Equations of Motion of a Free Bell and Clapper

The equation for a bell is:

$$I_b\ddot{\theta} + K_b\dot{\theta} + Mga\sin\theta = f(\theta, t)$$

The equation for a clapper is:

$$I_c(\ddot{\theta} + \ddot{\phi}) + K_c\dot{\phi} + mgb\sin(\theta + \phi) + mrb\sin(\phi - \alpha)\dot{\theta}^2 + mrb\cos(\phi - \alpha)\ddot{\theta} = 0$$

Where:

$$\begin{aligned}
t &= \text{time} \\
\theta &= \text{angular offset of bell from the downward vertical} \\
\phi &= \text{angular offset of clapper from the central axis of bell} \\
\alpha &= \text{angular measure of odd-struckness (see below)} \\
I_b, I_c &= \text{moments of inertia of bell and clapper} \\
K_b, K_c &= \text{friction coefficients for bell and clapper} \\
M, m &= \text{masses of bell and clapper} \\
g &= \text{acceleration due to gravity} \\
a &= \text{distance of C of G of bell from its swing axis} \\
b &= \text{distance of C of G of clapper from its swing axis} \\
r &= \text{separation of the two swing axes (usually positive)} \\
f &= \text{couple exerted on bell via rope and wheel}
\end{aligned}$$

When the bell is at rest mouth downwards, the clapper swing axis is taken to be $r\cos\alpha$ below the bell swing axis and displaced $r\sin\alpha$ to one side.
Of course, the clapper swing is limited and the second equation applies only when the clapper is moving relative to the bell. There has to be a constraint such as:

$$-\phi_1 \leqslant \phi \leqslant \phi_2$$

The equations can be massaged into a more convenient form by taking:

$$l_b = \frac{I_b}{Ma} \qquad l_c = \frac{I_c}{mb} \qquad k_b = \frac{K_b}{Ma} \qquad k_c = \frac{K_c}{mb}$$

Note that l_b and l_c are the simple equivalent lengths of the bell and clapper respectively.
The equations become:

$$\ddot{\theta} + \frac{k_b}{l_b}\dot{\theta} + \frac{g}{l_b}\sin\theta = F(\theta, t)$$

and:

$$(\ddot{\theta} + \ddot{\phi}) + \frac{k_c}{l_c}\dot{\phi} + \frac{g}{l_c}\sin(\theta + \phi) + \frac{r}{l_c}\sin(\phi - \alpha)\dot{\theta}^2 + \frac{r}{l_c}\cos(\phi - \alpha)\ddot{\theta} = 0$$

Note that $F = f/I_b$ but, if all you are interested in is simulating a freely swinging bell, you can take $F = 0$ and, if you ignore friction and have no odd-struckness, the equations become:

$$\ddot{\theta} + \frac{g}{l_b} \sin\theta = 0$$

and:

$$(\ddot{\theta} + \ddot{\phi}) + \frac{g}{l_c} \sin(\theta + \phi) + \frac{r}{l_c} \sin\phi\, \dot{\theta}^2 + \frac{r}{l_c} \cos\phi\, \ddot{\theta} = 0$$

While making simplifying assumptions, you can note that the behaviour is fairly insensitive to clapper throw and you can normally take:

$$\phi_1 = \phi_2 = \frac{\pi}{8}$$

It is these latest equations that are probably the best to work with but you might note that you can simplify even further if you scale time t to pseudo-time τ where:

$$\tau = \frac{t}{\sqrt{l_b}}$$

The equations now become:

$$\ddot{\theta} + g \sin\theta = 0$$

and:

$$(\ddot{\theta} + \ddot{\phi}) + g\frac{l_b}{l_c} \sin(\theta + \phi) + \frac{r}{l_c} \sin\phi\, \dot{\theta}^2 + \frac{r}{l_c} \cos\phi\, \ddot{\theta} = 0$$

Differentiation in these equations is with respect to τ (which of course doesn't have the dimension of time).

The beauty of this form is that you can see that just two independent parameters, l_b/l_c and r/l_c describe any bell. Moreover, it is pretty easy to determine l_b, l_c and r from a real bell. You just time small swings for the first two and use the equation for the period of a simple pendulum; for r you need a tape measure.

If you really want to take friction into account then you can determine the values of k_b/l_b and k_c/l_c (see the equations on the first page) for a real bell from:

$$\frac{k_b}{l_b} = \sqrt{\frac{g}{l_b}}\, \frac{\log_e 2}{n_b \pi} \qquad \frac{k_c}{l_c} = \sqrt{\frac{g}{l_c}}\, \frac{\log_e 2}{n_c \pi}$$

Where:

$n_b =$ number of swings for the bell amplitude to fall by a half

$n_c =$ number of swings for the clapper amplitude to fall by a half

Even in the simplified form, the equations give results which compare very well with observations on real bells.

F.H. King
The University Bellringer
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