1 Discrete Maths Seminars—Week 1

Exercise 1.1 Course-of-values induction

To prove a property $A(x)$ for all nonnegative numbers x it suffices to show that

$$[\forall k \ (0 \leq k < x).\ A(k)] \Rightarrow A(x),$$

for all nonnegative numbers x.

(i) Explain how to derive the above principle, course-of-values induction, from mathematical induction.

(ii) Prove by course-of-values induction that every natural number $n \geq 2$ can be written as a product of prime numbers.

(iii)[Harder] Prove by course-of-values induction that every natural number $n \geq 2$ can be written *uniquely* as a product of prime numbers in ascending order. [A product of primes in ascending order is a product $p_1^{r_1} \cdots p_k^{r_k}$ where $p_1, \cdots, p_k$ are primes such that $p_1 < \cdots < p_k$ and $r_1, \cdots, r_k$ are natural numbers.] □

Exercise 1.2 Venn diagrams

(i) Describe the set $A \cup B \cup C$ as a union of 7 disjoint sets (*i.e.*, so each pair of sets has empty intersection).

(ii) In a college of 100 students, 35 play football, 36 row and 24 play tiddlywinks. 13 play football and row, 2 play football and tiddlywinks but never row, 12 row and play tiddlywinks, while 4 practice all three activities. How many students participate in none of the activities of football, rowing and tiddlywinks?

(iii) A standard notation for the size, *i.e.* number of elements, of a finite set B is $|B|$.

(a) Explain why $|A \cup B| = |A| + |B| - |A \cap B|$, for finite sets A and B.

(b) From part (a) derive a similar expression for $|A \cup B \cup C|$, the size of a union of finite sets A, B and C, in terms of the sizes of A, B and C, and their intersections.

□

Exercise 1.3 Validity

(i) By constructing its truth table, verify that *Pierce's law*, $((A \Rightarrow B) \Rightarrow A) \Rightarrow A$, is a tautology for all Boolean propositions A and B.

(ii) Prove for any subsets X and Y of U that

$$X^c \cup Y = U \iff X \subseteq Y \ .$$

(iii) Prove directly, without appeal to Proposition 1.14, that Pierce's law is valid, *i.e.*: for any Boolean propositions A and B

$$[\![((A \Rightarrow B) \Rightarrow A) \Rightarrow A]\!]_{\mathcal{M}} = U_{\mathcal{M}} \ ,$$

for any model $\mathcal{M}$. □

Exercise 1.4 Structural induction

We define the length of a Boolean proposition by structural induction as follows:

$$\begin{aligned}
&l(a) = 1, \quad l(\mathbf{true}) = 1, \quad l(\mathbf{false}) = 1, \\
&l(A \wedge B) = l(A) + l(B) + 1, \\
&l(A \vee B) = l(A) + l(B) + 1, \quad l(\neg A) = l(A) + 1 \ .
\end{aligned}$$

We define a translation which eliminates disjunction from Boolean expressions by the following structural induction:

$$\begin{aligned}
&tr(a) = a, \quad tr(\mathbf{true}) = \mathbf{true}, \quad tr(\mathbf{false}) = \mathbf{false}, \\
&tr(A \wedge B) = tr(A) \wedge tr(B), \\
&tr(A \vee B) = \neg(\neg tr(A) \wedge \neg tr(B)), \quad tr(\neg A) = \neg tr(A) \ .
\end{aligned}$$

Prove by structural induction on Boolean propositions that

$$l(tr(A)) \leq 3.l(A) - 1 \ ,$$

for all Boolean propositions A. □