3 Discrete Maths Seminars

In the following exercises you may use any results of the lecture notes provided you state them clearly.

Exercise 3.1

(i) Show that the set of all finite subsets of $\mathbb{N}$ is countable.

(ii) Show that the set $\mathbb{Q}$ of all rational numbers is countable.

(ii) Show that the set of irrational numbers is uncountable.

Exercise 3.2

Which of the following sets are finite, which are infinite but countable, and which are uncountable?

(i) $\{ f : \mathbb{N} \to \{0,1\} \mid \forall n \in \mathbb{N}. f(n) \leq f(n + 1) \}$

(ii) $\{ f : \mathbb{N} \to \{0,1\} \mid \forall n \in \mathbb{N}. f(2n) \neq f(2n + 1) \}$

(iii) $\{ f : \mathbb{N} \to \{0,1\} \mid \forall n \in \mathbb{N}. f(n) \neq f(n + 1) \}$

(iv) $\{ f : \mathbb{N} \to \mathbb{N} \mid \forall n \in \mathbb{N}. f(n) \leq f(n + 1) \}$

(v) $\{ f : \mathbb{N} \to \mathbb{N} \mid \forall n \in \mathbb{N}. f(n) \geq f(n + 1) \}$

Exercise 3.3

(i) Show that if a nonempty set $A$ is countable, then there is a surjection $f : \mathbb{N} \to A$.

(ii) Show that if there is a surjection $f : \mathbb{N} \to A$, where $A$ is a set, then $A$ is countable. [Hint: Produce an injection from $A$ to $\mathbb{N}$ by considering the least $n$ for which $f(n) = a$.]

Exercise 3.4

(i) Show that $\mathbb{Q} \times \mathbb{Q}$ is countable.

(ii) Deduce that any set of disjoint discs (i.e. circular areas which may or may not include their perimeter) in the plane $\mathbb{R} \times \mathbb{R}$ is countable. [You may assume that the rational numbers are dense in the real numbers in the sense that for any reals $r_1 < r_2$ there is a rational $q$ such that $r_1 < q < r_2$.]