2 Discrete Maths Seminars

Exercise 2.1 Products

(i) Prove

(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(b) $(A \times B) \cap (C \times D) = (A \cap C) \times (B \cap D)$

(c) $(A \times B) \cup (C \times D) \subseteq (A \cup C) \times (B \cup D)$
[Show the converse inclusion does not hold in general by exhibiting sets where it fails.]

(ii) Show that a set $\{\{a\}, \{a, b\}\}$ behaves as an ordered pair (a, b) should, *i.e.*

$$\{\{a\}, \{a, b\}\} = \{\{a'\}, \{a', b'\}\} \iff a = a' \ \& \ b = b' \ .$$

[Consider the two cases $a = b$ and $a \neq b$.] $\square$

Exercise 2.2 Relations and functions

(i) Let A and B be finite sets with m and n elements respectively.

(a) How many relations are there from A to B?

(b) How many functions are there from A to B?

(c) How many partial functions are there from A to B?

(ii) Let $A = \{1, 2, 3, 4\}$, $B = \{a, b, c, d\}$ and $C = \{x, y, z\}$. Let $R = \{(1, a), (2, d), (3, a), (3, b), (3, d)\}$ and $S = \{(b, x), (b, z), (c, y), (d, z)\}$. So $R \subseteq A \times B$ and $S \subseteq B \times C$. What is their composition $S \circ R$?

(iii) Show that the composition of relations is associative. $\square$

Exercise 2.3 Inverse image and direct image of a function

(i) Suppose $f : X \to Y$ is a function. Show f^{-1} preserves the Boolean operations of union, intersection and complement, *i.e.* for all $B, C \subseteq Y$,

$$\begin{array}{ll} f^{-1}(B \cup C) = (f^{-1}B) \cup (f^{-1}C) \ , & f^{-1}\emptyset = \emptyset \ , \\ f^{-1}(B \cap C) = (f^{-1}B) \cap (f^{-1}C) \ , & f^{-1}Y = X \ , \\ f^{-1}(B^c) = (f^{-1}B)^c \ . & \end{array}$$

(ii) What analogous properties hold of the direct image under a function f? Justify your answers by providing a proof or a counter-example. □

Exercise 2.4 Bijections and equivalence relations

(i) Show that the composition of injective functions is injective. Show that the composition of surjective functions is surjective. Deduce that the composition of bijective functions is bijective.

(ii) Let $\cong$ be a relation on a set of sets S such that $A \cong B$ iff the sets A and B in S are in bijective correspondence. Show that $\cong$ is an equivalence relation. □