Coalgebraic Trace Semantics for the Ball Monad

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Coalgebras form a uniform framework for studying various kinds of dynamical systems.

Outline:

- Introduction to the theory of coalgebras.
- Trace semantics as a tool to describe the behaviour of non-deterministic systems.
- Trace semantics for systems with transitions given by complex probabilities.
What is a system?

A system consists of states and dynamics. This can be formalized as a coalgebra.

**Definition**

Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor. A coalgebra for $F$ is a morphism $X \xrightarrow{c} F(X)$ in $\mathcal{C}$. The object $X$ is called the *state space* of the coalgebra. The morphism $c$ is called the *dynamics*. 
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Let \( F : \mathbb{C} \rightarrow \mathbb{C} \) be an endofunctor. A coalgebra for \( F \) is a morphism \( X \xrightarrow{c} F(X) \) in \( \mathbb{C} \). The object \( X \) is called the state space of the coalgebra. The morphism \( c \) is called the dynamics.

Coalgebras for \( F \) form a category with commutative squares

\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
\downarrow{c} & & \downarrow{d} \\
F(X) & \xrightarrow{F(f)} & F(Y)
\end{array}
\]

as morphisms.
Consider the endofunctor $F(X) = A \times X$ on Sets.

An $F$-coalgebra is a map $X \xrightarrow{\langle f, g \rangle} A \times X$.
Consider the endofunctor $F(X) = A \times X$ on **Sets**.

An $F$-coalgebra is a map $\langle f, g \rangle : X \rightarrow A \times X$.

What is the observable behaviour of this coalgebra?

\[
\begin{align*}
X & \xrightarrow{g} X & X & \xrightarrow{g} X & X & \xrightarrow{g} \cdots \\
\downarrow f & \quad & \downarrow f & \quad & \downarrow f & \\
A & \quad & A & \quad & A & \\
\end{align*}
\]

\[
(f(x), \ f(g(x)), \ f(g^2(x)), \ \ldots) \in A^\mathbb{N}
\]
The set of possible behaviours $A^\mathbb{N}$ also forms an $F$-coalgebra:

$$
\begin{array}{c}
A^\mathbb{N} \\
\langle \text{head}, \text{tail} \rangle
\end{array} \xrightarrow{\text{beh}}

\begin{array}{c}
A \times A^\mathbb{N} \\
\langle a_0, (a_1, a_2, \ldots) \rangle
\end{array}

Elements of $A^\mathbb{N}$ are called streams over $A$. The coalgebra of streams is a final object in the category of $F$-coalgebras. The unique morphism into $A^\mathbb{N}$ is

$$
\begin{array}{c}
X \xrightarrow{\text{beh}}

\begin{array}{c}
A^\mathbb{N} \\
\langle \text{head}, \text{tail} \rangle
\end{array}
\end{array}

\begin{array}{c}
A \times X \xrightarrow{\text{beh}}

\begin{array}{c}
A \times A^\mathbb{N}
\end{array}
\end{array}

$$

where $\text{beh} : x \mapsto (f(x), f(g(x)), f(g^2(x)), \ldots)$. 

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Coalgebraic Trace Semantics for the Ball Monad
Formal languages

- An alphabet is a non-empty finite set of symbols.
- If $A$ is an alphabet, then $A^*$ denotes the set of all words over $A$. The set $A^*$ forms a free monoid.
  Operation: concatenation.
  Identity element: the empty word $\varepsilon$.
- A language over $A$ is a subset of $A^*$. 

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An example of a deterministic finite automaton:

The language recognized by this automaton is \(\{a^m(ba)^n | m, n \in \mathbb{N}\}\).
Second example: automata

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An automaton consists of:

- A set $X$ of states
- A transition function $\delta : X \times A \rightarrow X$
- An initial state $x_0 \in X$
- A set of accepting states $U \subseteq X$
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An automaton is a coalgebra for the functor $F(X) = X^A \times 2$. 
Second example: automata

An automaton is a coalgebra for $F(X) = X^A \times 2$. The underlying set of the final $F$-coalgebra is the set $\mathcal{P}(A^*)$ of all languages over $A$. The behaviour morphism assigns to a state $x_0$ of an automaton $X$ the language recognized by $X$ if $x_0$ is the initial state.
An automaton is a coalgebra for $F(X) = X^A \times 2$. The underlying set of the final $F$-coalgebra is the set $\mathcal{P}(A^*)$ of all languages over $A$. The behaviour morphism assigns to a state $x_0$ of an automaton $X$ the language recognized by $X$ if $x_0$ is the initial state. Coalgebra structure of $\mathcal{P}(A^*)$:

\[
\begin{align*}
\mathcal{P}(A^*) \times A & \rightarrow \mathcal{P}(A^*) \\
(L, a) & \mapsto L_a = \{ u \in A^* \mid au \in L \}
\end{align*}
\]

\[
\begin{align*}
U & \subseteq \mathcal{P}(A^*) \\
U & = \{ L \in \mathcal{P}(A^*) \mid \varepsilon \in L \}
\end{align*}
\]
Question: given a language $L$, what is the minimal automaton recognizing $L$?

Answer: The subcoalgebra of $P(A^*)$ generated by $L$: $\langle L \rangle = \{ L^u | u \in A^* \}$, with initial state $L$.

Proof. The coalgebra $\langle L \rangle$ recognizes $L$: the inclusion $\langle L \rangle \hookrightarrow \rightarrow P(A^*)$ is a coalgebra morphism. It is unique by finality of $P(A^*)$, so it must be the behaviour morphism. If $X$ is any automaton with initial state $x_0$ recognizing $L$, then $\text{beh}(x_0) = L$. Since $\text{beh}$ is a coalgebra morphism, $\text{beh}(\langle x_0 \rangle) = \langle \text{beh}(x_0) \rangle = \langle L \rangle$. Hence $\#X \geq \#\langle L \rangle$. 

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Minimal realization

Question: given a language $L$, what is the minimal automaton recognizing $L$?
Answer: The subcoalgebra of $\mathcal{P}(A^*)$ generated by $L$:

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Non-deterministic automata

This automaton recognizes the language

\[ \{ u \in \{a, b\}^* \mid u \text{ contains } bb \text{ as a subword} \} \]

Non-deterministic automata are coalgebras of the form

\[ X \rightarrow 2 \times \mathcal{P}(X)^A \cong \mathcal{P}(1 + A \times X) \]
Non-deterministic automata are coalgebras for the functor $F(X) = \mathcal{P}(1 + A \times X)$. This functor has no final coalgebra. This follows from:

**Lemma (Lambek)**

If $\chi \xrightarrow{c} F(X)$ is a final $F$-coalgebra, then $c$ is an isomorphism.

Nevertheless we would like to describe the semantics of non-deterministic automata coalgebraically.
We consider coalgebras of the form $X \to TFX$, where:

- $T$ is a monad on $\textbf{Sets}$.
- $F$ is a functor on $\textbf{Sets}$.

Non-deterministic automata form an example, with $T = \mathcal{P}$ and $F(X) = 1 + A \times X$. 

Given a monad \((T, \eta, \mu)\) on \textbf{Sets}, we define the \textbf{Kleisli category} \(K\ell(T)\) as follows:

- The objects of \(K\ell(T)\) are sets.
- A morphism \(X \rightarrow Y\) in \(K\ell(T)\) is a function \(X \rightarrow TY\).
- The identity on \(X\) is \(X \xrightarrow{\eta} TX\).
- The composition of \(X \xrightarrow{f} TY\) and \(Y \xrightarrow{g} TZ\) is

\[
\begin{align*}
X \xrightarrow{f} TY \xrightarrow{Tg} T^2Z \xrightarrow{\mu} TZ
\end{align*}
\]

We write \(f; g\) for the composition.
Adjunction for a Kleisli category

\[ \mathcal{Kl}(T) \]

\[ J: \text{Sets} \xrightarrow{\dashv} \mathcal{Kl}(T) \]

\[ J(X) = X \text{ on objects} \]

\[ J( X \xrightarrow{f} Y ) = ( X \xrightarrow{f} Y \xrightarrow{\eta} TY ) \text{ on arrows} \]

\[ T: \text{Sets} \xrightarrow{\dashv} \mathcal{Kl}(T) \]

\[ T(X) = TX \text{ on objects} \]

\[ T( X \xrightarrow{f} TY ) = ( TX \xrightarrow{Tf} T^2 Y \xrightarrow{\mu} TY ) \text{ on arrows} \]
Coalgebras of the form $X \to TFX$ are morphisms $X \to FX$ in $\mathcal{Kl}(T)$.

Idea: look for a final coalgebra in the Kleisli category.

We need to “lift” the functor $F : \mathbf{Sets} \to \mathbf{Sets}$ to a functor $\bar{F} : \mathcal{Kl}(T) \to \mathcal{Kl}(T)$. 
A **distributive law** is a natural transformation $\lambda : FT \Rightarrow TF$ for which the following diagrams commute:

\[
\begin{array}{ccc}
F & \xrightarrow{\eta_F} & TF \\
\downarrow F\eta & & \downarrow \lambda \\
FT & & \text{FT}
\end{array}
\quad \begin{array}{ccc}
FT^2 & \xrightarrow{\lambda_T} & TFT \\
\downarrow F\mu & & \downarrow T\lambda \\
FT & \xrightarrow{\lambda} & TF \\
\downarrow \mu_F & & \\
TF & & \text{TF}
\end{array}
\]
A distributive law \( \lambda : FT \Rightarrow TF \) induces a lifting of \( F : \text{Sets} \to \text{Sets} \) to \( \bar{F} : \mathcal{K}\ell(\mathcal{T}) \to \mathcal{K}\ell(\mathcal{T}) \):

On objects: \( \bar{F}(X) = F(X) \)

On morphisms: \( \bar{F}(X \xrightarrow{f} TY) = (FX \xrightarrow{Ff} FTY \xrightarrow{\lambda} TFY) \)

Coalgebras for \( TF \) in \( \text{Sets} \) correspond to coalgebras for \( \bar{F} \) in \( \mathcal{K}\ell(\mathcal{T}) \).
Trace semantics in dcpo-enriched Kleisli categories

**Theorem**

Suppose that:

1. $T(0) = 1$, which implies that $\mathcal{Kl}(T)$ has a zero object
2. $\mathcal{Kl}(T)$ is dcpo-enriched, in such a way that the zero maps are least elements in the Kleisli homsets
3. The functor $F$ has an initial algebra $FA \xrightarrow{\alpha} A$
4. The functor $\bar{F}$ is locally monotone:
   
   $$f \leq g \Rightarrow \bar{F}(f) \leq \bar{F}(g)$$

Then $J(\alpha^{-1}) : A \to FA$ is a final coalgebra for $\bar{F}$ in $\mathcal{Kl}(T)$. 
We look at non-deterministic automata $X \rightarrow \mathcal{P}(1 + A \times X)$. The Kleisli category $\mathcal{K}\ell(\mathcal{P})$ is isomorphic to $\textbf{Rel}$, the category of sets and relations. Relations, ordered by inclusion, form a dcpo, so the theorem can be applied. The initial algebra for $F(X) = 1 + A \times X$ is $A^*$. This is also the underlying set of the final coalgebra in $\mathcal{K}\ell(\mathcal{P})$. Hence, for each non-deterministic automaton $X \xrightarrow{c} \mathcal{P}(1 + A \times X)$, there is a unique coalgebra morphism $X \rightarrow \mathcal{P}(A^*)$. 

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Given a coalgebra $X \xrightarrow{\cdot} FX$ in $\mathcal{K}\ell(T)$, we have to find a unique “trace map” $\text{tr}_c$ making the diagram on the left commute. In other words, the operator $c; \bar{F}(-); J(\alpha^{-1}) : \text{Hom}(X, A) \to \text{Hom}(X, A)$ should have a unique fixed point. The fixed point exists by the dcpo fixed point theorem. It is unique since $\alpha$ is an initial algebra.
Analogy between dcpos and metric spaces

<table>
<thead>
<tr>
<th>Dcpo</th>
<th>Complete metric space</th>
</tr>
</thead>
<tbody>
<tr>
<td>Continuous function</td>
<td>Contraction</td>
</tr>
<tr>
<td>Dcpo fixed point theorem</td>
<td>Banach fixed point theorem</td>
</tr>
</tbody>
</table>

Does the trace semantics for dcpo-enriched Kleisli categories have an analogue for metric spaces?
Define the **ball monad** on objects as

\[ B(X) = \left\{ \varphi : X \to \mathbb{C} \mid \sum_{x \in X} |\varphi(x)| \leq 1 \right\} \]

An element \( \varphi \in B(X) \) can also be written as a formal sum \( \sum_i c_i x_i \) with \( c_i \in \mathbb{C} \) and \( x_i \in X \).

On morphisms,

\[ B(f)(\sum_i c_i x_i) = \sum_i c_i f(x_i) \]
The Kleisli category of the Ball monad

Let \textbf{Cms} be the category of complete metric spaces and non-expansive maps. \( f : X \to Y \) is non-expansive if

\[
d_Y(f(x), f(x')) \leq d_X(x, x')
\]

for all \( x, x' \in X \).

The set \( B(Y) \) is a complete metric space with \( \ell^1 \) metric

\[
d(\varphi, \psi) = \sum_{y \in Y} |\varphi(y) - \psi(y)|
\]

Hence the space of functions \( \text{Hom}_{\mathcal{K}l(B)}(X, Y) = X \to B(Y) \) also forms a complete metric space with supremum metric. Pre- and post-composition are non-expansive, so \( \mathcal{K}l(B) \) is enriched over \textbf{Cms}. 
Theorem

Let $F$ be a polynomial functor on $\textbf{Sets}$ with initial algebra $FA \xrightarrow{\alpha} A$, and let $\lambda : F \mathcal{B} \Rightarrow \mathcal{B}F$ be a distributive law. Then $J(\alpha^{-1}) : A \to FA$ is a final coalgebra for $\overline{F} : \mathcal{K}(\mathcal{B}) \to \mathcal{K}(\mathcal{B})$. 
We wish to prove that the map

$$\text{iter}_c = c; \tilde{F}(-); J(\alpha) : \text{Hom}(X, A) \to \text{Hom}(X, A)$$

has a unique fixed point, using Banach's theorem. Unfortunately, \(\text{iter}_c\) is not a contraction. Therefore we modify the metric on \(\text{Hom}(X, A)\).
Proof sketch

The initial algebra \( A \) can be obtained as a colimit:

\[
\begin{array}{cccc}
0 & \rightarrow & F0 & \rightarrow F^20 & \rightarrow & \cdots \\
& & \kappa_0 & \rightarrow & \kappa_1 & \rightarrow & \cdots \\
& & \kappa_1 & \rightarrow & \kappa_2 & \rightarrow & A
\end{array}
\]

Define \( \# : A \rightarrow \mathbb{N} \) by \( \#a = \max\{n \mid a \notin \text{im}(\kappa_n)\} \). Then define the metric on \( \text{Hom}(X, A) \) by

\[
d(\varphi, \psi) = \sup_{x \in X} \sum_{a \in A} \frac{1}{2^{\#a}} \cdot |\varphi(x)(a) - \psi(x)(a)|
\]

Apply Banach’s fixed point theorem to find the unique fixed point.
Coalgebras generalize several kinds of systems. The behaviour of a coalgebra is given by the unique morphism into the final coalgebra.

An initial algebra gives rise to a final coalgebra in a Kleisli category, provided the Kleisli category is dcpo-enriched.

This also works for the Ball monad, using enrichment in complete metric spaces.