1 Discrete Maths Seminars—Week 1

Exercise 1.1 Course-of-values induction

To prove a property $A(x)$ for all nonnegative numbers $x$ it suffices to show that

$$\forall k \ (0 \leq k < x). \ A(k) \Rightarrow A(x),$$

for all nonnegative numbers $x$.

(i) Explain how to derive the above principle, course-of-values induction, from mathematical induction.

(ii) Prove by course-of-values induction that every natural number $n \geq 2$ can be written as a product of prime numbers.

(iii)[Harder] Prove by course-of-values induction that every natural number $n \geq 2$ can be written uniquely as a product of prime numbers in ascending order. [A product of primes in ascending order is a product $p_1^{r_1} \cdots p_k^{r_k}$ where $p_1, \cdots, p_k$ are primes such that $p_1 < \cdots < p_k$ and $r_1, \cdots, r_k$ are natural numbers.]

Exercise 1.2 Venn diagrams

(i) Describe the set $A \cup B \cup C$ as a union of 7 disjoint sets (i.e., so each pair of sets has empty intersection).

(ii) In a college of 100 students, 35 play football, 36 row and 24 play tiddly-winks. 13 play football and row, 2 play football and tiddlywinks but never row, 12 row and play tiddlywinks, while 4 practice all three activities. How many students participate in none of the activities of football, rowing and tiddlywinks?

(iii) A standard notation for the size, i.e. number of elements, of a finite set $B$ is $|B|$.

(a) Explain why $|A \cup B| = |A| + |B| - |A \cap B|$, for finite sets $A$ and $B$.

(b) From part (a) derive a similar expression for $|A \cup B \cup C|$, the size of a union of finite sets $A$, $B$ and $C$, in terms of the sizes of $A$, $B$ and $C$, and their intersections.
Exercise 1.3 Validity

(i) By constructing its truth table, verify that Pierce’s law, 
\[(A \Rightarrow B) \Rightarrow A\] is a tautology for all Boolean propositions \(A\) and \(B\).

(ii) Prove for any subsets \(X\) and \(Y\) of \(U\) that
\[X^c \cup Y = U \iff X \subseteq Y\,.

(iii) Prove directly, without appeal to Proposition 1.14, that Pierce’s law is valid, i.e.: for any Boolean propositions \(A\) and \(B\)
\[\text{[(}(A \Rightarrow B) \Rightarrow A\text{)]}_\mathcal{M} = U_\mathcal{M}\,.

Exercise 1.4 Structural induction

We define the length of a Boolean proposition by structural induction as follows:
\[
l(a) = 1, \quad l(\text{true}) = 1, \quad l(\text{false}) = 1, \\
l(A \land B) = l(A) + l(B) + 1, \\
l(A \lor B) = l(A) + l(B) + 1, \quad l(\neg A) = l(A) + 1.
\]

We define a translation which eliminates disjunction from Boolean expressions by the following structural induction:
\[
tr(a) = a, \quad tr(\text{true}) = \text{true}, \quad tr(\text{false}) = \text{false}, \\
tr(A \land B) = tr(A) \land tr(B), \\
tr(A \lor B) = \neg(\neg tr(A) \land \neg tr(B)), \quad tr(\neg A) = \neg tr(A)\,.
\]

Prove by structural induction on Boolean propositions that
\[l(tr(A)) \leq 3.l(A) - 1\,,

for all Boolean propositions \(A\).

\[\square\]