm{Con} \text{ for all } e \in E,\\
&Y \subseteq X \in \mathrm{Con} \Rightarrow Y \in \mathrm{Con}, \quad \text{and}\\
&X \in \mathrm{Con} \;\&\; e \leq e' \in X \Rightarrow X \cup \{e\} \in \mathrm{Con}
\end{aligned}
$$

The *configurations*, $\mathcal{C}^\infty(E)$, of an event structure E consist of those subsets $x \subseteq E$ which are

Consistent: $\forall X \subseteq x.\ X$ is finite $\Rightarrow X \in \mathrm{Con}$, and
Down-closed: $\forall e, e'.\ e' \leq e \in x \Rightarrow e' \in x$.

Often we shall be concerned with just the finite configurations of an event structure. We write $\mathcal{C}(E)$ for the *finite* configurations of an event structure E.

Two events which are both consistent and incomparable w.r.t. causal dependency in an event structure are regarded as *concurrent*. In games the relation of *immediate* dependency $e \rightarrowtriangle e'$, meaning e and e' are distinct with $e \leq e'$

and no event in between, will play an important role. For $X \subseteq E$ we write $[X]$ for $\{e \in E \mid \exists e' \in X.\ e \leq e'\}$, the down-closure of X; note if $X \in \mathrm{Con}$, then $[X] \in \mathrm{Con}$ is a configuration and in particular each event e is associated with a *prime* configuration $[e]$.

Notation 1 Let E be an event structure. We use $x \mathrel{-\!\!\!\subset} y$ to mean y covers x in $\mathcal{C}^\infty(E)$, *i.e.* $x \subset y$ in $\mathcal{C}^\infty(E)$ with nothing in between, and $x \stackrel{e}{\longrightarrow\!\!\!\!\subset} y$ to mean $x \cup \{e\} = y$ for $x, y \in \mathcal{C}^\infty(E)$ and event $e \notin x$. We use $x \stackrel{e}{\longrightarrow\!\!\!\!\subset}$, expressing that event e is enabled at configuration x, when $x \stackrel{e}{\longrightarrow\!\!\!\!\subset} y$ for some y.

Definition 1. *Let E and E' be event structures. A* (partial) map *of event structures $f : E \to E'$ is a partial function on events $f : E \rightharpoonup E'$ such that for all $x \in \mathcal{C}(E)$ its direct image $fx \in \mathcal{C}(E')$ and $\forall e_1, e_2 \in x,\ f(e_1) = f(e_2)$ (with both defined) $\Rightarrow e_1 = e_2$.*

Maps of event structures compose as partial functions, with identity maps given by identity functions. We will say the map is *total* if the function f is total.

Definition 2 (Process operations).

- ***Products.*** *The category of event structures with partial maps has* products *$A \times B$ with projections Π_1 to A and Π_2 to B. The effect is to introduce arbitrary synchronisations between events of A and events of B in the manner of process algebra.*
- ***Restriction.*** *The restriction of an event structure E to a subset of events R, written $E \restriction R$, is the event structure with events $E' = \{e \in E \mid [e] \subseteq R\}$ and causal dependency and consistency induced by E.*

Using these two operations, we can obtain a notion of **synchronized composition**. Synchronized compositions play a central role in process algebra, in such seminal work as Milner's CCS and Hoare's CSP. Synchronized compositions of event structures A and B are obtained as restrictions $A \times B \restriction R$. We obtain *pullbacks* as a special case. Let $f : A \to C$ and $g : B \to C$ be maps of event structures. Defining P to be $A \times B \restriction \{p \in A \times B \mid f\Pi_1(p) = g\Pi_2(p) \text{ with both defined}\}$, we obtain a pullback square

$$\begin{tikzcd} & P \arrow[dl, "\Pi_1"'] \arrow[dr, "\Pi_2"] \arrow[dd, phantom, very near start, "\vee"] & \\ A \arrow[dr, "f"'] & & B \arrow[dl, "g"] \\ & C & \end{tikzcd}$$

in the category of event structures. When f and g are total the same construction gives the pullback in the category of event structures with *total* maps.

Definition 3 (Projection). *Let $(E, \leq, \mathrm{Con})$ be an event structure. Let $V \subseteq E$ be a subset of 'visible' events. Define the* projection *of E on V, to be $E{\downarrow}V =_{\mathrm{def}} (V, \leq_V, \mathrm{Con}_V)$, where $v \leq_V v'$ iff $v \leq v'$ & $v, v' \in V$ and $X \in \mathrm{Con}_V$ iff $X \in \mathrm{Con}$ & $X \subseteq V$.*

2.2 Concurrent strategies

Event structures with polarity Both a game and a strategy in a game are to be represented using event structures with polarity, which comprise (E, pol) where E is an event structure with a polarity function $pol : E \to \{+, -\}$ ascribing a polarity $+$ (Player) or $-$ (Opponent) to its events. The events correspond to (occurrences of) moves. Maps of event structures with polarity are maps of event structures which preserve polarities.

Definition 4 (Basic operations).

- ***Dual.*** *The* dual*, $E^\perp$, of an event structure with polarity E comprises the same underlying event structure E but with a reversal of polarities.*
- ***Simple parallel composition.*** *Let A and B be event structures with polarity. The operation $A \| B$ simply juxtaposes disjoint copies of A and B, maintaining their causal dependency and specifying a finite subset of events as consistent if it restricts to consistent subsets of A and B. Polarities are unchanged.*

All the constructions previously introduced for event structures generalize directly in the presence of polarities.

Pre-strategies Let A be an event structure with polarity, thought of as a game; its events stand for the possible occurrences of moves of Player and Opponent and its causal dependency and consistency relations the constraints imposed by the game. A pre-strategy represents a nondeterministic play of the game—all its moves are moves allowed by the game and obey the constraints of the game; the concept will later be refined to that of *strategy.* A *pre-strategy in A* is defined to be a total map $\sigma : S \to A$ from an event structure with polarity S. Two pre-strategies $\sigma : S \to A$ and $\tau : T \to A$ in A will be essentially the same when they are isomorphic, *i.e.* there is an isomorphism $\theta : S \cong T$ such that $\sigma = \tau\theta$; then we write $\sigma \cong \tau$.

Let A and B be event structures with polarity. Following Joyal [10], a *pre-strategy from A to B* is a pre-strategy in $A^\perp \| B$, so a total map $\sigma : S \to A^\perp \| B$. We write $\sigma : A \mathrel{\rightarrow\mkern-14mu+} B$ to express that σ is a pre-strategy from A to B. Note that a pre-strategy σ *in* a game A, *e.g.* $\sigma : S \to A$, coincides with a pre-strategy *from* the empty game $\emptyset$ *to* the game A, *i.e.* $\sigma : \emptyset \mathrel{\rightarrow\mkern-14mu+} A$.

Composing pre-strategies We present the composition of pre-strategies via pullbacks. Given two pre-strategies $\sigma : S \to A^\perp \| B$ and $\tau : T \to B^\perp \| C$, ignoring polarities we can consider the maps on the underlying event structures, *viz.* $\sigma : S \to A \| B$ and $\tau : T \to B \| C$. Viewed this way we can form the pullback in the category of event structures as shown below

$$\begin{tikzcd}
 & A \parallel T \arrow[dr, "\mathrm{id}_A \parallel \tau"] & & \\
P \arrow[ur, "\Pi_2"] \arrow[dr, "\Pi_1"'] \arrow[r, phantom, pos=0.1, "\rangle"] & & A \parallel B \parallel C \arrow[r] & A \parallel C \\
 & S \parallel C \arrow[ur, "\sigma \parallel \mathrm{id}_C"'] & &
\end{tikzcd}$$

where the map $A\|B\|C \to A\|C$ is undefined on B and acts as identity on A and C. The partial map from P to $A\|C$ given by the diagram above (either way round the pullback square) factors as the composition of the partial map $P \to P \downarrow V$, where V is the set of events of P at which the map $P \to A\|C$ is defined, and a total map $P \downarrow V \to A\|C$. The resulting total map gives us the composition $\tau \odot \sigma : P \downarrow V \to A^{\perp}\|C$ once we reinstate polarities.

Identities w.r.t. composition are given by copy-cat strategies. Let A be an event structure with polarity. The copy-cat strategy from A to A is an instance of a pre-strategy, so a total map $\gamma_A : \mathbb{C}\!\!\mathbb{C}_A \to A^{\perp}\|A$. It describes a concurrent strategy based on the idea that Player moves, of positive polarity, always copy previous corresponding moves of Opponent, of negative polarity. For $c \in A^{\perp}\|A$ we use $\overline{c}$ to mean the corresponding copy of c, of opposite polarity, in the alternative component. Define $\mathbb{C}\!\!\mathbb{C}_A$ to comprise the event structure with polarity $A^{\perp}\|A$ together with the extra causal dependencies generated by $\overline{c} \leq_{\mathbb{C}\!\!\mathbb{C}_A} c$ for all events c with $pol_{A^{\perp}\|A}(c) = +$. The *copy-cat* pre-strategy $\gamma_A : A \rightarrow\!\!\!\!\!+\; A$ is defined to be the map $\gamma_A : \mathbb{C}\!\!\mathbb{C}_A \to A^{\perp}\|A$ where γ_A acts as the identity function on the common set of events.

Interaction In this paper, we will be particularly interested in the results of the interaction between a strategy $\sigma : S \to B$ and a counter-strategy $\tau : T \to B^{\perp}$ in order to determine the resulting payoff. Unlike the composition $\tau \odot \sigma$ where the interaction of σ and τ are hidden, it is the status of the configurations in $\mathcal{C}^{\infty}(B)$ their full interaction induces which decides the resulting payoff. Ignoring polarities, we have total maps of event structures $\sigma : S \to B$ and $\tau : T \to B$. Form their pullback,

$$\begin{tikzcd} & P \arrow[dl, "\Pi_1"'] \arrow[dr, "\Pi_2"] \arrow[dd, phantom, "\lrcorner" rotate=-45, near start] & \\ S \arrow[dr, "\sigma"'] & & T \arrow[dl, "\tau"] \\ & B\,, & \end{tikzcd}$$

to obtain the event structure P resulting from the interaction of σ and τ. Because σ or τ may be nondeterministic there can be more than one maximal configuration z in $\mathcal{C}^{\infty}(P)$. Define the set of results of the interaction of σ and τ to be

$$\langle \sigma, \tau \rangle =_{\text{def}} \{\sigma \Pi_1 z \mid z \text{ is maximal in } \mathcal{C}^{\infty}(P)\}\,.$$

Strategies The main result of [17] is that two conditions on pre-strategies, *receptivity* and *innocence*, are necessary and sufficient for copy-cat to behave as identity w.r.t. the composition of pre-strategies. Receptivity ensures an openness to all possible moves of Opponent. innocence restricts the behaviour of Player; Player may only introduce new relations of immediate causality of the form $\ominus \rightarrowtriangle \oplus$ beyond those imposed by the game.

Definition 5. – ***Receptivity.*** *A pre-strategy σ is* receptive *iff $\sigma x \overset{a}{\longrightarrow\!\!\subset}$ & $pol_A(a) = -\Rightarrow \exists! s \in S.\ x \overset{s}{\longrightarrow\!\!\subset}$ & $\sigma(s) = a$.*

– **innocence.** *A pre-strategy* σ *is* innocent *when it is both*
+-innocent: *if* $s \twoheadrightarrow s'$ & $pol(s) = +$ *then* $\sigma(s) \twoheadrightarrow \sigma(s')$*, and*
−-innocent: *if* $s \twoheadrightarrow s'$ & $pol(s') = -$ *then* $\sigma(s) \twoheadrightarrow \sigma(s')$*.*
*A **strategy** is a receptive and innocent pre-strategy.*

Theorem 1 (from [17]). *Let* $\sigma : A \rightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow B$ *be pre-strategy. Copy-cat behaves as identity w.r.t. composition, i.e.* $\sigma \circ \gamma_A \cong \sigma$ *and* $\gamma_B \circ \sigma \cong \sigma$*, iff* σ *is receptive and innocent. Copy-cat pre-strategies* $\gamma_A : A \rightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow A$ *are receptive and innocent.*

Theorem 1 motivated the definition of a *strategy* as a pre-strategy which is receptive and innocent. In fact, we obtain a bicategory, **Games**, in which the objects are event structures with polarity—the games, the arrows from A to B are strategies $\sigma : A \rightarrow\!\!\!\!\!+\!\!\!\!\!\rightarrow B$ and the 2-cells are maps of spans. The vertical composition of 2-cells is the usual composition of maps of spans. Horizontal composition is given by the composition of strategies $\odot$ (which extends to a functor on 2-cells via the universality of pullback).

3 Concurrent games with payoff

We begin the core of the paper, the treatment of payoff in concurrent games. $\overline{\mathbb{R}}$ denotes $\mathbb{R} \cup \{-\infty, +\infty\}$, the reals extended with a minimum and maximum.

Definition 6. *A **concurrent game with payoff** is a triple* $(A, \kappa_A^+, \kappa_A^-)$*, where* A *is a concurrent game and* $\kappa_A^+, \kappa_A^- : \mathcal{C}^\infty(A) \to \overline{\mathbb{R}}$ *are **payoff functions**.*

In all of this paper, we will only consider **zero-sum** concurrent games, *i.e.* for all $z \in \mathcal{C}^\infty(A)$, $\kappa_A^-(z) = -\kappa_A^+(z)$. It follows that our games with payoff will be described by a concurrent game and its payoff function $\kappa_A = \kappa_A^+ : \mathcal{C}^\infty(A) \to \overline{\mathbb{R}}$. We extend the usual constructions on concurrent games to games with payoff.

Definition 7 (Constructions).

– ***Dual.*** *If* A *is a concurrent game with payoff, then the payoff function on* $A^\perp$ *is defined by* $\kappa_{A^\perp}(x) = -\kappa_A(x)$*, for* $x \in \mathcal{C}^\infty(A^\perp)$*.*
– ***Parallel composition.*** *If* A, B *are concurrent games with payoff, then the payoff function on* $A \parallel B$ *is defined by* $\kappa_{A\parallel B}(x) = \kappa_A(x_1) + \kappa_A(x_2)$*, where* $x_1 \in \mathcal{C}^\infty(A)$ *is the projection of* x *on* A *and* $x_2 \in \mathcal{C}^\infty(B)$ *is the projection of* x *on* B*.*

We now turn to the definitions leading to the value of a game. Since games and strategies are nondeterministic, these definitions come in two variants: the *optimistic* describing the outcome of a game if all the nondeterministic choices are in favour of Player, and the *pessimistic* describing the dual case, when all of those choices are in favour of Opponent. One of the main result of the paper will be that for race-free well-founded games (to be defined below), the two corresponding notions of value coincide.

Definition 8. *We define the* ***optimistic*** *(↑) and* ***pessimistic*** *(↓)* ***results*** *of an interaction, and* ***values*** *of a strategy and of a game, as follows. Here, σ is a strategy on A and τ is a counter-strategy (a strategy on $A^\perp$), and the notation $\sigma : A$ signifies a strategy $\sigma : S \to A$.*

$$\begin{array}{ll} r^\uparrow(\sigma,\tau) = \sup_{x\in\langle\sigma,\tau\rangle} \kappa_A(x) & r^\downarrow(\sigma,\tau) = \inf_{x\in\langle\sigma,\tau\rangle} \kappa_A(x) \\ v^\uparrow(\sigma) = \inf_{\tau:A^\perp} r^\uparrow(\sigma,\tau) & v^\downarrow(\sigma) = \inf_{\tau:A^\perp} r^\downarrow(\sigma,\tau) \\ v^\uparrow(A) = \sup_{\sigma:A} v^\uparrow(\sigma) & v^\downarrow(A) = \sup_{\sigma:A} v^\downarrow(\sigma) \end{array}$$

We say that a game A **has a value** if $v^\uparrow(A) = v^\downarrow(A) = -v^\downarrow(A^\perp) = -v^\uparrow(A^\perp)$: the optimistic and pessimistic values coincide, and commute with $(-)^\perp$. The commutation with $(-)^\perp$ is a form of minimax property, since the order of quantification on strategies is reversed in $v(A)$ and $-v(A^\perp)$, whereas the coincidence of the optimistic and pessimistic value deals with nondeterminism. Note that not all games have a value:

Example 1. Take the game $A = \ominus \rightsquigarrow \oplus$ with two events of opposite polarities conflicting with each other, along with $\kappa(\emptyset) = 0$, $\kappa(\{\oplus\}) = 1$ and $\kappa(\{\ominus\}) = -2$. Then it is easy to prove that $v^\uparrow(A) = 1$, $v^\downarrow(A) = -2$, $v^\uparrow(A^\perp) = 2$ and $v^\downarrow(A^\perp) = -1$.

The example above suggests a simple relationship between $v^\downarrow(A)$ and $v^\uparrow(A^\perp)$ but this is not always the case. For example, consider the infinite game A comprising the event structure with polarity

$$\ominus \qquad \oplus_1 \longrightarrow \oplus_2 \longrightarrow \oplus_3 \longrightarrow \cdots \longrightarrow \oplus_n \longrightarrow \cdots$$

where $\kappa(\emptyset) = 0$, $\kappa(\{\oplus_1,\ldots,\oplus_n\}) = n$, $\kappa(\{\oplus_1,\ldots,\oplus_n\}\cup\{\ominus\}) = -n$, $\kappa(\{\oplus_1,\ldots\}) = -\infty$ and $\kappa(\{\oplus_1,\ldots\}\cup\{\ominus\}) = +\infty$. Then one can check that the optimistic and pessimistic values coincide, in fact this is always the case when games do not have conflict. A direct analysis of the available strategies for Player and Opponent shows that $v(A) = 0$, whereas $v(A^\perp) = +\infty$.

The first example features a *race*, where both players compete for the same resource, whereas the second example is not *well-founded* : the game allows infinite configurations. These brings us to the two following notions, that will be crucial to get the value theorem.

Definition 9. *A game A is* ***race-free*** *if for all $x \in \mathcal{C}(A)$ such that $x \xrightarrow{a}\subset$ and $x \xrightarrow{a'}\subset$ with $pol(a) = -$ and $pol(a') = +$, we have $x \cup \{a, a'\} \in \mathcal{C}(A)$.*

A game A is ***well-founded*** *if every configuration in $\mathcal{C}^\infty(A)$ is finite.*

Definition 10. *Let A be a concurrent game with payoff, and $x \in \mathcal{C}^\infty(A)$. Let A/x be the* ***residual of*** *A* ***after*** *x, comprising*

- *events, $\{a \in A \setminus x \mid x \cup [a]_A \in \mathcal{C}^\infty(A)\}$,*
- *consistency relation, $X \in \mathrm{Con}_{A/x} \Leftrightarrow X \subseteq_f A/x \ \&\ x \cup [X]_A \in \mathcal{C}^\infty(A)$,*
- *causal dependency, the restriction of that on A.*

Define $\kappa_{A/x} : \mathcal{C}^\infty(A/x) \to \overline{\mathbb{R}}$ by taking $\kappa_{A/x}(y) = \kappa_A(x \cup y)$. Finally, define $(A, \kappa_A)/x = (A/x, \kappa_{A/x})$. When x is a singleton $\{a\}$, we shall generally write A/a instead of $A/\{a\}$.

4 The value theorem

In this section, we prove the value theorem on concurrent games. The proof proceeds in two steps. First, we exhibit key constructions on strategies and the study the results of their interactions. This analysis will allow us to characterize the values of all positions of the game. Exploiting well-foundedness of the game, we will deduce the sought-for value theorem.

4.1 Constructions on strategies

In "glueing" strategies together it is helpful to assume that all the initial negative moves of the strategies are exactly the same, and indeed coincide with the initial negative moves of the game:

Lemma 1. *Let $\sigma : S \to A$ be a strategy, then there exists a strategy $\sigma' : S' \to A$ with $\sigma' \cong \sigma$, for which*

$$\forall s' \in S'.\ pol_{S'}[s']_{S'} = \{-\} \Rightarrow \sigma'(s') = s'\,. \qquad (\dagger)$$

Henceforth we will assume all strategies satisfy the property ($\dagger$). In particular, its adoption facilitates the definition of a 'sum' of strategies in a game.

Proposition 1. *Let $\sigma_i : S_i \to A$, for $i \in I$, be strategies (assumed to satisfy ($\dagger$)). W.l.og. we may assume that whenever indices $i, j \in I$ are distinct then so are those events of S_i and S_j which causally depend on a positive event (otherwise we could tag such events by their respective indices). Define S to be the event structure with events $\bigcup_{i \in I} S_i$, causal dependency $s \le_S e'$ iff $s \le_{S_i} e'$, for some $i \in I$, and consistency $X \in \mathrm{Con}_S$ iff $X \in \mathrm{Con}_{S_i}$, for some $i \in I$. Defining $\|_{i\in I}\, \sigma_i(s) = \sigma_i(s)$ if $s \in S_i$ yields a strategy $\|_{i\in I}\, \sigma_i : S \to A$.*

The next construction takes a strategy σ on a game A/a, where a is an initial positive event of game A, and creates a strategy on A that starts by playing a, then resumes as σ.

Proposition 2. *Suppose A is a race-free game such that $\emptyset \stackrel{a}{\longrightarrow\!\!\!\!\subset}$ with $pol(a) = +$. Then for any strategy $\sigma : S \to A/a$, where w.l.o.g. $a \notin S$, there is a strategy $\mathrm{play}_a(\sigma) : S' \to A$: the event structure S' comprises*

- *events, $S \cup \{a\}$,*
- *causal dependency, that on S extended by $a \le_{S'} s$, for $s \in S$, whenever $a \le_A \sigma(s)$,*
- *with consistency, $X \in \mathrm{Con}_{S'}$ iff $X \cap S \in \mathrm{Con}_S$,*

and $\mathrm{play}_a(\sigma)(s) = \sigma(s)$, for $s \in S$, with $\mathrm{play}_a(\sigma)(a) = a$.

Given a strategy on σ on a residual game A/a, where a is an initial negative event of A, we can create a strategy on A that awaits a, then resumes as σ.

Proposition 3. *Suppose A is a game such that $\emptyset \overset{a}{\longrightarrow\!\!\subset}$ with $pol(a) = -$. Then for any strategy $\sigma : S \to A/a$, where w.l.o.g. $a \notin S$, there is a strategy $\mathrm{wait}_a(\sigma) : S' \to A$: the event structure S' comprises*

- *events, $S \cup A_-$, where $A_- =_{\mathrm{def}} \{a' \in A \mid pol_A[a']_A \subseteq \{-\}\}$,*
- *causal dependency, that on S and A_- extended by $a \leq_{S'} s$, for $s \in S$, whenever $a \leq_A \sigma(s)$ or $pol(s) = +$,*
- *with consistency, $X \in \mathrm{Con}_{S'}$ iff $X \cap S \in \mathrm{Con}_S$ & $\mathrm{wait}_a(\sigma)X \in \mathrm{Con}_A$,*

where $\mathrm{wait}_a(\sigma)(s')$ is defined to be $\sigma(s')$ if $s' \in S$, otherwise s'.

It is useful to extend the notion of residual from games to strategies:

Definition 11. *Let $\sigma : S \to A$ be a strategy and $x \in \mathcal{C}^\infty(S)$. Define the function $\sigma/x : S/x \to A/\sigma x$ to be the restriction of σ. In the case where x is a singleton $\{s\}$, we shall generally write σ/s instead of $\sigma/\{s\}$.*

Proposition 4. *For $\sigma : S \to A$ a strategy and $x \in \mathcal{C}^\infty(S)$, the function $\sigma/s : S/s \to A/\sigma(s)$ is a strategy.*

Let A be a game with payoff κ_A and $\sigma : S \to A$ and $\tau : T \to A^\perp$ be strategies. The set of values resulting from their interaction is given by $\{\kappa_A x \mid x \in \langle\sigma, \tau\rangle\}$, which we generally write as $\kappa\langle\sigma, \tau\rangle$ when the game is clear from the context. We use $\langle\sigma, \tau\rangle^+ =_{\mathrm{def}} \{x \in \langle\sigma, \tau\rangle \mid + \in pol\ x\}$ for the configurations in $\langle\sigma, \tau\rangle$ containing events of positive polarity. We will make crucial use of the following analysis of the interactions between strategies.

Lemma 2. *Let A be a well-founded race-free game with payoff. Let σ and σ_i, for $i \in I$, be strategies in A, and τ a strategy in $A^\perp$. Then,*

$$\begin{array}{cc} \kappa\langle\sigma, \tau\rangle = \{-v \mid v \in \kappa\langle\tau, \sigma\rangle\} & \kappa\langle\mathrm{play}_a(\sigma), \tau\rangle = \kappa\langle\sigma, \tau/a\rangle \\ \kappa\langle\|_{i\in I}\, \sigma_i, \tau\rangle \subseteq \bigcup_{i\in I} \kappa\langle\sigma_i, \tau\rangle & \kappa\langle\|_{i\in I}\, \sigma_i, \tau\rangle^+ = \bigcup_{i\in I} \kappa\langle\sigma_i, \tau\rangle^+ \\ \kappa\langle\mathrm{wait}_a(\sigma), \tau\rangle \supseteq \bigcup_{t:\tau(t)=a} \kappa\langle\sigma, \tau/t\rangle & \kappa\langle\mathrm{wait}_a(\sigma), \tau\rangle^+ = \bigcup_{t:\tau(t)=a} \kappa\langle\sigma, \tau/t\rangle^+ \end{array}$$

From this follow two important corollaries. Firstly, if a is an initial positive event of A we have $\kappa\langle\mathrm{play}_a(\sigma), \mathrm{wait}_a(\tau)\rangle = \kappa\langle\sigma, \tau\rangle$; two strategies, one playing a move and the other waiting for the move, synchronise. This immediately follows from the lemma above and the observation that $\mathrm{wait}_a(\sigma)/a = \sigma$. Secondly, the following additional construction will be crucial. For $(e_i)_{i\in I}$ the family of negative minimal events of A and strategies $\sigma_i : S_i \to A/e_i$, we define $\mathrm{case}_{i\in I}\sigma_i =_{\mathrm{def}} \|_{i\in I}\, \mathrm{wait}_{e_i}\sigma_i$. Roughly, this strategy waits for an input e_i and then proceeds as σ_i; though the full story is subtle as two distinct events e_i and e_j may be consistent with each other and the strategies σ_i and σ_j overlap. From the lemma we can prove that for all $\tau : T \to A^\perp$ such that T has a minimal $+$-event, then

$$\kappa\langle\mathrm{case}_{i\in I}\sigma_i, \tau\rangle \subseteq \bigcup_{i\in I, t:\tau(t)=a_i} \kappa\langle\sigma_i, \tau/t\rangle \qquad \kappa\langle\mathrm{case}_{i\in I}\sigma_i, \tau\rangle^+ = \bigcup_{i\in I, t:\tau(t)=a_i} \kappa\langle\sigma_i, \tau/t\rangle^+$$

In Lemma 2 and the observation above, in all the cases where we have inclusions instead of equalities this is by necessity. For instance with the case construction above, a configuration in $\langle \sigma_i, \tau\rangle$, by definition a maximal configuration of the pullback of σ_i and τ, although it reappears as a configuration of the pullback of $\mathrm{case}_{i\in I}\sigma_i$ and τ, it may no longer be maximal so fail to contribute to $\langle \mathrm{case}_{i\in I}\sigma_i, \tau\rangle$.

4.2 Values of these constructions

Lemma 3. *For any race-free well-founded game A, we have:*

$$\begin{array}{rlrl} v^{\uparrow}(\mathrm{play}_a(\sigma)) \le v^{\uparrow}(\sigma) & & v^{\downarrow}(\mathrm{play}_a(\sigma)) \le v^{\downarrow}(\sigma) \\ v^{\uparrow}(\sigma) \le v^{\uparrow}(\sigma/a) & & v^{\downarrow}(\sigma) \le v^{\downarrow}(\sigma/a) \end{array}$$

Proof. Direct consequence of Lemma 2.

Lemma 4. *Suppose A is race-free and well-founded and $\sigma : S \to A$ is a strategy with a minimal $+$-event. Let $(f_j)_{j\in J}$ be the family of minimal $+$-events of A. Let $\sigma : S \to A$ be a strategy such that there is a minimal $+$-event $s \in S$. Then, $v^{\downarrow}(\sigma) \le \sup_{j\in J} v^{\downarrow}(\{f_j\})$ and $v^{\uparrow}(\sigma) \le \sup_{j\in J} v^{\uparrow}(\{f_j\})$.*

Proof. The pessimistic case follows from Lemma 2. *Optimistic case.* Suppose that the inequality is false, *i.e.* $\sup_{j\in J} v^{\uparrow}(\{f_j\}) < v^{\uparrow}(\sigma)$. This implies that there is $\alpha \in \mathbb{R}$ such that $\sup_{j\in J} v^{\uparrow}(\{f_j\}) < \alpha$ and $v^{\uparrow}(\sigma) > \alpha$. The first inequality implies $\forall j \in J,\ \forall \sigma : A'/f_j,\ \exists \tau' : A^{\perp}/f_j,\ \forall z' \in \langle \sigma', \tau'\rangle,\ \kappa(z') < \alpha$, which is easily shown to imply

$$\forall (\sigma_k)_{k\in K},\ \exists (\tau_j)_{j\in\sigma K},\ \forall k \in K,\ \forall z' \in \langle \sigma_k, \tau_{\sigma k}\rangle,\ \kappa(z') < \alpha \tag{1}$$

where K is the set of positive minimal events in S. Applying this property to the family of strategies obtained by $\sigma_k = \sigma/k$, we get a family of counter-strategies $(\tau_j)_{j\in\sigma K}$. We extend this family to J by setting τ_j to be the empty strategy (closed under receptivity) whenever $e_j \notin \sigma K$. Thus, we get a family $(\tau_j)_{j\in J}$. Similarly, the second inequality implies that

$$\forall \tau : A^{\perp},\ \exists z \in \langle \sigma, \tau\rangle,\ \kappa(z) > \alpha\,.$$

Applied to $\tau = \mathrm{case}_{j\in J}\tau_j$, we get $z \in \langle \sigma, \mathrm{case}_{j\in J}\tau_j\rangle$ such that $\kappa(z) > \alpha$. By our observation on the interaction with case, there is $k_0 \in K$, and $z' \in \langle \sigma/k_0, \tau_{\sigma k_0}\rangle$ such that $\kappa(z') = \kappa(z) > \alpha$. However, applying (1) to k_0 also shows that $\kappa(z') < \alpha$, contradiction. Hence, the required inequality is true.

Lemma 5. *Let A be a race-free well-founded game and $(e_i)_{i\in I}$ the family of its negative minimal events. Then,*

$$\begin{array}{l} \min(\kappa(\emptyset), \inf_{i\in I} \sup_{\sigma : A/e_i} v^{\downarrow}(\sigma)) \le v^{\downarrow}(A) \\ \min(\kappa(\emptyset), \inf_{i\in I} \sup_{\sigma : A/e_i} v^{\uparrow}(\sigma)) \le v^{\uparrow}(A) \end{array}$$

Proof. For as long as possible, we do not distinguish the optimistic and pessimistic cases. If the inequality is false, there is $\alpha \in \mathbb{R}$ such that $\min(\kappa(\emptyset), \inf_{i\in I} \sup_{\sigma:A/e_i} v(\sigma)) > \alpha > v(A)$, which in turn implies the following three propositions:

$$\kappa(\emptyset) > \alpha \tag{2}$$

$$\forall i \in I,\ \exists \sigma_i : A/e_i,\ \forall \tau : A^\perp/e_i,\ r(\sigma_i, \tau) > \alpha \tag{3}$$

$$\forall \sigma : A,\ \exists \tau : A^\perp,\ r(\sigma, \tau) < \alpha \tag{4}$$

In particular, (3) gives a family $(\sigma_i)_{i\in I}$. Instantiating (4) to $\text{case}_{i\in I}\sigma_i$, we get $\tau : T \to A^\perp$ such that $r(\text{case}_{i\in I}\sigma_i, \tau) < \alpha$.

$$\kappa(\emptyset) > \alpha \tag{5}$$

$$\forall i \in I,\ \forall t,\ \tau(t) = e_i \Rightarrow r(\sigma_i, \tau/t) > \alpha \tag{6}$$

$$r(\text{case}_{i\in I}, \tau) < \alpha \tag{7}$$

Pessimistic case. Since $r(\text{case}_{i\in I}\sigma_i, \tau) < \alpha$, there must be $y \in \langle \text{case}_{i\in I}\sigma_i, \tau\rangle$ such that $\kappa(y) < \alpha$. If T has no minimal +-event, then necessarily we have $y = \emptyset$, therefore $\kappa_A(y) = \kappa_A(\emptyset) > \alpha$, contradiction. Therefore, T has a minimal +-event. Then by our analysis of interactions for case, there is a minimal +-event $t \in T$ and $\tau(t) = e_{i_0}$ and $y' \in \langle \sigma_{i_0}, \tau/t\rangle$ such that $\kappa(y') = \kappa(y) < \alpha$. But this is absurd by (6), so we have found a contradiction.
Optimistic case. By (7) instantiated to the pessimistic case we have that for all $y \in \langle \text{case}_{i\in I}, \tau\rangle$, $\kappa(y) < \alpha$. Take one such $y \in \langle \text{case}_{i\in I}, \tau\rangle$ ($\langle \text{case}_{i\in I}, \tau\rangle$ is non-empty by Zorn's lemma). As above, y cannot be empty as that would cause a contradiction, and T must have a minimal +-event. Therefore, there is a minimal +-event $t \in T$ and $\tau(t) = e_{i_0}$ and $y' \in \langle \sigma_{i_0}, \tau/t\rangle$ such that $\kappa(y') = \kappa(y) < \alpha$, contradicting (6).

4.3 Value theorem

Let A be a fixed well-founded and race-free game.

Lemma 6. *Let $x \in \mathcal{C}(A)$. Let $(e_i)_{i\in I}$ be the family of extensions of x of negative polarity, and $(f_j)_{j\in J}$ be the family of extensions of x of positive polarity. Then,*

$$v^\uparrow(x) = \max(\min(\kappa(x), \inf_{i\in I} v^\uparrow(x \cup \{e_i\})), \sup_{j\in J} v^\uparrow(x \cup \{f_j\}))$$
$$v^\downarrow(x) = \max(\min(\kappa(x), \inf_{i\in I} v^\downarrow(x \cup \{e_i\})), \sup_{j\in J} v^\downarrow(x \cup \{f_j\}))$$

Where the ***value*** *$v(x)$ of a configuration $x \in \mathcal{C}(A)$ is defined as $v(A/x)$.*

Proof. The reasoning is the same in the optimistic and pessimistic cases, so we do not distinguish them.

$\leq$. Let $\sigma : S \to A/x$ be a strategy. If there is a minimal event $s \in S$ with $pol(s) = +$, then $v(x) \leq v(\sigma) \leq \sup_{j\in J} v(x \cup \{f_j\})$ by Lemma 4. Otherwise, there is no such minimal $s \in S$. Then $v(\sigma) \leq \kappa(x)$. Indeed, letting $\tau : T \to A/x$ be the empty strategy closed by receptivity, we have $\langle\sigma,\tau\rangle = \{\emptyset\}$ and $r(\sigma,\tau) = \kappa(x)$. Similarly taking $i_0 \in I$, by Lemma 3 we have $v(\sigma) \leq v(\sigma/e_{i_0})$, and therefore $v(\sigma) \leq \inf_{i\in I} v(x \cup \{e_i\})$.

$\geq$. Let us prove that $\sup_{j\in J} v(x \cup \{f_j\}) \leq v(x)$, taking $j_0 \in J$ and $\sigma : A/(x \cup \{f_{j_0}\})$. Then by Lemma 3 we have $v(\mathrm{play}_{f_{j_0}}\sigma) \leq v(\sigma)$ and $v(\sigma) \leq v(x)$. Finally, we need to prove that $\min(\kappa(x), \inf_{i\in I} v(x \cup \{e_i\})) \leq v(x)$, but this is Lemma 5.

Theorem 2. *If A is well-founded and race-free then A has a value,* i.e. *we have:*

$$v^{\uparrow}(A) = v^{\downarrow}(A) \qquad v(A) = -v(A^{\perp})$$

(Note that the second equality only makes sense because by the first, we can talk in a non-ambiguous way of the ***value*** *$v(A)$ of a game A.)*

Proof. Relatively direct consequence of Lemma 6.

We say that a strategy $\sigma : S \to A$ is **optimal** when its *pessimistic value* is equal to the value of the game. Note that it also implies that the optimistic value is equal to the value of the game, since for all $\sigma : S \to A$ we must have $v^{\downarrow}(\sigma) \leq v^{\uparrow}(\sigma) \leq v(A)$. It also follows that for optimal strategies, the pessimistic and optimistic values coincide. When σ is optimal, we will therefore sometimes just write $v(\sigma)$ for its value.

Example 2. Any well-founded race-free game has a value. However this value is not necessarily reached: there are games without optimal strategies. Consider the game A with events $\{\oplus_i \mid i \in \mathbb{N}\}$, pairwise inconsistent, with $\kappa(\emptyset) = 0$ and $\kappa(\{\oplus_i\}) = i$. Its value is $+\infty$ since each positive natural number can be reached, but no strategy σ satisfies $v^{\downarrow}(\sigma) = +\infty$ (though the strategy that plays a nondeterministic choice of natural number satisfies $v^{\uparrow}(\sigma) = +\infty$).

5 Compositionality of optimal strategies

Finally we study how payoff relates to the composition of strategies. We hope that thinking compositionally about values and optimal strategies can be helpful in computing values and optimal strategies for complex games from smaller ones. There are two main kinds of composition of strategies. The first is the categorical composition $\tau\odot\sigma$ of $\sigma : S \to A^{\perp} \parallel B$ and $\tau : T \to B^{\perp} \parallel C$. The second is parallel composition $\sigma \parallel \tau : S \parallel T \to A \parallel B$.

We start this section with the observation that for any strategy $\sigma : S \to A$ we have that $v^{\downarrow}(\sigma) = \inf\{\kappa(\sigma x) \mid x \in \mathcal{C}(S) \;\; +\text{-maximal}\}$, since the definition of pessimistic value quantifies at the same time over Opponent strategies and resulting interactions. From this, we get:

Proposition 5. *For strategies $\sigma : S \to A^{\perp} \parallel B$ and $\tau : T \to B^{\perp} \parallel C$, we have $v^{\downarrow}(\tau \odot \sigma) \geq v^{\downarrow}(\tau) + v^{\downarrow}(\sigma)$. Likewise for $\sigma : S \to A$ and $\tau : T \to B$, we have $v(\sigma \parallel \tau) = v(\sigma) + v(\tau)$.*

For categorical composition, $v^{\downarrow}(\tau \odot \sigma) \leq v^{\downarrow}(\tau) + v^{\downarrow}(\sigma)$ does not hold in general, and neither do the two inequalities in the optimistic case. However, the situation is different for *optimal strategies*. To establish this, we first note the following:

Proposition 6. *For any race-free, well-founded games A and B, $v(A \parallel B) = v(A) + v(B)$.*

Proof. By the value theorem, it does not matter whether we work on the optimistic or pessimistic cases. By simplicity, let us pick the pessimistic one. Firstly, we prove that $v(A \parallel B) \geq v(A) + v(B)$. Indeed, let $\sigma : S \to A$ and $\tau : T \to B$ be strategies. Then, as needed we have $v^{\downarrow}(\sigma \parallel \tau) \geq v^{\downarrow}(\sigma) + v^{\downarrow}(\tau)$ by Proposition 5.

Moreover, this inequality also holds for $A^{\perp}$ and $B^{\perp}$, therefore $v(A^{\perp} \parallel B^{\perp}) \geq v(A^{\perp}) + v(B^{\perp})$, from which it follows that $v(A \parallel B) \leq v(A) + v(B)$ by the value theorem and the definition of the dual of games with payoff.

Theorem 3. *If $\sigma : S \to A^{\perp} \parallel B$ and $\tau : T \to B^{\perp} \parallel C$ are optimal strategies, so is $\tau \odot \sigma$. Moreover copycat is optimal, therefore there is a bicategory of concurrent games with payoff and optimal strategies.*

Proof. Suppose σ and τ optimal. We reason as follows:

$$\begin{aligned} v^{\downarrow}(\tau \odot \sigma) &\geq v^{\downarrow}(\sigma) + v^{\downarrow}(\tau) \\ &= v(A^{\perp} \parallel B) + v(B^{\perp} \parallel C) \\ &= v(A^{\perp} \parallel C) \end{aligned}$$

This implies that $v^{\downarrow}(\tau \odot \sigma) = v(A^{\perp} \parallel C)$, since a strict inequality would contradict the definition of $v(A^{\perp} \parallel C)$.

Copycat is optimal: take a $+$-maximal $x \in \mathcal{C}(A^{\perp} \parallel A)$. Necessarily, x has the form $y \cup \overline{y}$, where $y \in \mathcal{C}(A)$. Moreover, $\kappa_{A^{\perp} \parallel A}(x) = \kappa_A(y) - \kappa_A(y) = 0$, therefore we have $v^{\downarrow}(\gamma_A) = 0$. However we also have $v(A^{\perp} \parallel A) = v(A) - v(A) = 0$, therefore copycat is optimal.

We finish this section by remarking that from the theorem above it follows that when σ and τ are optimal, we have $v(\tau \odot \sigma) = v(\sigma) + v(\tau)$, since both sides are forced by optimality to coincide with the value of the game.

6 Conclusion

We have proved a value theorem for race-free well-founded concurrent games. Note that this theorem is not an equilibrium theorem since the value is not always reached. However it is always reached in finite games. In fact our constructions on strategies give an algorithm to compute the value and optimal strategies for finite games. In future we plan to investigate the existence and computation of equilibria in the non-zero-sum case. This will require the extension of our framework to deal with probabilistic strategies, and should allow to formulate a better connection with the concurrent games of [3, 7].

We proved that optimal strategies are stable under composition, forming a bicategory. This compositional structure is worth investigating further. We hope that it can be extended to a cartesian-closed category of payoff games and optimal strategies, thus providing the basis for a *concurrent programming language* based on the simply-typed λ-calculus and concurrent operations on strategies, for which typable terms always describe optimal strategies.

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A Preliminaries : stable families and composition

The detailed proofs often rely on stable families. Constructions such as product and pullback are often done most conveniently in the category of stable families. There is a full and faithful embedding of the category of event structures into the category of stable families. It has a right adjoint which translates limits such as pullback and product of stable families to the corresponding universal constructions in the category of event structures.[3]

Definition 12. *For $\sigma : S \to A$ and $\tau : T \to A^\perp$ we will write $[\sigma, \tau]$ for the set of* ***interactions*** *between σ and τ,* i.e. *maximal configurations of the stable family $\mathcal{C}(T) \odot \mathcal{C}(S)$. (It is the pullback of σ and τ in the category of stable families.)*

B Constructions on strategies

Lemma 7. *Let $\sigma : S \to A$ be a strategy. Then $\sigma \cong \sigma'$, a strategy $\sigma' : S' \to A$ for which*

$$\forall s' \in S'.\ pol_{S'}[s']_{S'} = \{-\} \Rightarrow \sigma'(s') = s'\,. \qquad (\dagger)$$

Moreover,

$$S'_- = A_-$$

where for an event structure with polarity E we write $E_- =_{\mathrm{def}} \{e \in E \mid pol[e] \subseteq \{-\}\}$.

Proof. As a consequence of receptivity and negative innocence [LICS11], whenever $\emptyset \subseteq^- y$ in $\mathcal{C}(A)$ there is a *unique* $x \in \mathcal{C}(S)$ so that $\emptyset \subseteq^- x \;\&\; \sigma x = y$. Consequently, the map σ induces an order isomorphism w.r.t. inclusion between configurations $x \in \mathcal{C}(S)$ where $\emptyset \subseteq^- x$ and $y \in \mathcal{C}(A)$ where $\emptyset \subseteq^- y$. The order isomorphism restricts to an order isomorphism between prime configurations. It follows that σ is bijective between events $s \in S_-$ and events $a \in A_-$. This bijection extends to a bijective renaming of events of S to those of S'.

The lemma permits us to assume strategies satisfy $(\dagger)$ in the following results.

Proposition 7. *Let $\sigma_i : S_i \to A$, for $i \in I$, be strategies (assumed to satisfy $(\dagger)$). W.l.og. we may assume that whenever indices $i, j \in I$ are distinct then so are those events of S_i and S_j which causally depend on a positive event (otherwise we could tag such events by their respective indices). Define S to be the event structure with events $\bigcup_{i\in I} S_i$, causal dependency $s \leq_S e'$ iff $s \leq_{S_i} e'$, for some $i \in I$, and consistency $X \in \mathrm{Con}_S$ iff $X \in \mathrm{Con}_{S_i}$, for some $i \in I$. Defining $\|_{i\in I}\, \sigma_i(s) = \sigma_i(s)$ if $s \in S_i$ yields a strategy $\|_{i\in I}\, \sigma_i : S \to A$.*

[3] A recent reference: Glynn Winskel. Event structures, stable families and games. Lecture notes, Comp Science Dept, Aarhus University, Available from `http://daimi.au.dk/~gwinskel`, 2011.

Proof. Pre-strategy. Follows from the observation that for any $x \in \mathcal{C}(\bigcup_{i\in I} S_i)$ there is $i \in I$ such that $x \in \mathcal{C}(S_i)$. Therefore preservation of configuration and local injectivity directly follow from those properties for the σ_is.

Receptivity. Trivial, since (†) is preserved by union and implies receptivity.

Innocence. For any $s_1, s_2 \in \bigcup_{i\in I} S_i$, if $s_1 \rightarrowtriangle s_2$ then there is $i \in I$ such that $s_1, s_2 \in S_i$ and $s_1 \rightarrowtriangle s_2$ in S_i as well. Therefore if $pol(s_1) \neq -$ or $pol(s_2) \neq +$ then by innocence of σ_i we have $\sigma_i(s_1) \rightarrowtriangle \sigma_i(s_2)$ as well, therefore $(\|_{i\in I} \sigma_i)(s_1) \rightarrowtriangle (\|_{i\in I} \sigma_i)(s_2)$ and $\|_{i\in I} \sigma_i$ is innocent.

Proposition 8. *Suppose A is a race-free game such that $\emptyset \overset{a}{\longrightarrow\!\!\subset}$ with $pol(a) = +$. Then for any strategy $\sigma : S \to A/a$, where w.l.o.g. $a \notin S$, there is a strategy $\mathrm{play}_a(\sigma) : S' \to A$: the event structure S' comprises*

- *events, $S \cup \{a\}$,*
- *causal dependency, that on S extended by $a \leq_{S'} s$, for $s \in S$, whenever $a \leq_A \sigma(s)$,*
- *with consistency, $X \in \mathrm{Con}_{S'}$ iff $X \cap S \in \mathrm{Con}_S$,*

and $\mathrm{play}_a(\sigma)(s) = \sigma(s)$, for $s \in S$, with $\mathrm{play}_a(\sigma)(a) = a$.

Proof. It is easy to check that S' is an event structure and that $\mathrm{play}_a(\sigma)$ is a total map of event structures which preserves polarity. Innocence is inherited from σ. That it is receptive follows from the race-freedom of A: Let $x \in \mathcal{C}(S')$ and $\mathrm{play}_a(\sigma)x \overset{a'}{\longrightarrow\!\!\subset}$ where $a' \in A$ and $pol(a') = -$. If $a \in x$ then receptivity condition for $\mathrm{play}_a(\sigma)$ follows directly from that of σ. If $a \notin x$ then $x \in \mathcal{C}(S)$ and $\mathrm{play}_a(\sigma)x = \sigma x$. From the race-fredom of A we deduce that $\sigma x \overset{a'}{\longrightarrow\!\!\subset}$ in A/a. Again the receptivity condition for $\mathrm{play}_a(\sigma)$ follows from that of σ.

Proposition 9. *Suppose A is a game such that $\emptyset \overset{a}{\longrightarrow\!\!\subset}$ with $pol(a) = -$. Then for any strategy $\sigma : S \to A/a$, where w.l.o.g. $a \notin S$, there is a strategy $\mathrm{wait}_a(\sigma) : S' \to A$: the event structure S' comprises*

- *events, $S \cup A_-$, where $A_- =_{\mathrm{def}} \{a' \in A \mid pol_A[a']_A \subseteq \{-\}\}$,*
- *causal dependency, that on S and A_- extended by $a \leq_{S'} s$, for $s \in S$, whenever $a \leq_A \sigma(s)$ or $pol(s) = +$,*
- *with consistency, $X \in \mathrm{Con}_{S'}$ iff $X \cap S \in \mathrm{Con}_S$ & $\mathrm{wait}_a(\sigma)X \in \mathrm{Con}_A$,*

where $\mathrm{wait}_a(\sigma)(s')$ is defined to be $\sigma(s')$ if $s' \in S$, otherwise s'.

Proof. By innocence, the causal dependencies on S and A_- agree where they overlap. As $a \notin S$, by assumption, we obtain a partial order $\leq_{S'}$ from the definition above. It is routine to check that S' is an event structure.

Observe that if $\sigma(s) \in A_-$ then $s \in S_-$, for all $s \in S$: otherwise there would be a maximal positive event on which s causally depended, contradicting $-$-innocence of σ.

In checking that $\mathrm{wait}_a(\sigma)$, clearly a total function, is a map of event structures it is straightforward to show that the image of a configuration $x \in \mathcal{C}(S')$ is down-closed in A. By definition $\mathrm{wait}_a(\sigma)$ preserves consistency, so $\mathrm{wait}_a(\sigma)x$ is also consistent and in $\mathcal{C}(A)$. Suppose now $s_1, s_2 \in x$ with $\mathrm{wait}_a(\sigma)(s_1) = \mathrm{wait}_a(\sigma)(s_2)$. If both $s_1, s_2 \in S$ then $\sigma(s_1) = \sigma(s_2)$ so $s_1 = s_2$ as σ is map of event structures. Otherwise, either $s_1 \notin S$ or $s_2 \notin S$. If both $s_1 \notin S$ and $s_2 \notin S$, then $s_1 = s_2$, directly from the definition of $\mathrm{wait}_a(\sigma)$. Otherwise, w.l.o.g. suppose $s_1 \in S$ and $s_2 \notin S$. Then $\sigma(s_1) = s_2$ and $s_2 \in A_-$. By the observation above, $s_1 \in S_-$. But σ is assumed to satisfy (†), so $\sigma(s_1) = s_1 = s_2$. The function $\mathrm{wait}_a(\sigma)$ is indeed a map of event structures.

The map $\mathrm{wait}_a(\sigma)$ clearly preserves polarity. The construction preserves the innocence inherited from σ. We show receptivity. Suppose $x \in \mathcal{C}(S)$ and $\mathrm{wait}_a(\sigma)x \overset{a'}{\longrightarrow\!\!\subset}$ in A where a' has negative polarity. Consider first the case when $a' \in A_-$. Then it can be checked that $x \cup \{a'\} \in \mathcal{C}(S')$. This yields $x \overset{a'}{\longrightarrow\!\!\subset}$ with $\mathrm{wait}_a(\sigma)(a') = a'$. To show uniquesness, assume $\mathrm{wait}_a(\sigma)(s') = a'$. If $s' \notin S$ we obtain $\mathrm{wait}_a(\sigma)(s') = s' = a'$. If on the other hand, $s' \in S$ we obtain $\mathrm{wait}_a(\sigma)(s') = \sigma(s') = a' \in A_-$. By the observation, $s' \in S_-$ and $\sigma(s') = s'$ as σ satisfies (†), and again $s' = a'$.

In the case where $a' \notin A_-$ there must be $a_1 \leq_A a'$ with $pol(a_1) = +$. Hence there is $s_1 \in x$, with $pol(s_1) = +$, such that $\sigma(s_1) = a_1$. From the causal dependency of S' we must have $a \in x$. It follows that $x \setminus \{a\} \in \mathcal{C}(S)$ and $\sigma(x \setminus \{a\}) \overset{a'}{\longrightarrow\!\!\subset}$ in A/a, whereupon receptivity of σ ensures the required receptivity condition for $\mathrm{wait}_a(\sigma)$.

Proposition 10. *For $\sigma : S \to A$ a strategy and $x \in \mathcal{C}^\infty(S)$, the function $\sigma/s : S/s \to A/\sigma(s)$ is a strategy.*

Proof. A straightforward check.

Lemma 8. *For all $\sigma : S \to A$ and $\tau : T \to A^\perp$, then*

$$\kappa\langle\sigma, \tau\rangle = \{-v \mid v \in \kappa\langle\tau, \sigma\rangle\}$$

Proof. Straightforward.

For all the forthcoming lemmas, the well-founded hypothesis is not strictly necessary. However we keep it as it simplifies the proofs, and these lemmas will only be applied on well-founded games in order to get the theorem.

Lemma 9. *If the game A is well-founded and race-free,*

$$\kappa\langle \mathop{\|}_{i\in I} \sigma_i, \tau\rangle \subseteq \bigcup_{i\in I} \kappa\langle\sigma_i, \tau\rangle$$
$$\kappa\langle \mathop{\|}_{i\in I} \sigma_i, \tau\rangle^+ = \bigcup_{i\in I} \kappa\langle\sigma_i, \tau\rangle^+$$

Proof. First, we prove that $\kappa\langle\|_{i\in I} \sigma_i, \tau\rangle \subseteq \bigcup_{i\in I} \kappa\langle\sigma_i, \tau\rangle$. Take $y \in \langle\|_{i\in I} \sigma_i, \tau\rangle$. Necessarily, there is $z \in [\|_{i\in I} \sigma_i, \tau]$ such that $\sigma\Pi_1 z = y$. By definition of $\|_{i\in I} \sigma_i$, there is $i \in I$ such that $\Pi_1 z \in S_i$. It follows that $z \in [\sigma_i, \tau]$, therefore $y \in \langle\sigma_i, \tau\rangle$ as well.

Likewise if $y \in \langle \sigma_i, \tau \rangle$ with a positive event, take its witness $z \in [\sigma_i, \tau]$. Obviously $z \in \mathcal{C}(T) \odot \mathcal{C}(S')$ (where $\|_{i \in I} \sigma_i : S' \to A$). Maximality follows from that of z in $\mathcal{C}(T) \odot \mathcal{C}(S_i)$: indeed since y has a $+$-event this event is only consistent with events in S_i, hence any extension of z must be compatible with S_i.

Lemma 10. *If A is race-free and well-founded, then,*

$$\kappa\langle \mathrm{play}_a(\sigma), \tau \rangle = \kappa\langle \sigma, \tau/a \rangle$$

Proof. First we prove that $\kappa\langle \mathrm{play}_a(\sigma), \tau \rangle \subseteq \kappa\langle \sigma, \tau/a \rangle$. Take $y \in \langle \mathrm{play}_a(\sigma), \tau \rangle$ and its witness $z \in [\mathrm{play}_a(\sigma), \tau]$ such that $y = \mathrm{play}_a(\sigma) z$. The difficult part of the proof consist in proving that $a \in y$, let us start with that. Suppose that $a \notin y$. Obviously if $y \cup \{a\} \in \mathrm{Con}_A$, we have a contradiction with the maximality of z. Otherwise if $a \notin y$ but $y \cup \{a\} \notin \mathrm{Con}_A$, then consider a subconfiguration $y' \subseteq y$ that is minimal such that $y' \cup \{a\} \notin \mathrm{Con}_A$, that is, all the subconfigurations of y' are compatible with a. Necessarily y' is non-empty, otherwise it would be compatible with a, take $y'' {-\!\!\subset} y'$, write e the event such that $y'' \cup \{e\} = y'$. If $pol(e) = -$ then by race-freedom of A we have $y' \cup \{a\} \in \mathrm{Con}_A$ as well, contradiction, therefore $pol(e) = +$. Consider the witnesses $z', z'' \in [\mathrm{play}_a(\sigma), \tau]$ corresponding to y', y'', and $u' = \Pi_1 z'$, $u'' = \Pi_1 z''$, with $u' {-\!\!\subset} u''$ and $u' \cup \{s\} = u''$, with $\mathrm{play}_a(\sigma)(s) = e$. Then take $u'' \cap S$, it is still a configuration of S' (with $\mathrm{play}_a(\sigma) : S' \to A$). Necessarily we have $(\mathrm{play}_a(\sigma)(u'' \cap S)) \cup \{a\} \in \mathcal{C}(A)$ since σ is a strategy on A/a, and we also have $u'' \cap S \subseteq^- u''$ where $(\mathrm{play}_a(\sigma) u'') \cup \{a\} \notin \mathcal{C}(A)$, but this is forbidden by race-freedom of A, contradiction.

Therefore, $a \in y$. Then we have $(a, \overline{a}) \in z$. Set $z' = z \setminus \{(a, \overline{a})\}$, it is straightforward to check that $z' \in [\sigma, \tau/a]$ and $\sigma \Pi_1 z' = y \setminus \{a\}$, therefore $\kappa(\sigma \Pi_1 z') = \kappa(y)$ by definition of κ on A/a.

We now turn to the other inequality. Take $y \in \kappa\langle \sigma, \tau/a \rangle$ along with its witness $z \in [\sigma, \tau/a]$. Then it is straightforward to check that $z' = z \cup \{(a, \overline{a})\} \in [\mathrm{play}_a(\sigma), \tau]$ and $\kappa(\mathrm{play}_a(\sigma) z') = \kappa(y)$ by definition of κ on A/a.

Lemma 11. *We have the following* equalities *between strategies:*

$$\mathrm{play}_a(\sigma)/a = \sigma$$
$$\mathrm{wait}_a(\sigma)/a = \sigma$$

Proof. Trivial.

Lemma 12. *If A is well-founded and race-free, then,*

$$\kappa\langle \mathrm{play}_a(\sigma), \mathrm{wait}_a(\tau) \rangle = \kappa\langle \sigma, \tau \rangle$$

Proof. Trivial using Lemmas 10 and 11.

Lemma 13. *If A is well-founded and race-free, then,*

$$\kappa\langle \mathrm{wait}_a(\sigma), \tau\rangle \supseteq \bigcup_{t:\tau(t)=a} \kappa\langle \sigma, \tau/t\rangle$$

$$\kappa\langle \mathrm{wait}_a(\sigma), \tau\rangle^+ = \bigcup_{t:\tau(t)=a} \kappa\langle \sigma, \tau/t\rangle^+$$

Proof. We start with the left-to-right inclusion, take $y \in \langle \mathrm{wait}_a(\sigma), \tau\rangle$ (supposed to have positive events) along with its witness $z \in [\mathrm{wait}_a(\sigma), \tau]$. Since y has positive events it must contain a, as positive events in $\mathrm{wait}_a(\sigma) : S' \to A$ are set to depend on a. Therefore there is some $t \in T$ such that $\tau(t) = a$ and $(a, t) \in z$. Defining $z' = z \setminus (a, t)$, it is straightforward to prove that $z' \in [\sigma, \tau/t]$, and $\kappa(\sigma\pi_1 z) = \kappa(y)$ by definition of κ on A/a.

Reciprocally take $t \in T$ such that $\tau(t) = a$, and $y \in \langle \sigma, \tau/t\rangle$ with its witness $z \in [\sigma, \tau/t]$. Then it is straightforward to prove that $z' = z \cup (a, t) \in [\mathrm{wait}_a(\sigma), \tau]$, and $\kappa((\mathrm{wait}_a(\sigma))\pi_1 z') = \kappa(y)$ by definition of κ on A/a. Take $x \in \mathcal{C}(T)$ +-maximal and such that *pol* $x \subseteq \{+\}$ with $a \notin x$, then define $z = \{(\overline{e}, e) \mid e \in x\}$. Then it is straightforward to check that $z \in \mathcal{C}(T) \odot \mathcal{C}(S')$, and z is maximal: indeed $\pi_2 z = y$ is +-maximal, and $\pi_1 z$ is +-maximal as well by definition of $\mathrm{wait}_a(\sigma)$ since $a \notin x$. It follows that $z \in [\mathrm{wait}_a(\sigma), \tau]$ with as required $\kappa(\mathrm{wait}_a(\sigma))\pi_1 z = \kappa\overline{\tau x}$.

Corollary 1. *Setting* $\mathrm{case}_{i\in I}\sigma_i = \|_{i\in I} \mathrm{wait}_{a_i}(\sigma_i)$, *and if $\tau : T \to A^\perp$ is such that T has a minimal +-event, then.*

$$\kappa\langle \mathrm{case}_{i\in I}\sigma_i, \tau\rangle \subseteq \bigcup_{i\in I} \bigcup_{t:\tau(t)=a_i} \kappa\langle \sigma_i, \tau/t\rangle$$

$$\kappa\langle \mathrm{case}_{i\in I}\sigma_i, \tau\rangle^+ = \bigcup_{i\in I} \bigcup_{t:\tau(t)=a_i} \kappa\langle \sigma_i, \tau/t\rangle^+$$

If T has no +-minimal event, then $\kappa\langle \mathrm{case}_{i\in I}\sigma_i, \tau\rangle = \{\kappa(\emptyset)\}$.

Proof. We apply the following reasoning, putting all the previous lemmas together:

$$\begin{aligned}
\kappa\langle \mathrm{case}_{i\in I}\sigma_i, \tau\rangle &= \kappa\langle \|_{i\in I} \mathrm{wait}_{a_i}(\sigma_i), \tau\rangle \\
&\subseteq \bigcup_{i\in I} \kappa\langle \mathrm{wait}_{a_i}(\sigma_i), \tau\rangle \\
&\subseteq \bigcup_{i\in I}(\bigcup_{t:\tau(t)=a_i} \kappa\langle \sigma_i, \tau/t\rangle
\end{aligned}$$

All these inclusions become equalities when restricted to configurations with a positive event.

C Results of these constructions

Lemma 14. *For any well-founded race-free game A and $a \in A$ with $pol(a) = +$ such that $\emptyset \overset{a}{\longrightarrow\!\!\!\subset}$, for any strategy $\sigma : S \to A/a$, we have:*

$$v^{\uparrow}(\mathrm{play}_a(\sigma)) \leq v^{\uparrow}(\sigma)$$
$$v^{\downarrow}(\mathrm{play}_a(\sigma)) \leq v^{\downarrow}(\sigma)$$

Proof. First inequality:

$$v^{\uparrow}(\mathrm{play}_a(\sigma) \leq v^{\uparrow}(\sigma)$$

Let $\tau : T \to A^{\perp}/a$, and $z \in \langle \mathrm{play}_a \sigma, \mathrm{wait}_a \tau \rangle$. By Lemma 12, there is $z' \in \langle \sigma, \tau \rangle$ such that $\kappa(z) = \kappa(z')$.

Second inequality:

$$v^{\downarrow}(\mathrm{play}_a \sigma) \leq v^{\downarrow}(\sigma)$$

Let $\tau : T \to A^{\perp}/a$ and $z \in \langle \sigma, \tau \rangle$. Then by Lemma 12 there is $z' \in \langle \mathrm{play}_a \sigma, \mathrm{wait}_a \tau \rangle$ such that $\kappa(z) = \kappa(z')$.

Lemma 15. *For any well-founded race-free game A, $a \in A$ with $pol(a) = -$ such that $x \overset{a}{\longrightarrow\!\!\!\subset}$, for all strategy $\sigma : S \to A/x$, we have:*

$$v^{\uparrow}(\sigma) \leq v^{\uparrow}(\sigma/a)$$
$$v^{\downarrow}(\sigma) \leq v^{\downarrow}(\sigma/a)$$

Proof. First inequality:

$$v^{\uparrow}(\sigma) \leq v^{\uparrow}(\sigma/a)$$

Let $\tau : T \to A^{\perp}/(x \cup \{a\})$, and $z \in \langle \sigma, \mathrm{play}_a \tau \rangle$. By Lemma 10 there is $z' \in \langle \sigma/a, \tau \rangle$ such that $\kappa(z) = \kappa(z')$.

Second inequality:

$$v^{\downarrow}(\sigma) \leq v^{\downarrow}(\sigma/a)$$

Let $\tau : T \to A^{\perp}/(x \cup \{a\})$, and $z \in \langle \sigma/a, \tau \rangle$. Then by Lemma 10 there is $z' \in \langle \sigma, \mathrm{play}_a \tau \rangle$ such that $\kappa(z) = \kappa(z')$.

Lemma 16. *Suppose A is race-free, $x \in \mathcal{C}^{\infty}(A)$. Let $(f_j)_{j \in J}$ be the family of minimal $+$-events of A. Let $\sigma : S \to A$ be a strategy such that there is a minimal $+$-event $s \in S$. Then,*

$$v^{\downarrow}(\sigma) \leq \sup_{j \in J} v^{\downarrow}(\{f_j\})$$
$$v^{\uparrow}(\sigma) \leq \sup_{j \in J} v^{\uparrow}(\{f_j\})$$

Proof. Pessimistic case. Necessarily there must be $j_0 \in J$ such that $\sigma(s) = f_{j_0}$. Then, we are going to prove that

$$v^{\downarrow}(\sigma) \leq v^{\downarrow}(\sigma/s)$$

Indeed, take $\tau : A^{\perp}/f_{j_0}$, and $z \in \langle \sigma/s, \tau \rangle$. By Lemma 13, there is $z' \in \langle \sigma, \mathrm{wait}_{f_{j_0}}(\tau) \rangle$ such that $\kappa(z) = \kappa(z')$.

Optimistic case. Suppose that the inequality is false, *i.e.*

$$\sup_{j \in J} v^{\uparrow}(\{f_j\}) < v^{\uparrow}(\sigma)$$

This implies that there is $\alpha \in \mathbb{R}$ such that $\sup_{j \in J} v^{\uparrow}(\{f_j\}) < \alpha$ and $v^{\uparrow}(\sigma) > \alpha$. The first inequality implies:

$$\forall j \in J,\ \forall \sigma : A'/f_j,\ \exists \tau' : A^{\perp}/f_j,\ \forall z' \in \langle \sigma', \tau' \rangle,\ \kappa(z') < \alpha$$

which is easily shown to imply:

$$\forall (\sigma_k)_{k \in K},\ \exists (\tau_j)_{j \in \sigma K},\ \forall k \in K,\ \forall z' \in \langle \sigma_k, \tau_{\sigma k} \rangle,\ \kappa(z') < \alpha \tag{8}$$

where K is the set of positive minimal events in S. Applying this property to the family of strategies obtained by $\sigma_k = \sigma/k$, we get a family of counter-strategies $(\tau_j)_{j \in \sigma K}$. We extend this family to J by setting τ_j as the empty strategy (closed under receptivity) whenever $e_j \notin \sigma K$. Thus, we get a family $(\tau_j)_{j \in J}$.

Likewise, the second inequality implies that:

$$\forall \tau : A^{\perp},\ \exists z \in \langle \sigma, \tau \rangle,\ \kappa(z) > \alpha$$

Let us apply it to $\tau = \mathrm{case}_{j \in J} \tau_j$, we get $z \in \langle \sigma, \mathrm{case}_{j \in J} \tau_j \rangle$ such that $\kappa(z) > \alpha$. By Corollary 1, there is $k_0 \in K$, and $z' \in \langle \sigma/k_0, \tau_{\sigma k_0} \rangle$ such that $\kappa(z') = \kappa(z) > \alpha$. However, applying (1) to k_0 also shows that $\kappa(z') < \alpha$, contradiction. Hence, the required inequality is true.

Lemma 17. *Let A be a game, $(e_i)_{i \in I}$ the family of its negative minimal events. Then,*

$$\min(\kappa(\emptyset), \inf_{i \in I} \sup_{\sigma : A/e_i} v^{\downarrow}(\sigma)) \leq v^{\downarrow}(A)$$
$$\min(\kappa(\emptyset), \inf_{i \in I} \sup_{\sigma : A/e_i} v^{\uparrow}(\sigma)) \leq v^{\uparrow}(A)$$

Proof. For as long as possible, we do not distinguish the optimistic and pessimistic cases. Suppose that the inequality is false. It implies that there is $\alpha \in \mathbb{R}$ such that

$$\min(\kappa(\emptyset), \inf_{i \in I} \sup_{\sigma : A/e_i} v(\sigma)) > \alpha$$
$$v(A) < \alpha$$

which imply the following three propositions:

$$\kappa(\emptyset) > \alpha \tag{9}$$
$$\forall i \in I,\ \exists \sigma_i : A/e_i,\ \forall \tau : A^\perp/e_i,\ r(\sigma_i, \tau) > \alpha \tag{10}$$
$$\forall \sigma : A,\ \exists \tau : A^\perp,\ r(\sigma, \tau) < \alpha \tag{11}$$

In particular, (10) gives a family $(\sigma_i)_{i\in I}$. Instanciating (11) with $\mathrm{case}_{i\in I}\sigma_i$, we get $\tau : T \to A^\perp$ such that $r(\mathrm{case}_{i\in I}\sigma_i, \tau) < \alpha$.

$$\kappa(\emptyset) > \alpha \tag{12}$$
$$\forall i \in I,\ \forall t,\ \tau(t) = e_i \Rightarrow r(\sigma_i, \tau/t) > \alpha \tag{13}$$
$$r(\mathrm{case}_{i\in I}, \tau) < \alpha \tag{14}$$

Let us now distinguish the optimistic and pessimistic cases.

Pessimistic case. Since $r(\mathrm{case}_{i\in I}\sigma_i, \tau) < \alpha$, there must be $y \in \langle \mathrm{case}_{i\in I}\sigma_i, \tau\rangle$ such that $\kappa(y) < \alpha$. If T has no minimal $+$-event, then necessarily we have $y = \emptyset$, therefore $\kappa_A(y) = \kappa_A(\emptyset) > \alpha$, contradiction. Therefore, T has a minimal $+$-event. Then by Corollary 1 there is a minimal $+$-event $t \in T$ and $\tau(t) = e_{i_0}$ and $y' \in \langle \sigma_{i_0}, \tau/t\rangle$ such that $\kappa(y') = \kappa(y) < \alpha$. But this is absurd by (13), so we have found a contradiction.

Optimistic case. By (14) instanciated in the pessimistic case we have that for all $y \in \langle \mathrm{case}_{i\in I}, \tau\rangle$, $\kappa(y) < \alpha$. Take one such $y \in \langle \mathrm{case}_{i\in I}, \tau\rangle$ ($\langle \mathrm{case}_{i\in I}, \tau\rangle$ is non-empty by Zorn's lemma). As above, y cannot be empty as that would cause a contradiction, and T must have a minimal $+$-event. Therefore by Corollary 1 there is a minimal $+$-event $t \in T$ and $\tau(t) = e_{i_0}$ and $y' \in \langle \sigma_{i_0}, \tau/t\rangle$ such that $\kappa(y') = \kappa(y) < \alpha$, contradicting (13).

D Proof of the value theorem

Let A be a fixed well-founded and race-free game.

Lemma 18. *Let $x \in \mathcal{C}(A)$. Let $(e_i)_{i\in I}$ be the family of extensions of x of negative polarity, and $(f_j)_{j\in J}$ be the family of extensions of x of positive polarity. Then,*

$$v^\uparrow(x) = \max(\min(\kappa(x), \inf_{i\in I} v^\uparrow(x \cup \{e_i\})), \sup_{j\in J} v^\uparrow(x \cup \{f_j\}))$$
$$v^\downarrow(x) = \max(\min(\kappa(x), \inf_{i\in I} v^\downarrow(x \cup \{e_i\})), \sup_{j\in J} v^\downarrow(x \cup \{f_j\}))$$

Proof. The reasoning is the same in the optimistic and pessimistic cases, hence we do not distinguish them. We prove the first unequality:

$$v(x) \le \max(\min(\kappa(x), \inf_{i\in I} v(x \cup \{e_i\})), \sup_{j\in J} v(x \cup \{f_j\}))$$

Let $\sigma : S \to A/x$ be a strategy. If there is a minimal event $s \in S$ with $pol(s) = +$, then $v(x) \le v(\sigma) \le \sup_{j\in J} v(x \cup \{f_j\})$ by Lemma 16. Otherwise, there is no such minimal $s \in S$. Then $v(\sigma) \leq \kappa(x)$. Indeed, let $\tau : T \to A/x$ be the empty strategy closed by receptivity, we have $\langle \sigma, \tau \rangle = \{\emptyset\}$ and $r(\sigma, \tau) = \kappa(x)$. Likewise take $i_0 \in I$, by Lemma 15 we have $v(\sigma) \leq v(\sigma/e_{i_0})$, therefore $v(\sigma) \leq \inf_{i \in I} v(x \cup \{e_i\})$.

We now prove the other inequality:

$$\max(\min(\kappa(x), \inf_{i \in I} v(x \cup \{e_i\})), \sup_{j \in J} v(x \cup \{f_j\})) \leq v(x)$$

Let us prove that $\sup_{j \in J} v(x \cup \{f_j\}) \leq v(x)$, taking $j_0 \in J$ and $\sigma : A/(x \cup \{f_{j_0}\})$. Then by Lemma 14 we have $v(\mathrm{play}_{f_{j_0}} \sigma) \leq v(\sigma)$ and $v(\sigma) \leq v(x)$. Finally, we need to prove that $\min(\kappa(x), \inf_{i \in I} v(x \cup \{e_i\})) \leq v(x)$, but this is Lemma 17.

Theorem 4. *If A is well-founded and race-free, then the optimistic and pessimistic values coincide:*

$$v^{\uparrow}(A) = v^{\downarrow}(A)$$

This justifies writing $v(A)$ for the value *of a game.*

Proof. It is obvious from the lemma above that there cannot be a maximal $x \in \mathcal{C}(A)$ maximal such that $v^{\uparrow}(A/x) \neq v^{\downarrow}(A/x)$. Since A is well-founded, that must be true for the empty configuration.

Theorem 5. *If A is well-founded and race-free, then we have:*

$$v(A) = -v(A^{\perp})$$

Proof. Let $x \in \mathcal{C}(A)$ be maximal such that $v(A/x) = -v(A^{\perp}/x)$. Let $(e_i)_{i \in I}$ be the family of negative extensions of x and $(f_j)_{j \in J}$ its family of positive extensions. Then,

$$\begin{aligned}
v(A/x) &= \max(\min(\kappa_A(x), \inf_{i \in I} v(A/(x \cup \{e_i\}))), \sup_{j \in J} v(A/(x \cup \{f_j\}))) \\
&= \max(\min(-\kappa_{A^{\perp}}(x), \inf_{i \in I} -v(A^{\perp}/(x \cup \{e_i\}))), \sup_{j \in J} -v(A^{\perp}/(x \cup \{f_j\}))) \\
&= -\min(\max(\kappa_{A^{\perp}}(x), \sup_{i \in I} v(A^{\perp}/(x \cup \{e_i\}))), \inf_{j \in J} v(A^{\perp}/(x \cup \{f_j\}))) \\
&= -\max(\min(\kappa_{A^{\perp}}(x), \inf_{j \in J} v(A^{\perp}/(x \cup \{f_j\}))), \\
&\qquad \min(\sup_{i \in I} v(A^{\perp}/(x \cup \{e_i\})), \inf_{j \in J} v(A^{\perp}/(x \cup \{f_j\}))))
\end{aligned}$$

But for all $i_0 \in I$, $v(A^{\perp}/(x \cup \{e_{i_0}\})) \leq v(A^{\perp})$ by Lemma 14 and for all $j_0 \in J$, we have $v(A^{\perp}) \leq v(A^{\perp}/(x \cup \{f_{j_0}\}))$ by Lemma 15, therefore $\sup_{i \in I} v(A^{\perp}/(x \cup \{e_i\})) \leq \inf_{j \in J} v(A^{\perp}/(x \cup \{f_j\}))$, and:

$$\begin{aligned}
v(A/x) &= -\max(\min(\kappa_{A^{\perp}}(x), \inf_{j \in J} v(A^{\perp}/(x \cup \{f_j\}))), \inf_{j \in J} v(A^{\perp}/(x \cup \{f_j\}))) \\
&= -v(A^{\perp}/x)
\end{aligned}$$

Contradiction. Therefore there is no such maximal x and the property is true for the empty configuration, thus $v(A) = -v(A^{\perp})$ since A is well-founded.

E Proofs on compositionality of optimal strategies

Proposition 11. *Let A be a game and $\sigma : S \to A$ a strategy. Then,*

$$v^{\downarrow}(\sigma) = \inf\{\kappa(\sigma x) \mid x \in \mathcal{C}(S) \ \ +\text{-maximal}\}$$

Proof. $\leq$. It suffices to show:

$$\forall x \in \mathcal{C}(S) \ \ +\text{-maximal}, \ \exists \tau : T \to A^{\perp}, \ \exists y \in \langle \sigma, \tau \rangle, \ \kappa(y) \leq \kappa(\sigma x)$$

Thus, let $x \in \mathcal{C}(S)$ be $+$-maximal. Set $T = (\sigma x)^{\perp}$ with $\tau : T \to A^{\perp}$ acting as the identity on events. τ is obviously innocent but not necessarily receptive, consider its closure $\tau' : T' \to A^{\perp}$ by receptivity. Then, define:

$$z = \{(e, \overline{\sigma e}) \mid e \in x\}$$

It is straightforward to check that $z \in \mathcal{C}(S) \odot \mathcal{C}(T')$, and it is maximal since x is $+$-maximal and by construction of τ'. It follows that $\sigma \Pi_1 z = \sigma x \in \langle \sigma, \tau' \rangle$.

$\geq$. It suffices to show that for all $\tau : T \to A^{\perp}/x$ and $y \in \langle \sigma, \tau \rangle$ there exists a $+$-maximal $x \in \mathcal{C}(S)$ such that $\kappa(\sigma x) \leq \kappa(y)$. But for all such y there is $z \in \mathcal{C}(S) \odot \mathcal{C}(T)$ maximal such that $y = \sigma \Pi_1 z$. Set $x = \Pi_1 z$, since z is maximal x must be $+$-maximal, and $\kappa(\sigma x) = \kappa(y)$.

Proposition 12. *For strategies $\sigma : S \to A^{\perp} \parallel B$ and $\tau : T \to B^{\perp} \parallel C$, we have $v^{\downarrow}(\tau \odot \sigma) \geq v^{\downarrow}(\tau) + v^{\downarrow}(\sigma)$. Likewise for $\sigma : S \to A$ and $\tau : T \to B$, we have $v(\sigma \parallel \tau) = v(\sigma) + v(\tau)$.*

Proof. Straightforward using Proposition 11. *TODO:* Add the proof of the optimistic parallel composition.

Proposition 13. *For any race-free, well-founded games A and B, we have $v(A \parallel B) = v(A) + v(B)$.*

Proof. By the value theorem, it does not matter whether we work on the optimistic or pessimistic cases. By simplicity, let us pick the pessimistic one. Firstly, we prove that $v(A \parallel B) \geq v(A) + v(B)$. Indeed, let $\sigma : S \to A$ and $\tau : T \to B$ be strategies. Then, as needed we have $v^{\downarrow}(\sigma \parallel \tau) \geq v^{\downarrow}(\sigma) + v^{\downarrow}(\tau)$ by Proposition 12.

Moreover, this inequality also holds for $A^{\perp}$ and $B^{\perp}$, therefore $v(A^{\perp} \parallel B^{\perp}) \geq v(A^{\perp}) + v(B^{\perp})$, from which it follows that $v(A \parallel B) \leq v(A) + v(B)$ by the value theorem and the definition of the dual of games with payoff.

Theorem 6. *If $\sigma : S \to A^{\perp} \parallel B$ and $\tau : T \to B^{\perp} \parallel C$ are optimal strategies, so is $\tau \odot \sigma$. Moreover copycat is optimal, therefore there is a bicategory of concurrent games with payoff and optimal strategies.*

Proof. Suppose σ and τ optimal. We reason as follows:

$$\begin{aligned} v^{\downarrow}(\tau \odot \sigma) &\geq v^{\downarrow}(\sigma) + v^{\downarrow}(\tau) \\ &= v(A^{\perp} \parallel B) + v(B^{\perp} \parallel C) \\ &= v(A^{\perp} \parallel C) \end{aligned}$$

This implies that $v^{\downarrow}(\tau \odot \sigma) = v(A^{\perp} \parallel C)$, since a strict inequality would contradict the definition of $v(A^{\perp} \parallel C)$.

Copycat is optimal: take a $+$-maximal $x \in \mathcal{C}(A^{\perp} \parallel A)$. Necessarily, x has the form $y \cup \overline{y}$, where $y \in \mathcal{C}(A)$. Moreover, $\kappa_{A^{\perp} \parallel A}(x) = \kappa_A(y) - \kappa_A(y) = 0$, therefore by Proposition 11, we have $v^{\downarrow}(\gamma_A) = 0$. However we also have $v(A^{\perp} \parallel A) = v(A) - v(A) = 0$, therefore copycat is optimal.