The Ball Monad and its Metric Trace Semantics

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Overview

- Trace semantics in order-enriched Kleisli categories
- Definition of the ball monad
- Metric trace semantics for the ball monad
An example of a non-deterministic automaton:

This automaton recognizes the language

\[ \{ u \in \{a, b\}^* \mid u \text{ contains } bb \text{ as a subword} \} \]
Non-deterministic automata

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Non-deterministic automata are coalgebras of the form

$$X \to 2 \times \mathcal{P}(X)^A \cong \mathcal{P}(1 + A \times X)$$

in the category **Sets**.
Non-deterministic automata are coalgebras for the functor $F(X) = \mathcal{P}(1 + A \times X)$ on $\textbf{Sets}$. This functor has no final coalgebra. Can we nevertheless model the language recognized by the automaton via finality?
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A coalgebra $X \to \mathcal{P}(1 + A \times X)$ in $\text{Sets}$ is the same as a coalgebra $X \to 1 + A \times X$ in $\text{Rel}$. We will seek for a final coalgebra in $\text{Rel}$. 
Non-deterministic automata

Theorem

The functor $1 + A \times X$ on $\text{Rel}$ has final coalgebra $A^*$. For each coalgebra $X \rightarrow 1 + A \times X$ in $\text{Rel}$, this gives a relation $X \rightarrow A^*$, hence a map $X \rightarrow \mathcal{P}(A^*)$.

Observe that $A^*$ is also the initial algebra for $1 + A \times X$. 
We consider coalgebras of the form $X \to TFX$, where:

- $T$ is a monad on $\text{Sets}$.
- $F$ is a functor on $\text{Sets}$.

Non-deterministic automata form an example, with $T = \mathcal{P}$ and $F(X) = 1 + A \times X$.

Coalgebras of the form $X \to TFX$ are morphisms $X \to FX$ in the Kleisli category $\mathcal{K}_{\ell}(T)$. We need to “lift” the functor $F : \text{Sets} \to \text{Sets}$ to a functor $\bar{F} : \mathcal{K}_{\ell}(T) \to \mathcal{K}_{\ell}(T)$. 
A distributive law $\lambda : FT \Rightarrow TF$ induces a lifting of $F : \text{Sets} \to \text{Sets}$ to $\bar{F} : \mathcal{K}\ell(T) \to \mathcal{K}\ell(T)$:

On objects: $\bar{F}(X) = F(X)$
On morphisms: $\bar{F}(X \overset{f}{\to} TY) = (FX \overset{Ff}{\to} FTY \overset{\lambda}{\to} TFY)$

Coalgebras for $TF$ in $\text{Sets}$ correspond to coalgebras for $\bar{F}$ in $\mathcal{K}\ell(T)$. 
Theorem

Suppose that:

- $T(0) = 1$, which implies that $\mathcal{K}ℓ(T)$ has a zero object
- $\mathcal{K}ℓ(T)$ is dcpo-enriched, in such a way that the zero maps are least elements in the Kleisli homsets
- The functor $F$ has an initial algebra $FA \xrightarrow{\alpha} A$
- The functor $\bar{F}$ is locally monotone:
  \[ f \leq g \Rightarrow \bar{F}(f) \leq \bar{F}(g) \]

Then $J(\alpha^{-1}) : A \rightarrow FA$ is a final coalgebra for $\bar{F}$ in $\mathcal{K}ℓ(T)$. 
Proof sketch

Given a coalgebra $X \xrightarrow{c} FX$ in $\mathcal{K}_\ell(T)$, we have to find a unique “trace map” $\text{tr}_c$ making the diagram on the left commute. In other words, the operator $c; \bar{F}(-); J(\alpha^{-1}) : \text{Hom}(X, A) \to \text{Hom}(X, A)$ should have a unique fixed point. The fixed point exists by the dcpo fixed point theorem. It is unique since $\alpha$ is an initial algebra.
### Analogy between dcpos and metric spaces

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Does the trace semantics for dcpo-enriched Kleisli categories have an analogue for metric spaces?
The Ball monad

Define the **ball monad** on objects as

\[
\mathcal{B}(X) = \left\{ \varphi : X \to \mathbb{C} \mid \sum_{x \in X} |\varphi(x)| \leq 1 \right\}
\]

An element \( \varphi \in \mathcal{B}(X) \) can also be written as a formal sum \( \sum_i c_i x_i \) with \( c_i \in \mathbb{C} \) and \( x_i \in X \).

On morphisms,

\[
\mathcal{B}(f)(\sum_i c_i x_i) = \sum_i c_i f(x_i)
\]
Let \textbf{Cms} be the category of complete metric spaces and non-expansive maps. \( f : X \to Y \) is non-expansive if

\[
d_Y(f(x), f(x')) \leq d_X(x, x')
\]

for all \( x, x' \in X \).

The set \( \mathcal{B}(Y) \) is a complete metric space with \( \ell^1 \) metric

\[
d(\varphi, \psi) = \sum_{y \in Y} |\varphi(y) - \psi(y)|
\]

Hence the space of functions \( \text{Hom}_{K\ell(\mathcal{B})}(X, Y) = X \to \mathcal{B}(Y) \) also forms a complete metric space with supremum metric. Pre- and post-composition are non-expansive, so \( K\ell(\mathcal{B}) \) is enriched over \textbf{Cms}. 
Theorem

Let $F$ be a polynomial functor on $\textbf{Sets}$ with initial algebra $FA \xrightarrow{\alpha} A$, and let $\lambda : FB \Rightarrow BF$ be a distributive law. Then $J(\alpha^{-1}) : A \to FA$ is a final coalgebra for $\overline{F} : K\ell(B) \to K\ell(B)$. 
We wish to prove that the map

\[ \text{iter}_c = c; \tilde{F}(-); J(\alpha) : \text{Hom}(X, A) \rightarrow \text{Hom}(X, A) \]

has a unique fixed point, using Banach's theorem. Unfortunately, \( \text{iter}_c \) is not a contraction. Therefore we modify the metric on \( \text{Hom}(X, A) \).
The initial algebra $A$ can be obtained as a colimit:

$$0 \longrightarrow F0 \longrightarrow F^20 \longrightarrow \cdots \longrightarrow A$$

Define $\# : A \to \mathbb{N}$ by $\#a = \max\{n \mid a \notin \text{im}(\kappa_n)\}$. Then define the metric on $\text{Hom}(X, A)$ by

$$d(\varphi, \psi) = \sup_{x \in X} \sum_{a \in A} \frac{1}{2^\#a} \cdot |\varphi(x)(a) - \psi(x)(a)|$$

Apply Banach’s fixed point theorem to find the unique fixed point.
Concluding remarks

- We have extended coalgebraic trace semantics to include the ball monad.
- This approach uses the fact that the Kleisli category of the ball monad is enriched over metric spaces, instead of partial orders.

Future research:
- Generalize this result to arbitrary metric-enriched Kleisli categories.
- Describe trace semantics for quantum computation.