Equations of Motion of a Free Bell and Clapper

The equation for a bell is:

\[ I_b \ddot{\theta} + K_b \dot{\theta} + M g a \sin \theta = f(\theta, t) \]

The equation for a clapper is:

\[ I_c (\ddot{\phi} + \dot{\phi}) + K_c \dot{\phi} + m g b \sin(\theta + \phi) + m r b \sin(\phi - \alpha) \dot{\phi}^2 + m r b \cos(\phi - \alpha) \dot{\phi} = 0 \]

Where:

- \( t \) = time
- \( \theta \) = angular offset of bell from the downward vertical
- \( \phi \) = angular offset of clapper from the central axis of bell
- \( \alpha \) = angular measure of odd-struckness (see below)
- \( I_b, I_c \) = moments of inertia of bell and clapper
- \( K_b, K_c \) = friction coefficients for bell and clapper
- \( M, m \) = masses of bell and clapper
- \( g \) = acceleration due to gravity
- \( a \) = distance of C of G of bell from its swing axis
- \( b \) = distance of C of G of clapper from its swing axis
- \( r \) = separation of the two swing axes (usually positive)
- \( f \) = couple exerted on bell via rope and wheel

When the bell is at rest mouth downwards, the clapper swing axis is taken to be \( r \cos \alpha \) below the bell swing axis and displaced \( r \sin \alpha \) to one side.

Of course, the clapper swing is limited and the second equation applies only when the clapper is moving relative to the bell. There has to be a constraint such as:

\[ -\phi_1 \leq \phi \leq \phi_2 \]

The equations can be massaged into a more convenient form by taking:

\[ l_b = \frac{I_b}{M a} \quad l_c = \frac{I_c}{m b} \quad k_b = \frac{K_b}{M a} \quad k_c = \frac{K_c}{m b} \]

Note that \( l_b \) and \( l_c \) are the simple equivalent lengths of the bell and clapper respectively.

The equations become:

\[ \ddot{\theta} + \frac{k_b}{l_b} \dot{\theta} + \frac{g}{l_b} \sin \theta = F(\theta, t) \]

and:

\[ (\ddot{\phi} + \dot{\phi}) + \frac{k_c}{l_c} \dot{\phi} + \frac{g}{l_c} \sin(\theta + \phi) + \frac{r}{l_c} \sin(\phi - \alpha) \dot{\phi}^2 + \frac{r}{l_c} \cos(\phi - \alpha) \dot{\phi} = 0 \]
Note that $F = f/I_b$ but, if all you are interested in is simulating a freely swinging bell, you can take $F = 0$ and, if you ignore friction and have no odd-struckness, the equations become:

$$\ddot{\theta} + \frac{g}{l_b} \sin \theta = 0$$

and:

$$(\ddot{\theta} + \dot{\phi}) + \frac{g}{l_c} \sin(\theta + \phi) + \frac{r}{l_c} \sin \phi \dot{\theta}^2 + \frac{r}{l_c} \cos \phi \dot{\theta} = 0$$

While making simplifying assumptions, you can note that the behaviour is fairly insensitive to clapper throw and you can normally take:

$$\phi_1 = \phi_2 = \frac{\pi}{8}$$

It is these latest equations that are probably the best to work with but you might note that you can simplify even further if you scale time $t$ to pseudo-time $\tau$ where:

$$\tau = \frac{t}{\sqrt{l_b}}$$

The equations now become:

$$\ddot{\theta} + g \sin \theta = 0$$

and:

$$(\ddot{\theta} + \dot{\phi}) + g\frac{l_b}{l_c} \sin(\theta + \phi) + \frac{r}{l_c} \sin \phi \dot{\theta}^2 + \frac{r}{l_c} \cos \phi \dot{\theta} = 0$$

Differentiation in these equations is with respect to $\tau$ (which of course doesn’t have the dimension of time).

The beauty of this form is that you can see that just two independent parameters, $l_b/l_c$ and $r/l_c$ describe any bell. Moreover, it is pretty easy to determine $l_b$, $l_c$ and $r$ from a real bell. You just time small swings for the first two and use the equation for the period of a simple pendulum; for $r$ you need a tape measure.

If you really want to take friction into account then you can determine the values of $k_b/l_b$ and $k_c/l_c$ (see the equations on the first page) for a real bell from:

$$\frac{k_b}{l_b} = \frac{\sqrt{g \log_e 2}}{l_b \frac{n_b\pi}{2}} \quad \frac{k_c}{l_c} = \frac{\sqrt{g \log_e 2}}{l_c \frac{n_c\pi}{2}}$$

Where:

$n_b = \text{number of swings for the bell amplitude to fall by a half}$

$n_c = \text{number of swings for the clapper amplitude to fall by a half}$

Even in the simplified form, the equations give results which compare very well with observations on real bells.

F.H. King
The University Bellringer
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