Computation Theory Supervision 2

1. Suppose that $A$ and $B$ are subsets of $\mathbb{N}$. Let $f : \mathbb{N} \to \mathbb{N}$ be a register machine computable function satisfying: for all $x \in \mathbb{N}$, $x$ is an element of $A$ if and only if $f(x)$ is an element of $B$. Show that $A$ is register machine decidable if and only if $B$ is.

2. Suppose that $f$ and $g$ are register machine computable partial functions $\mathbb{N} \rightarrow \mathbb{N}$, such that

$$\{x \in \mathbb{N} \mid f(x) \downarrow\} = \{x \in \mathbb{N} \mid g(x) \uparrow\}.$$

Show that $\{x \in \mathbb{N} \mid f(x) \downarrow\}$ is decidable.

3. Are the following subsets of $\mathbb{N}$ decidable? Justify your answer.

   (a) The set of numbers $e \in \mathbb{N}$ for which $17 \in \text{im}(\varphi_e)$.
   (b) The set of numbers $e \in \mathbb{N}$ for which, when executing the register machine $\text{prog}(e)$ with input 0, the content of register $R_5$ remains unchanged during the entire execution.
   (c) The set of codes $\langle e, e' \rangle$ of pairs of numbers $e$ and $e'$ satisfying $\varphi_e = \varphi_{e'}$.

4. Construct a Turing machine that ends in an accepting state if its input tape is of the form $\triangleright 0^n 1^n$ for some $n \in \mathbb{N}$, and ends in a rejecting state otherwise. You may assume that that the input tape always consists of the left endmarker $\triangleright$, followed by a string of zeroes and ones, followed by only blank symbols $\_$. 

5. Construct a Turing machine that computes the addition function $+: \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$.