The weighted spectral distribution; A graph metric with applications.

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A graph metric: motivation.

Graph theory ↔ statistical modelling

- Data point.
- Statistical model,
  - ARMA model, neural network etc.
- error/residual
- Sum squared error

- Observed graph at time *t*
- Topology generator.
  - BA model.
  - INET model etc.
- Weighted spectral distribution.
- Quadratic norm between WSD's (Weighted Spectral Distributions.)

- Inference: has the process generating the network changed over time?
- Parameter estimation: what parameters best fit the observed graph/data points.
- Model validation: does the proposed topology generator represent the data well?
- Clustering: Can we separate classes of graphs into their respective clusters?

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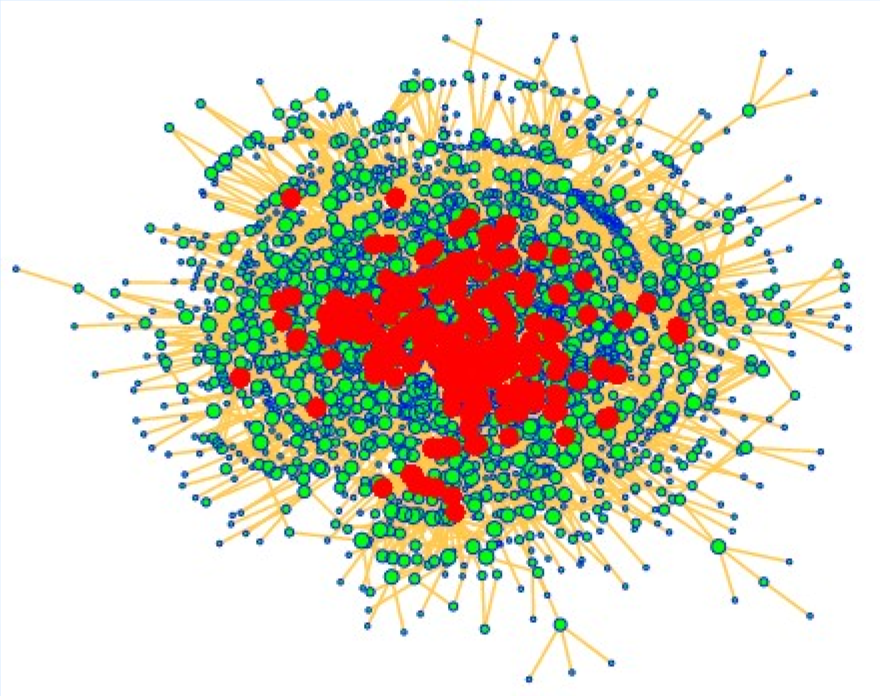
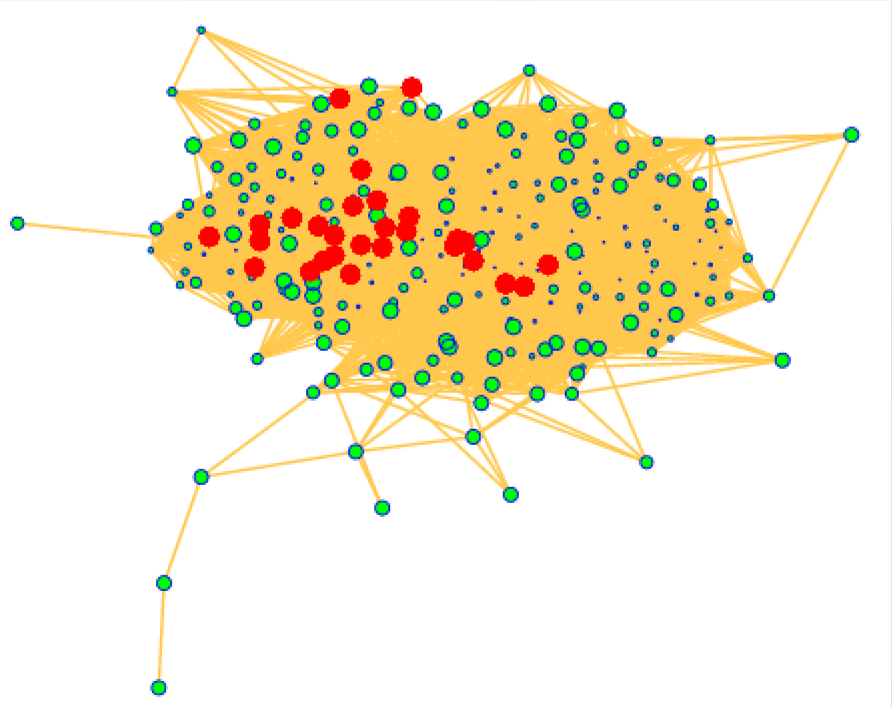
A metric for graph distance.

What is 'structure'?





- Both graphs share graph measures:
    - Clustering coefficient,
    - Degree distribution,
- There exists no other method for large networks.

Normalised Laplacian matrices.

Normalised Laplacian:

$$L(G)(u,v)=\begin{cases} 1 & if\ u=v, d_u \neq 0 \\ \frac{-1}{\sqrt{d_u d_v}} & u \neq v \\ 0 & otherwise \end{cases}$$

Alternatively using adjacency matrix and diagonal matrix D with degree of nodes:

$$L(G)=I-D^{-1/2}AD^{-1/2}$$

Expressing $L(G)$ using the eigenvalue decomposition:

$$L(G)=\sum_i \lambda_i e_i e_i^T$$

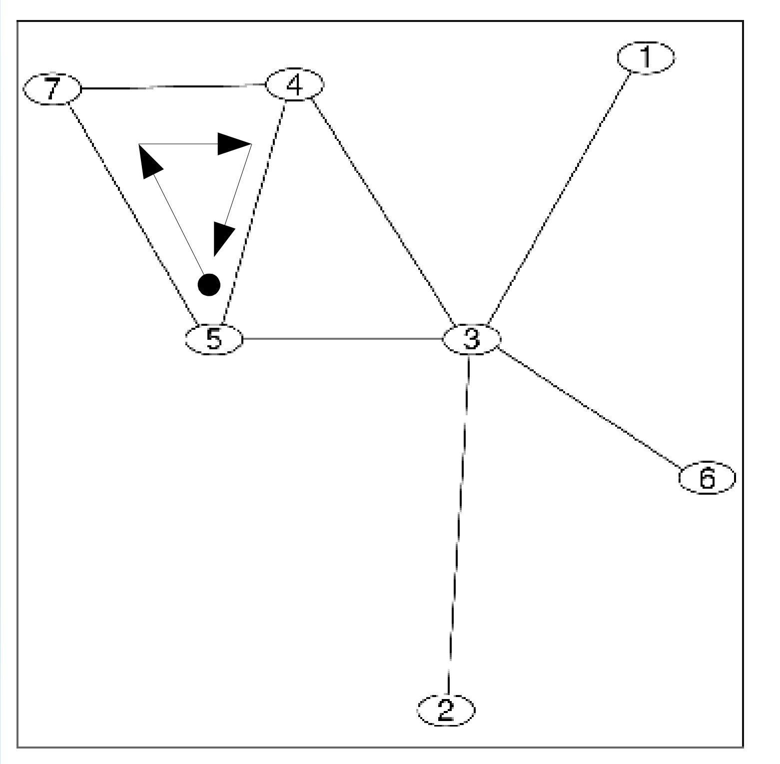
Note:

- $L(G)$ may be approximated using $\lambda_i\, e_i\, e_i^{\mathrm{T}}$ with approximation error proportion to $1\text{-}\lambda_i$
- e_i identifies i^{th} cluster in the data, assigning each node its importance to that cluster. (spectral clustering).
- Unlike spectral clustering we will use all eigenvalues.

Background theory.

- A *random walk cycle*: the probability of starting at a node taking a path of length N and returning to the original node.

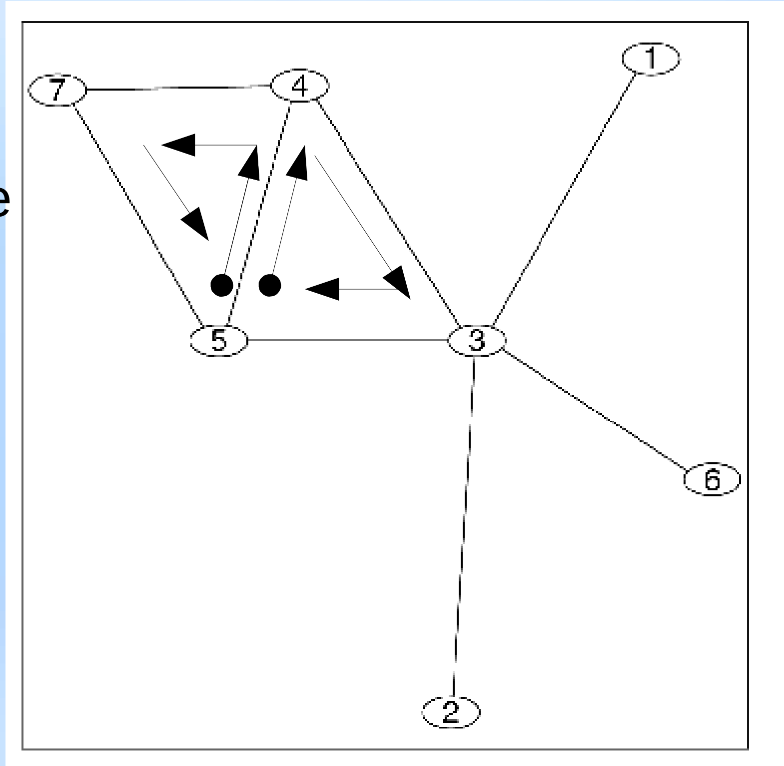
$\frac{1}{2} \times \frac{1}{3} \times \frac{1}{3} = 0.055$



Background theory.

- A *random walk cycle*: the probability of starting at a node taking a path of length *N* and returning to the original node.

- Several alternative 3-cycles available.
- The *N*-cycles are a measure of the local connectivity of a node.



Theory: random walk cycles.

$$L(G) = I - D^{-1/2} A D^{-1/2}$$

Defining the matrix on the right as B:

$$B = D^{-1/2} A D^{-1/2}$$

The elements of B may be expressed in terms of degrees and the adjacency matrix as:

$$B_{i,j} = \frac{A_{i,j}}{\sqrt{d_i}\sqrt{d_j}}$$

$(B^N)_{ij}$ is the sum of products of all paths of length N starting at node i and ending at node j As $1\text{-}\lambda_i$ are the eigenvalues of B :

$$\sum_i (1-\lambda_i)^N = tr(B^N)$$

We defining a random walk cycle to be a random walk of length N that starts and end at the same node (repeats included). This may be expressed in terms of B by noting :

$$B_{i,j} B_{j,k} \ldots B_{l,i} = \frac{A_{i,j}}{\sqrt{d_i}\sqrt{d_j}} \frac{A_{j,k}}{\sqrt{d_j}\sqrt{d_k}} \ldots \frac{A_{l,i}}{\sqrt{d_l}\sqrt{d_i}} = \frac{1}{d_i d_j \ldots d_k}$$

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Which simply results in the diagonal of B^N:

$$B_{i,j} B_{j,k} \dots B_{l,i} = \left(B^N\right)_{i,i}$$

Thus the eigenvalues of the normalised Laplacian can be related to random walk cycles in a graph via:

$$\sum_i \left(1-\lambda_i\right)^N = tr\left(B^N\right)$$

i.e. we may calculate the number of weighted *N*-cycles via:

$$\sum_i \left(1-\lambda_i\right)^N = tr\left(B^N\right) = \sum_C \frac{1}{d_i d_j \dots d_k}$$

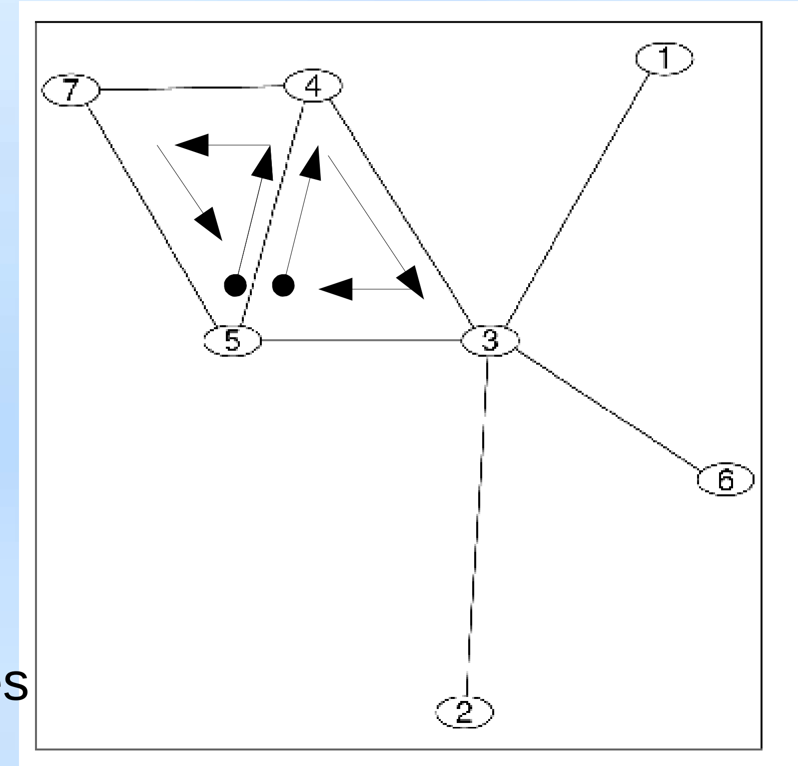
Where *C* is the set of *N*-cycles in the graph containing the *N* nodes *i*,*j*, ...*k*.

Background theory.

- A *random walk cycle*: the probability of starting at a node taking a path of length *N* and returning to the original node.

Theorem:

$$\sum_i \left(1-\lambda_i\right)^N = tr\left(B^N\right) = \sum_C \frac{1}{d_i d_j \ldots d_k}$$



- The right hand side is the sum of probabilities of all *N*-cycles in a graph.
- The left hand side relates this to the eigenvalues of the normalised Laplacian.
- We get a relationship between the eigenvalues of the normalised Laplacian (*global structure*) and the *N*-cycles (*local structure*) of a graph.

Theory: spectral distribution.

- Problem:
  - Estimating the eigenvalues is expensive and inexact, in addition we are really only interested in those near 0 or -2.
- Solution:
  - Using Sylvester's law of inertia and pivoting calculate the number of eigenvalues that fall in an interval => we are now looking at the distribution of eigenvalues, $f(\lambda)$.
  - The weighted spectral distribution can now be defined as:

$$WSD: G \rightarrow R^{|K|}\left\{k \in K : (1-k)^N f(\lambda = k)\right\}$$

Theory: metric definition.

Finally we may define a metric based on the quadratic norm between the weighted spectral distributions of two graphs, G_1 and G_2, as:

$$\Delta(G_{1,}G_{2,}N)=\sum_{k\in K}(1-k)^N\left(f_1(\lambda=k)-f_2(\lambda=k)\right)^2$$

- Notes:
  - The number of components in a graph is equal to the number of eigenvalues at 0.
    - This is given the highest structural weighting.
  - Eigenvalues in the spectral gap (i.e. close to 1) are given very low weighting
    - as the spectral gap is expected to hold little structural information (it is important for other things!).
  - All the eigenvalues are considered not just the first *k.*
  - *Δ* is a metric in the strict sense except for the identity law which is true almost certainly.

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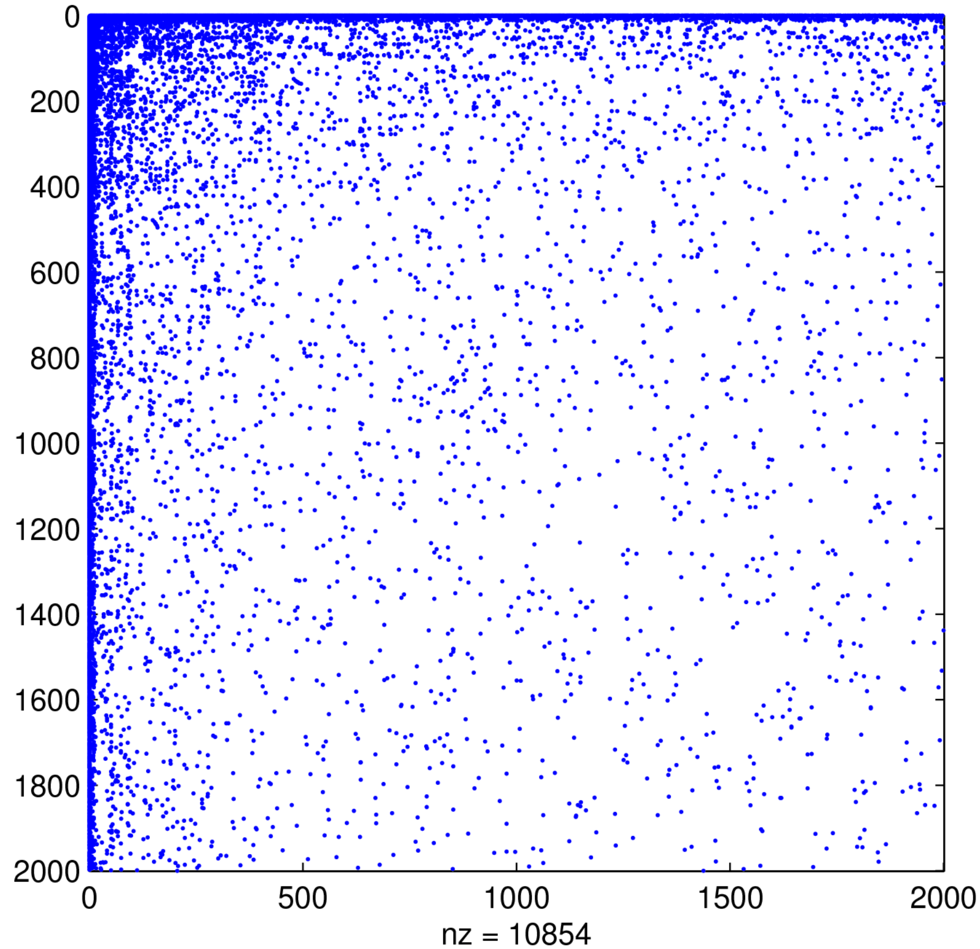
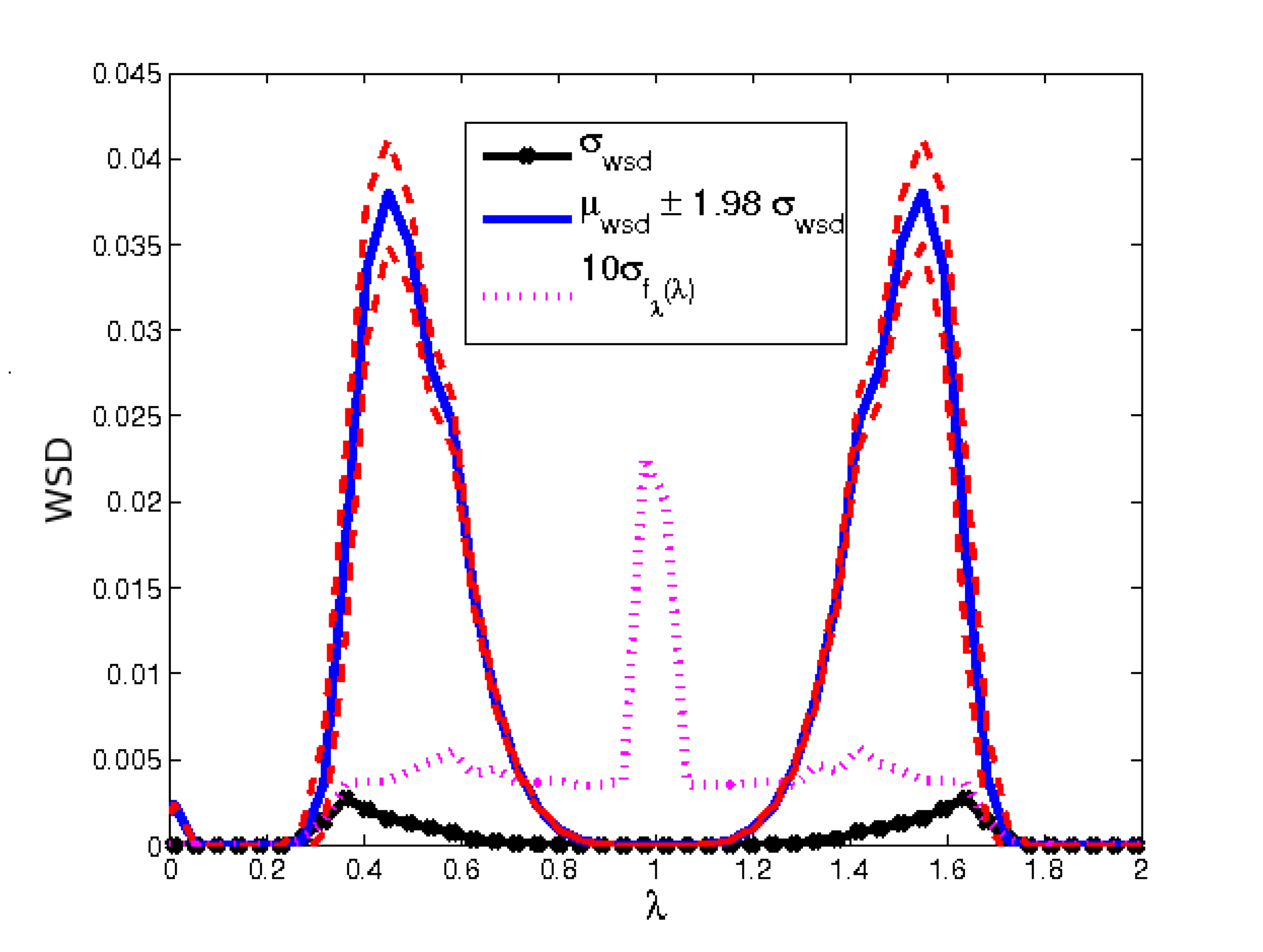
WSD example



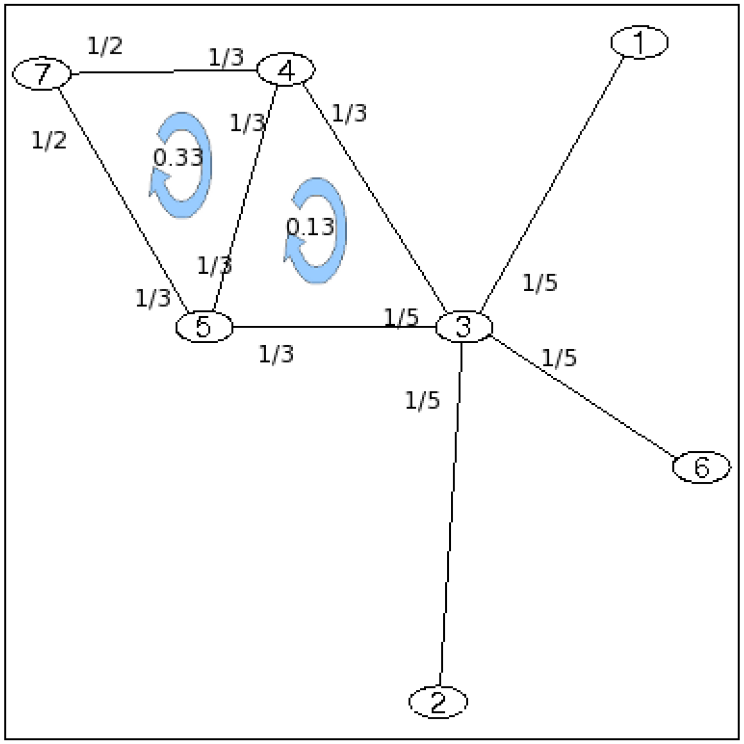
Adjacency matrix of an AB graph, 2000 nodes.

WSD example



WSD taken over 51 bins.

Simple example



Examine the number of 3-cycles in this graph.

- There are two 3-cycles
  - $\frac{1}{2} \times \frac{1}{3} \times \frac{1}{3} \times 6 = 0.333$
  - $\frac{1}{3} \times \frac{1}{3} \times 1/5 \times 6 = \underline{0.133}$

    0.466

(note: 6 from the 6 directional cycles in each loop).

Normalised Laplacian eigenvectors.

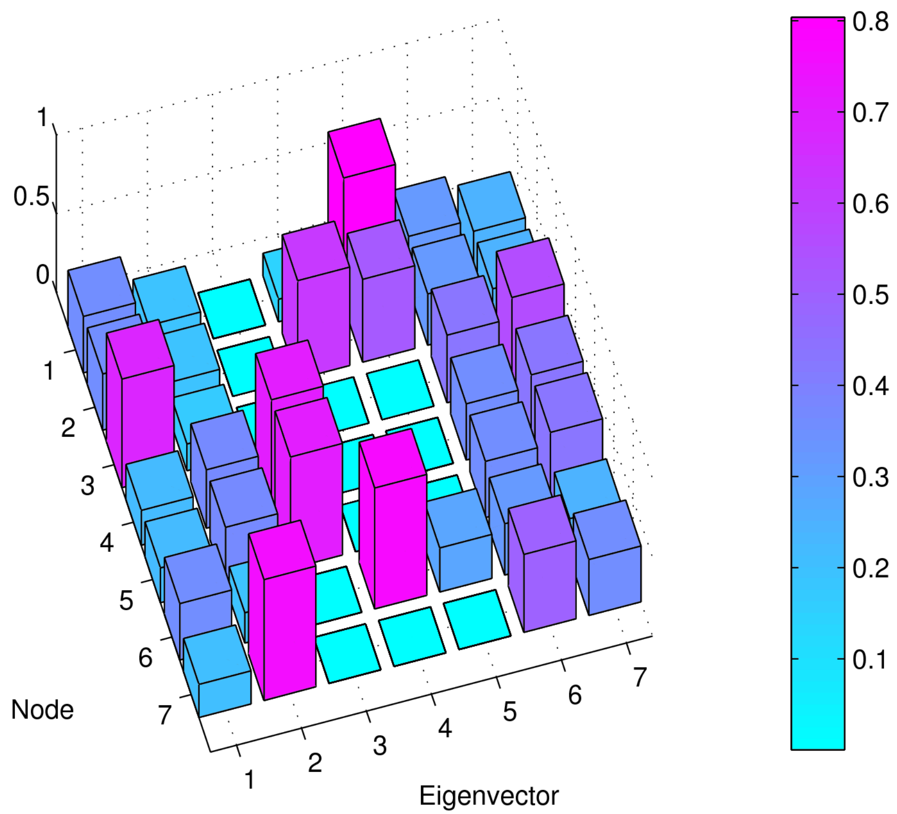
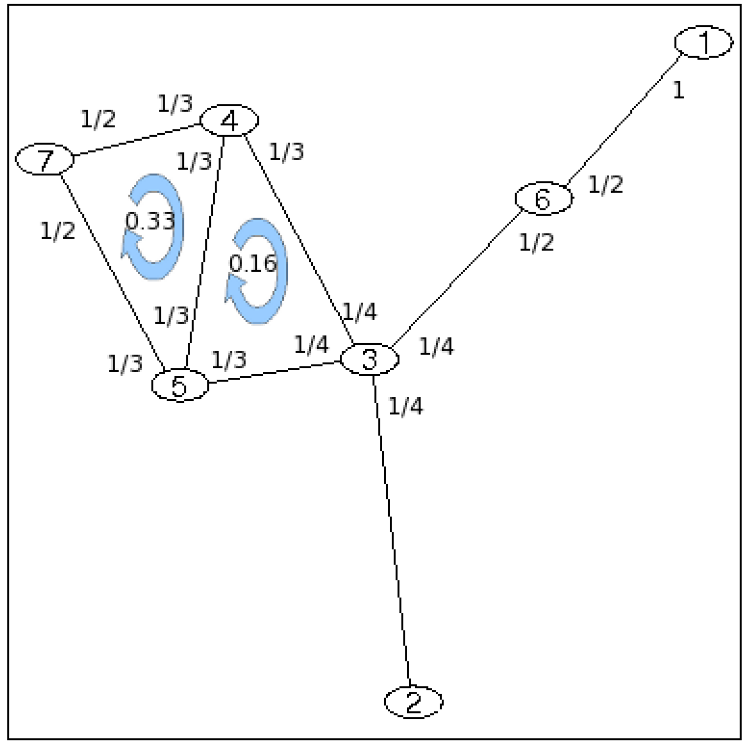


TABLE I

EIGENVALUES, WSD AND DOMINANT NODES OF EXAMPLE NETWORK.

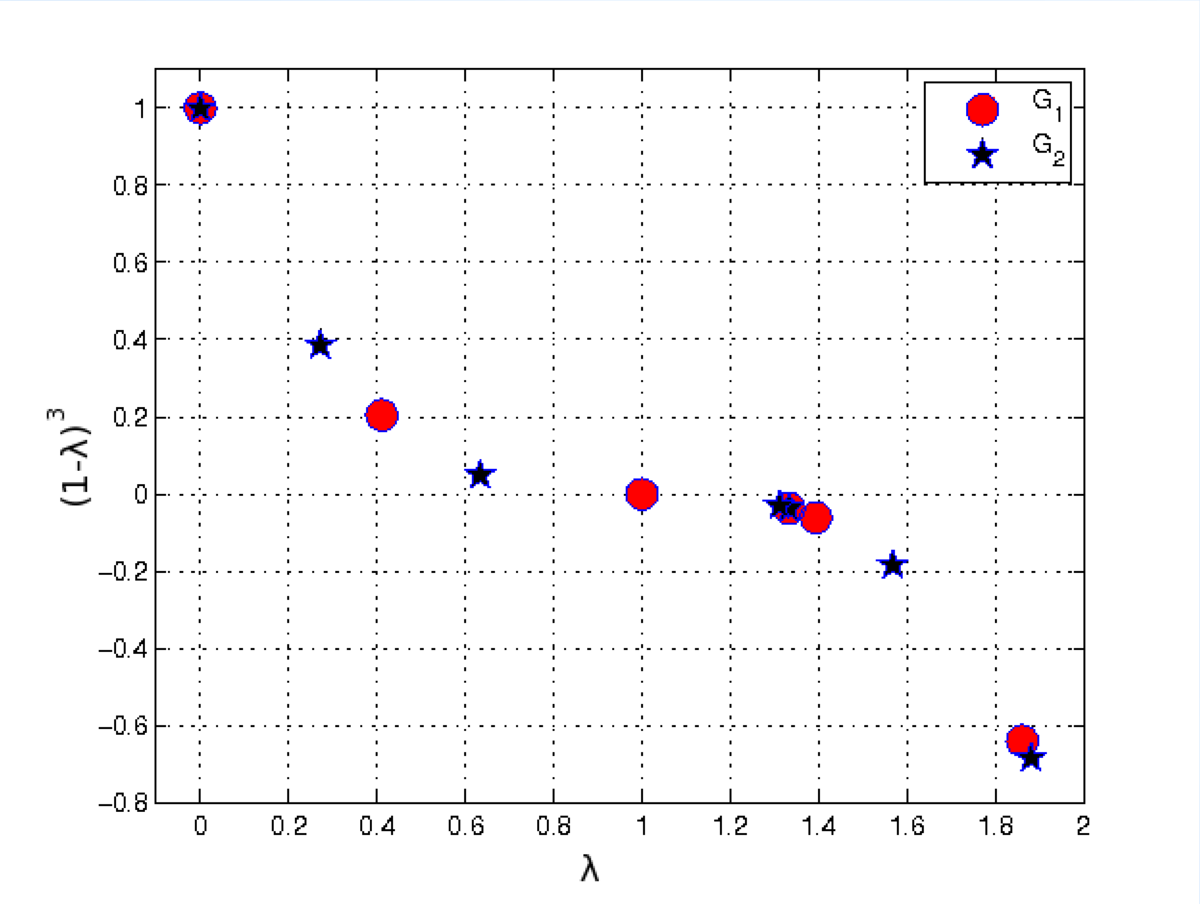
e_7	Eigenvector	λ	$1-\lambda$	$(1-\lambda)^3$	Dominant nodes
0.2500	1	1.8615	-0.8615	-0.6394	3,1,2,6
0.2500	2	1.3942	-0.3942	-0.0612	7,4,5
0.5590	3	1.3333	-0.3333	-0.0370	4,5
0.4330	4	1.0000	0.0000	0.0000	6,2
0.4330	5	1.0000	0.0000	0.0000	1,2,6
0.2500	6	0.4110	0.5890	0.2043	7,3
0.3536	7	0.0000	1.0000	1.0000	3,4,5,7
	$\sum_{i=1}^{7}(1-\lambda_i)^3$			0.4667	

Adjusting the network



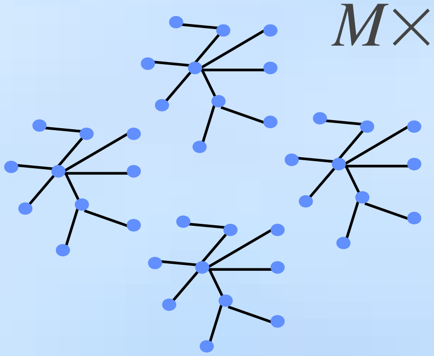
- Node 1 has been rewired from node 3 to link to node 6.
    - The loops are unchanged.
    - However, the random walk probabilities have changed.

WSD example



- The effect is to move the eigenvalues and thus the random walk cycle probabilities.
- Note: this is not the case when using the adjacency matrix.

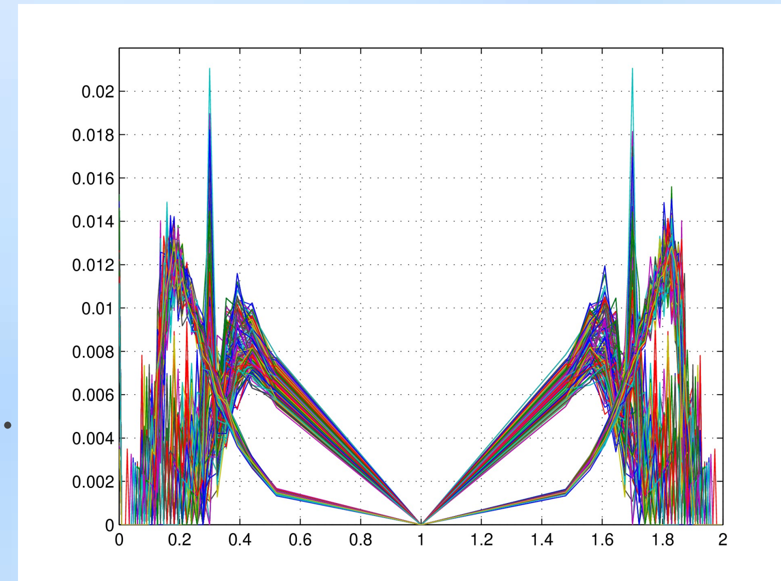
Clustering using the WSD.



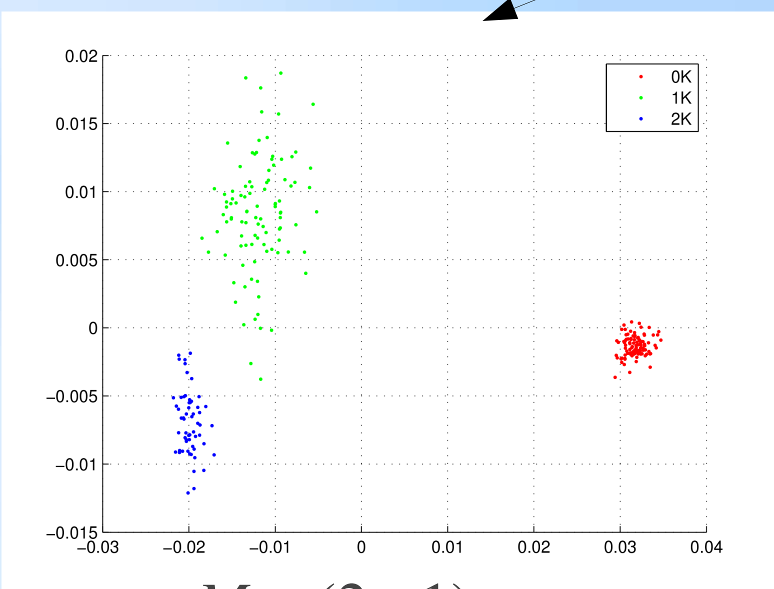
$M\times(N\times N)$

WSD

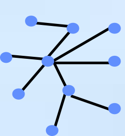
$M\times(K\times 1)$



Random Projection or Multidimensional scaling.



$M\times(2\times 1)$



M objects.
N- nodes.
K bins.
k co-ordinates

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Random Projection.

Random projection is a technique for data compression often used in compressed sensing.

Basic idea is very simple:

Given a matrix $A = M \times K$ we wish to produce a matrix of reduced dimension $M \times k$ *where* $k << K$.

We can form an approximation to A in k dimensions by randomly projecting A onto an $M \times k$ projection matrix T where $T \sim N(0,1)$.

i.e. we simply multiply the data by a matrix of appropriate size containing random numbers!.

Note: $E[T_{i,j}\ T_{k,l}] = 0\ \forall\ i,j \neq k,l$ => inner product of two rows of T is zero in expectation => T is *(nearly)* **orthogonal**.

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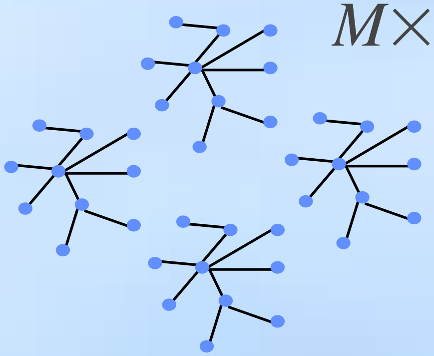
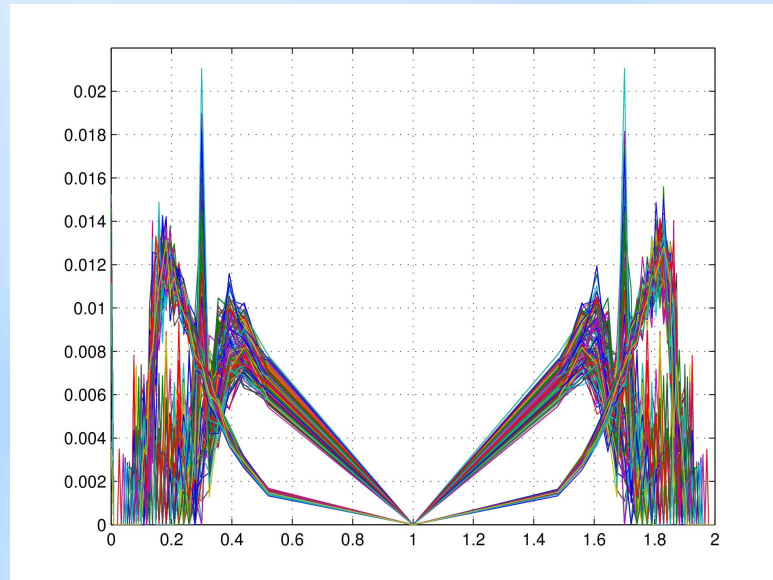
Random projection example.



$M \times (N \times N)$

WSD

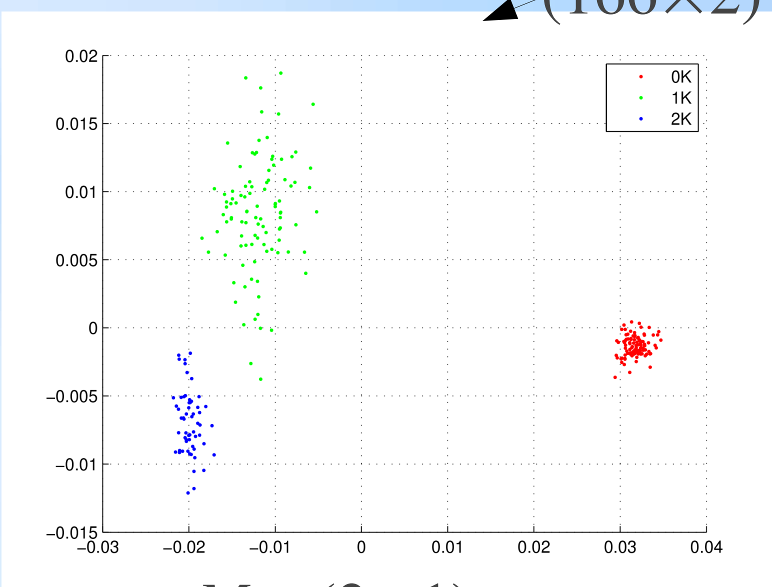
$M \times (K \times 1)$



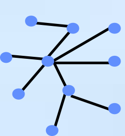
$\sim N(0,1)$

$(166 \times 71) \times (71 \times 2) =$

(166×2)



$M \times (2 \times 1)$



M objects.
N- nodes.
K bins.
k co-ordinates

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Multi-dimensional Scaling.

Given

- matrix $A = M \times K$,
- a metric defining the distance between each row of A,

Aim:

- produce a matrix of reduced dimension $M \times k$ *where* $k << K$.

First we construct the dissimilarity matrix:

$$\Delta(i,j) = \Delta(G_i, G_j)$$

T construct the Gram matrix by double centring the distances as:

$$H = -1/2 J \Delta^{\circ 2} J \qquad J = I_N - 1_N 1_N^T / N$$

A projection into k dimensions may then be constructed using the first k eigenpairs of H:

$$Y = [V]_{1:k} [\Lambda]_{1:k}^{1/2} \qquad H = V \Lambda V^T$$

Aside (by coincidence current research involves):

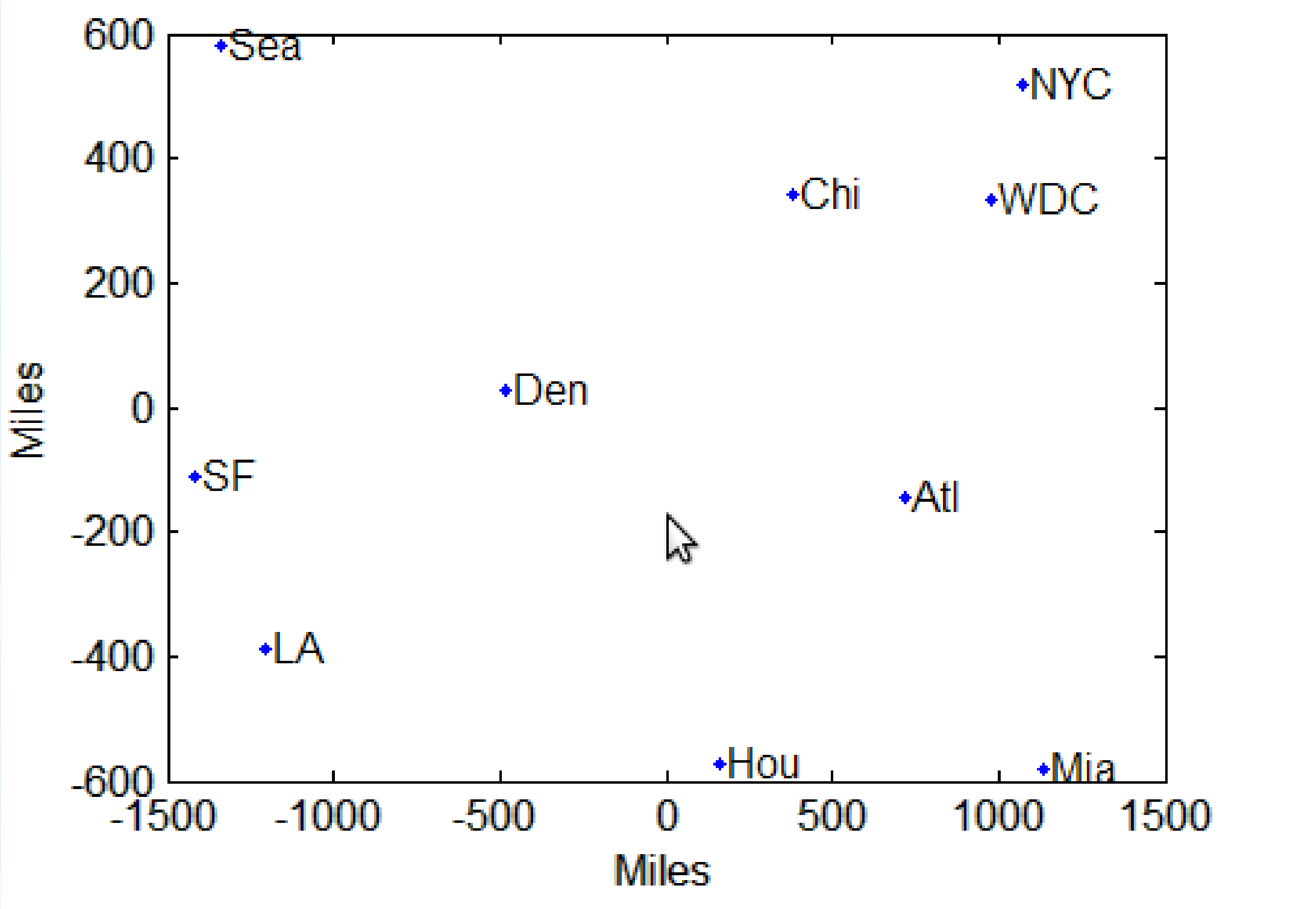
- MDS also forms the core of localisation and tracking techniques.
- If Δ is not complete several methods exist;
  - the Nystrom approximation for missing blocks;
  - Weighted MDS via SDP for missing elements.
  - Apply a particle filter to track movement and estimate distances and weights (error variance).

$\Delta(G_i, G_j)$ = quadratic norm between WSD's as defined earlier.

Example.

	Atlanta		Denver		LA					
Atlanta	0									
	587	0								
Denver	1212	920	0							
	701	940	879	0						
LA	1936	1745	831	1374	0					
	604	1188	1726	968	2339	0				
	748	713	1631	1420	2451	1092	0			
	2139	1858	949	1645	347	2594	2571	0		
	2182	1737	1021	1891	959	2734	2408	678	0	
	543	597	1494	1220	2300	923	205	2442	2329	0

example



600
400
200
0
-200
-400
-600
Miles
Sea
NYC
Chi
WDC
Den
SF
Atl
LA
Hou
Mia
-1500
-1000
-500
0
500
1000
1500
Miles

Example applications.

- Estimating the optimum parameters for a topology generator.
  - Comparing which topology generator produces a 'best' fit for the internet.
- Tracking evolution of the internet.

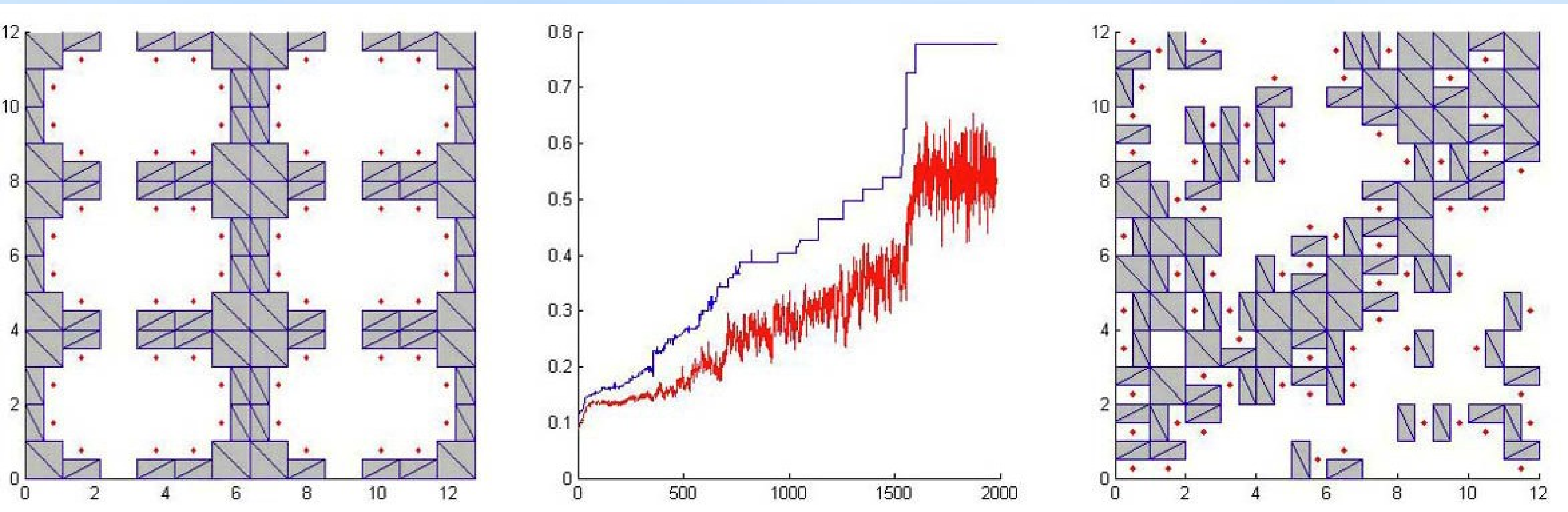
- Clustering applications:
  - Discriminating between topology generators.
  - Network application identification.
  - Orbis model analysis.

Internet AS topology models

- We compare 5 topology generators:
- The Waxman model
- The 2nd Barabasi and Albert Model (BA2)
- The Generalised Linear Preference model (GLP)
- The INET model
- Positive Feedback Preference model (PFP)
- To 2 data sets for the internet at AS level:
  - Skitter dataset (Traceroute based).
  - UCLA dataset (BGP looking glass server)

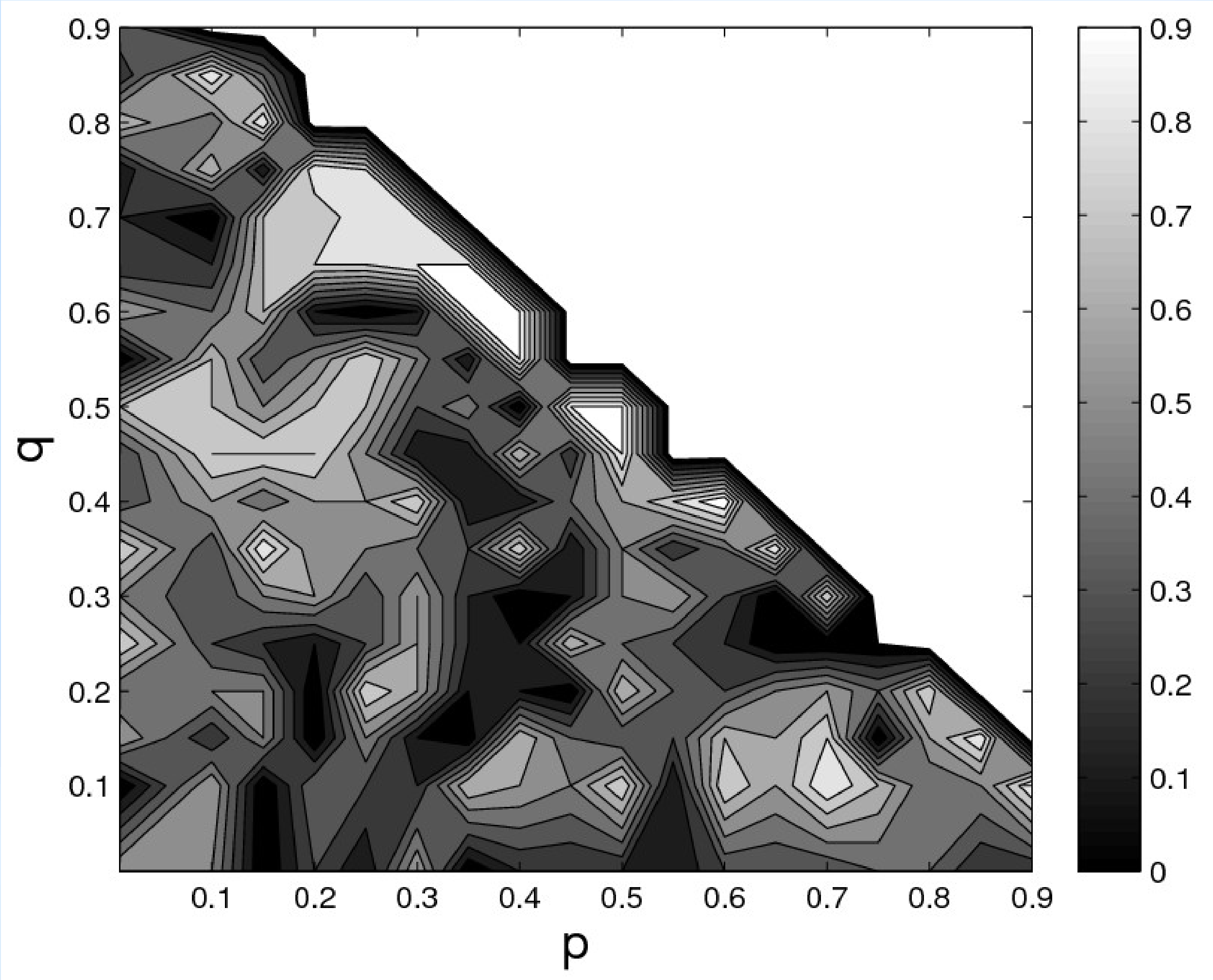
Related work

[3] S. Hanna, “Representation and generation of plans using graph spectra,” in *6th International Space Syntax Symposium*, Istanbul, (2007).



Application 1: Tuning topology generators.

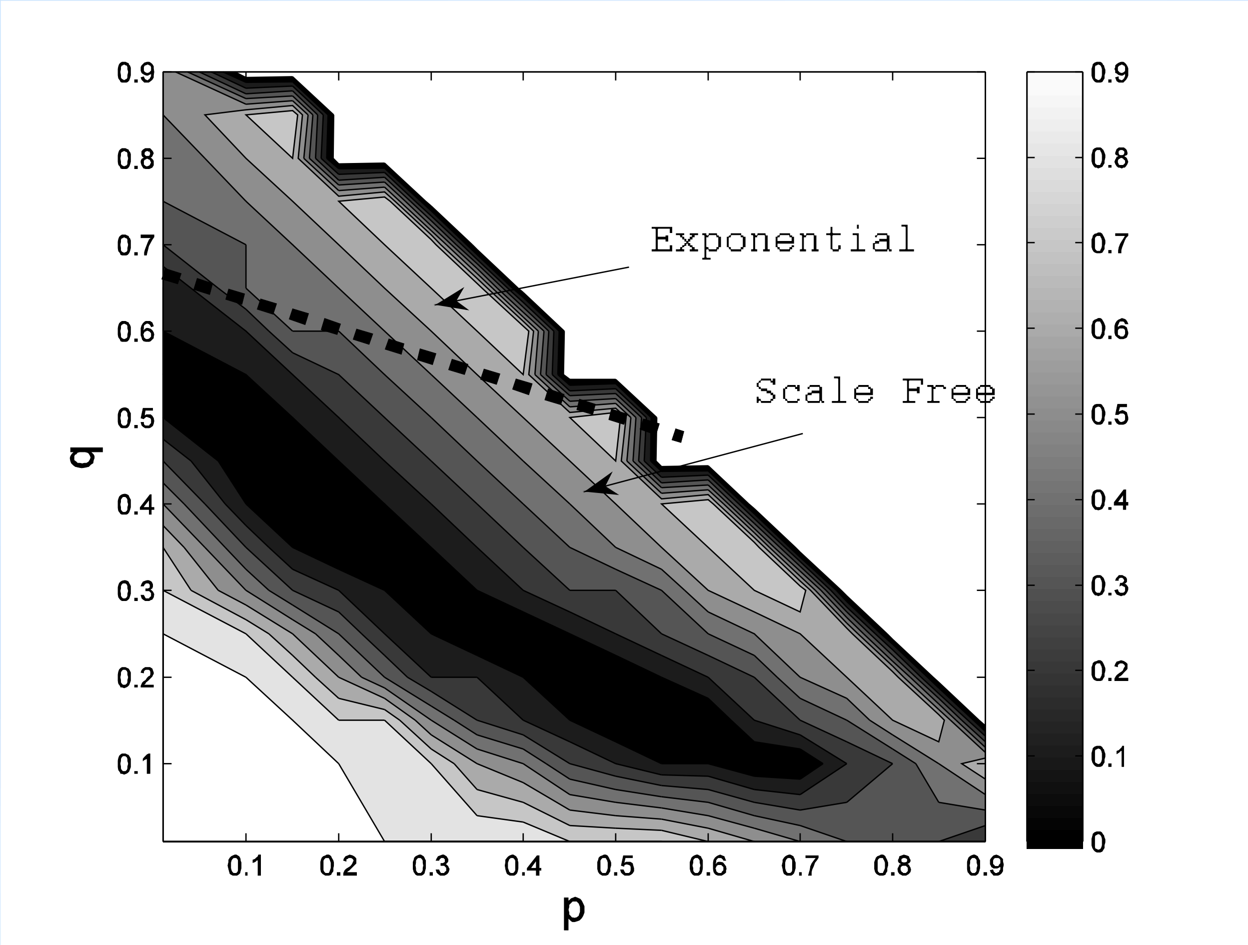
How NOT to select appropriate parameters for a topology generator.



Tuning an AB2 model using the (unweighted) spectral difference.

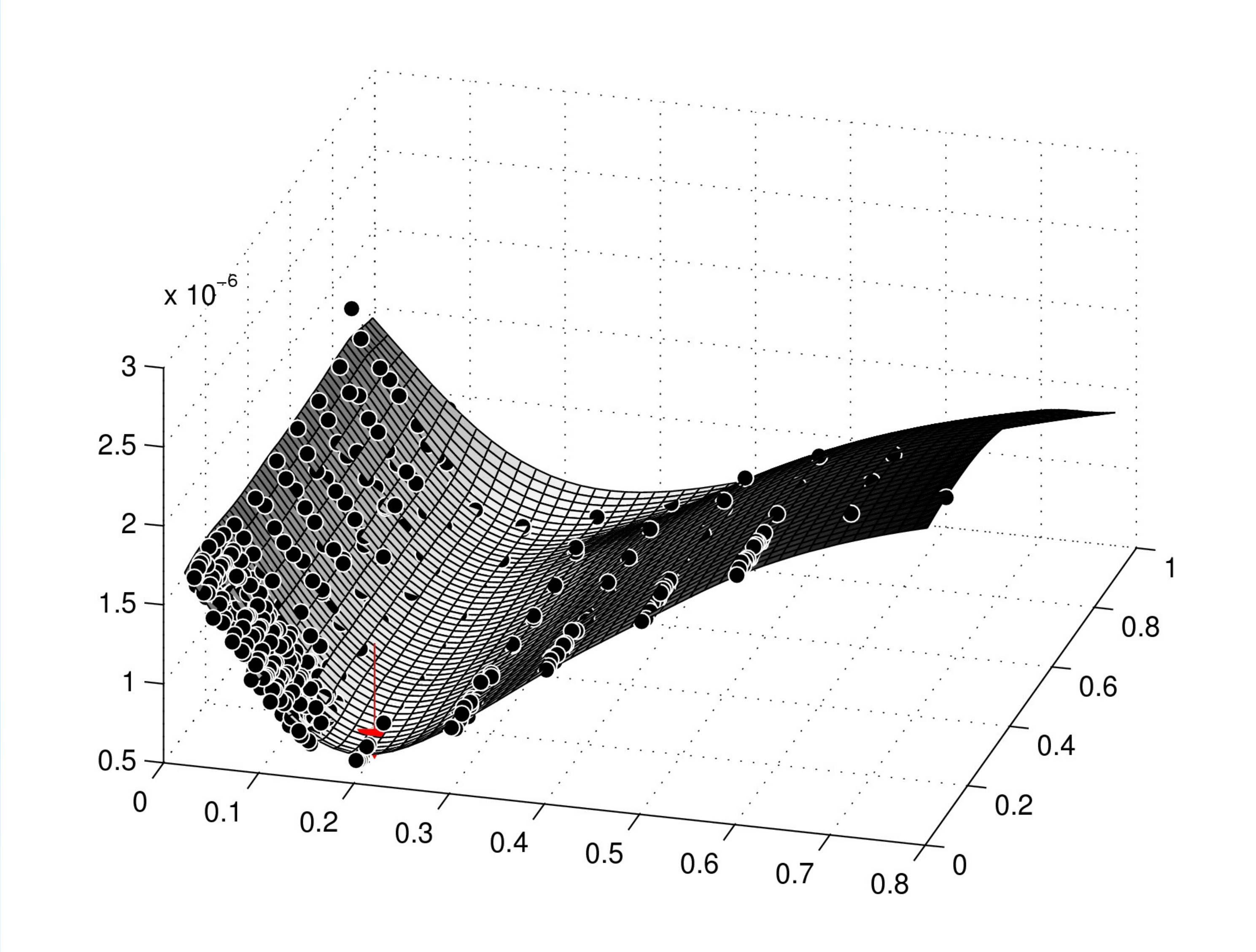
The WSD result.

How ~~NOT~~ to select appropriate parameters for a topology generator.



Tuning an AB model using a weighted spectral difference.

A 3-D view and example of speeding up the calculations



Tuning a GLP model using the WSD and active learning.

Application 1a: Comparing topology generators.

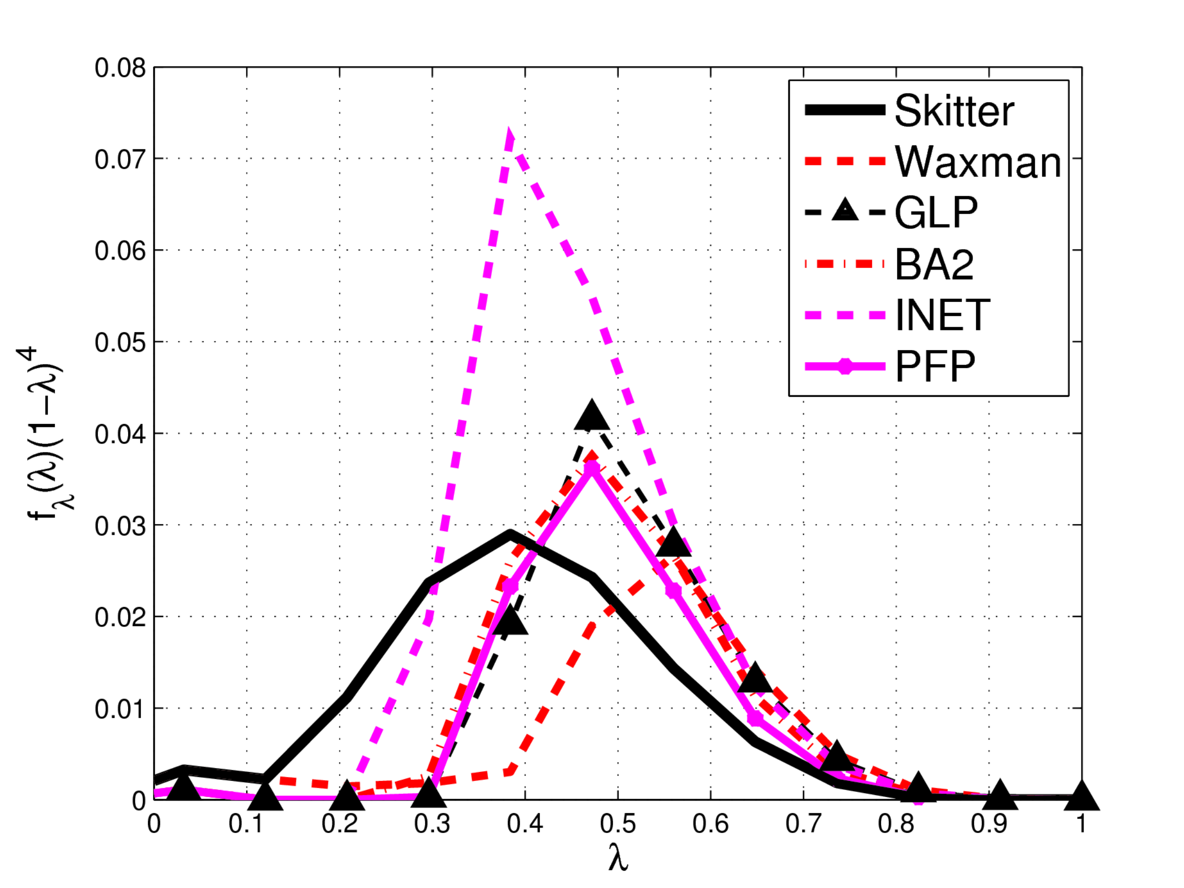


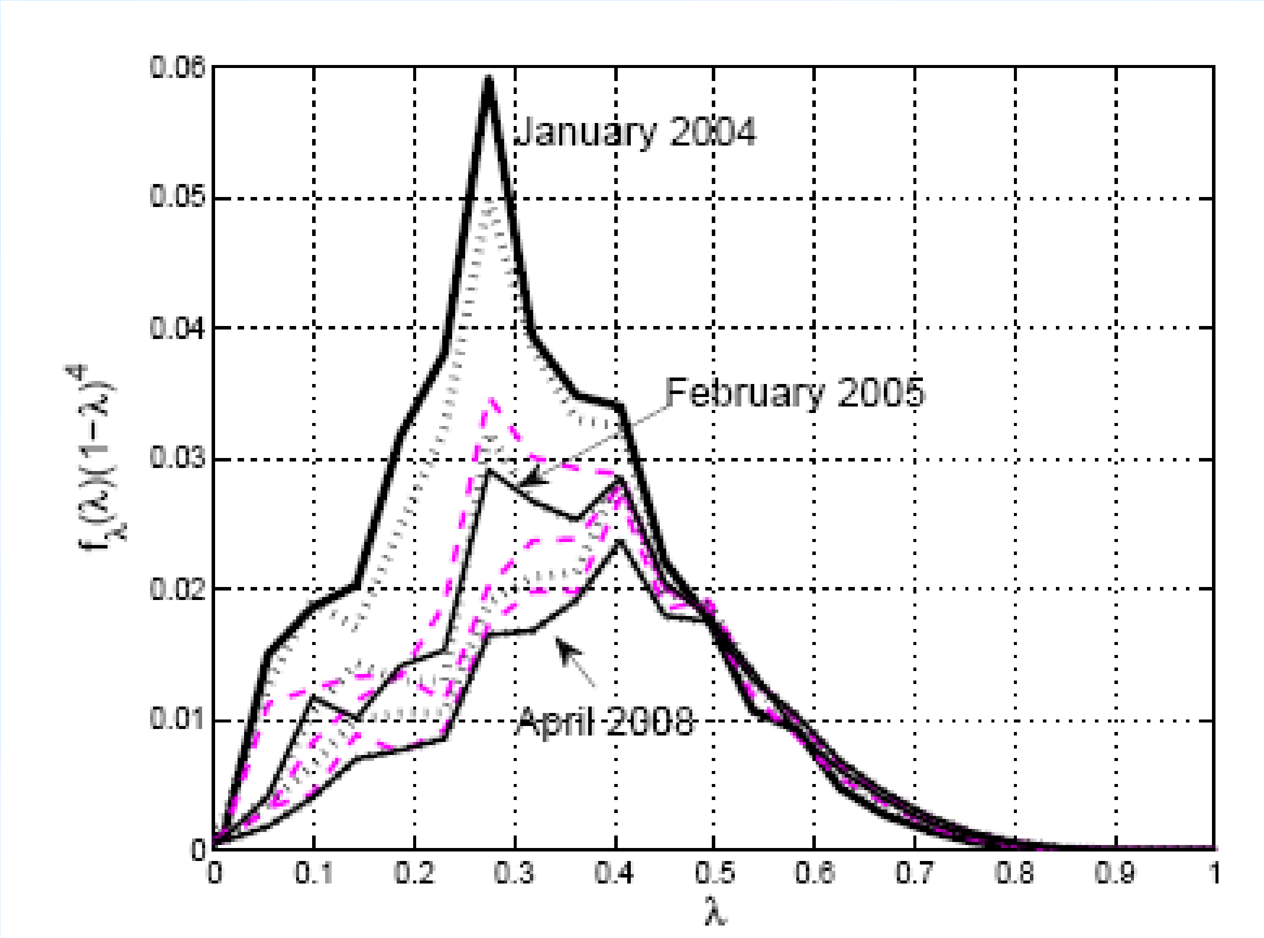
TABLE II

OPTIMUM PARAMETER VALUES FOR MATCHING SKITTER TOPOLOGY SAMPLED IN MARCH 2004.

Waxman	$\alpha = 0.08$ (default $= 0.15$)	$\beta = 0.08$ (default $= -0.2$)	$J(G_{Waxman}, G_{Skitter}) = 0.0026$
AB	$p = 0.2865$ (default $= 0.6$)	$q = 0.3145$ (default $= 0.3$)	$J(G_{AB}, G_{Skitter}) = 0.0014$
GLP	$p = 0.5972$ (default $= 0.45$)	$\beta = 0.1004$ (default $= 0.64$)	$J(G_{GLP}, G_{Skitter}) = 0.0021$
Inet	$\alpha = 0.1013$ (default $= 0.3$)	–	$J(G_{INET}, G_{Skitter}) = 0.0064$
PFP	–	–	$J(G_{PFP}, G_{Skitter}) = 0.0014$

Application 2: Tracking evolution of the internet.

We see quite clearly the change in the structure of the network.

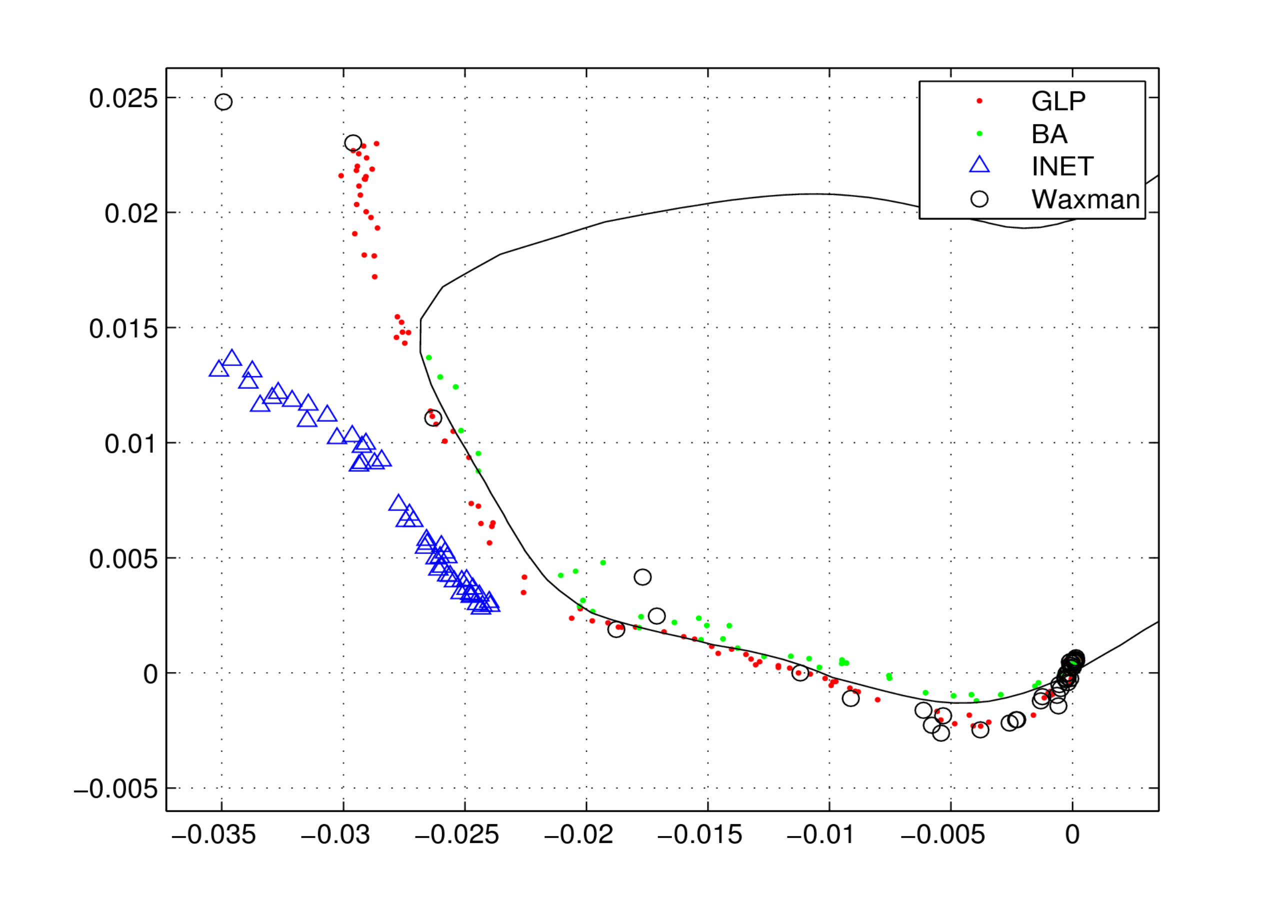


Example applications.

- ~~Estimating the optimum parameters for a topology generator.~~
  - ~~Comparing which topology generator produces a 'best' fit for the internet.~~
- ~~Tracking evolution of the internet.~~

- Clustering applications:
  - Discriminating between topology generators.
  - Network application identification.
  - Orbis model analysis.

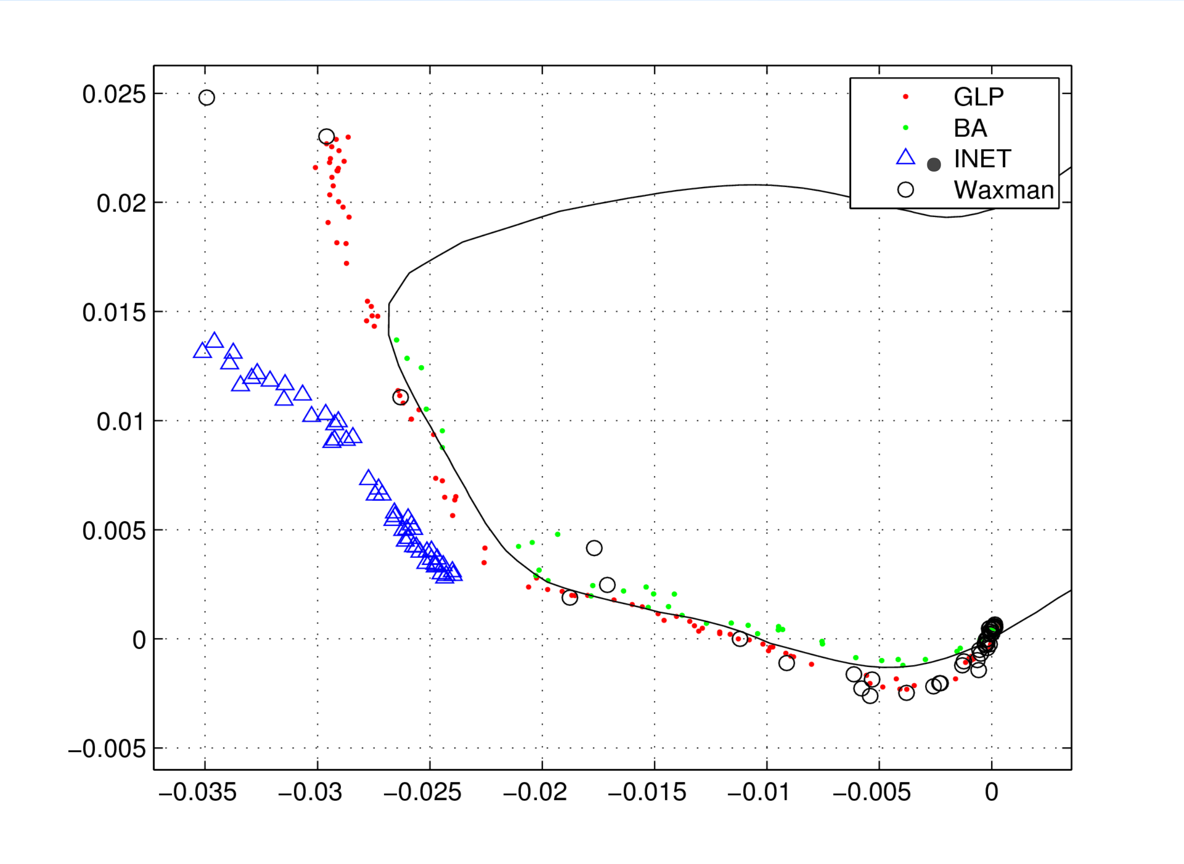
Application 3: Discriminating between topology generators.



500 graphs are generated of each:

- Waxman, GLP, AB, INET.
- Using random parameters.

Result



- The boundary between the AB and GLP models is pretty tight:

  A support vector machine is used (50-50 data split between training and test set); 11% misclassification error.

- INET occupies a different part of the projection space.
- Waxman maps to pretty much everywhere (not shown).
- Conclusion: The WSD+RP separates topologies with different structures.

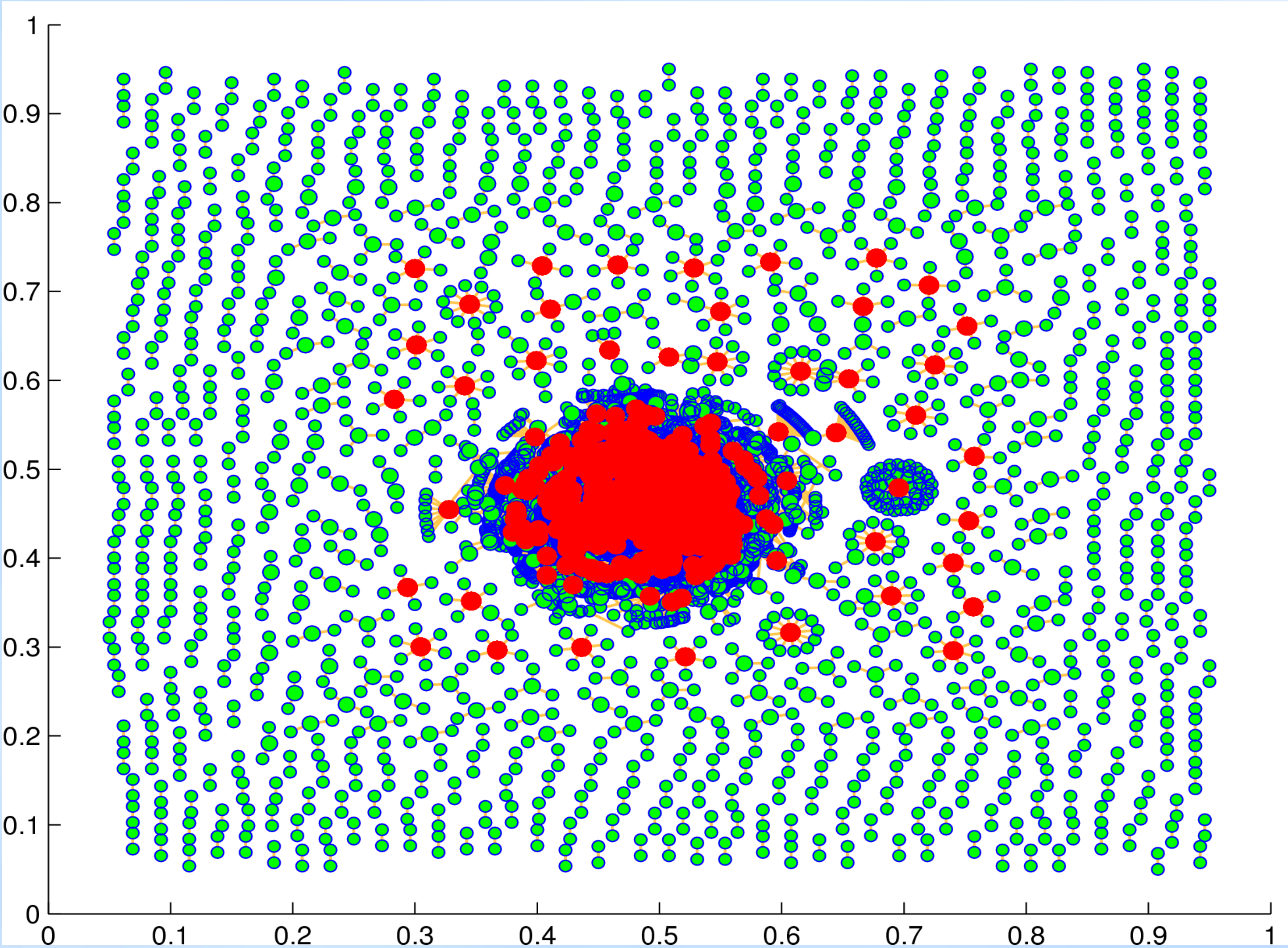
Application 4: Network application identification.

CAIDA data*

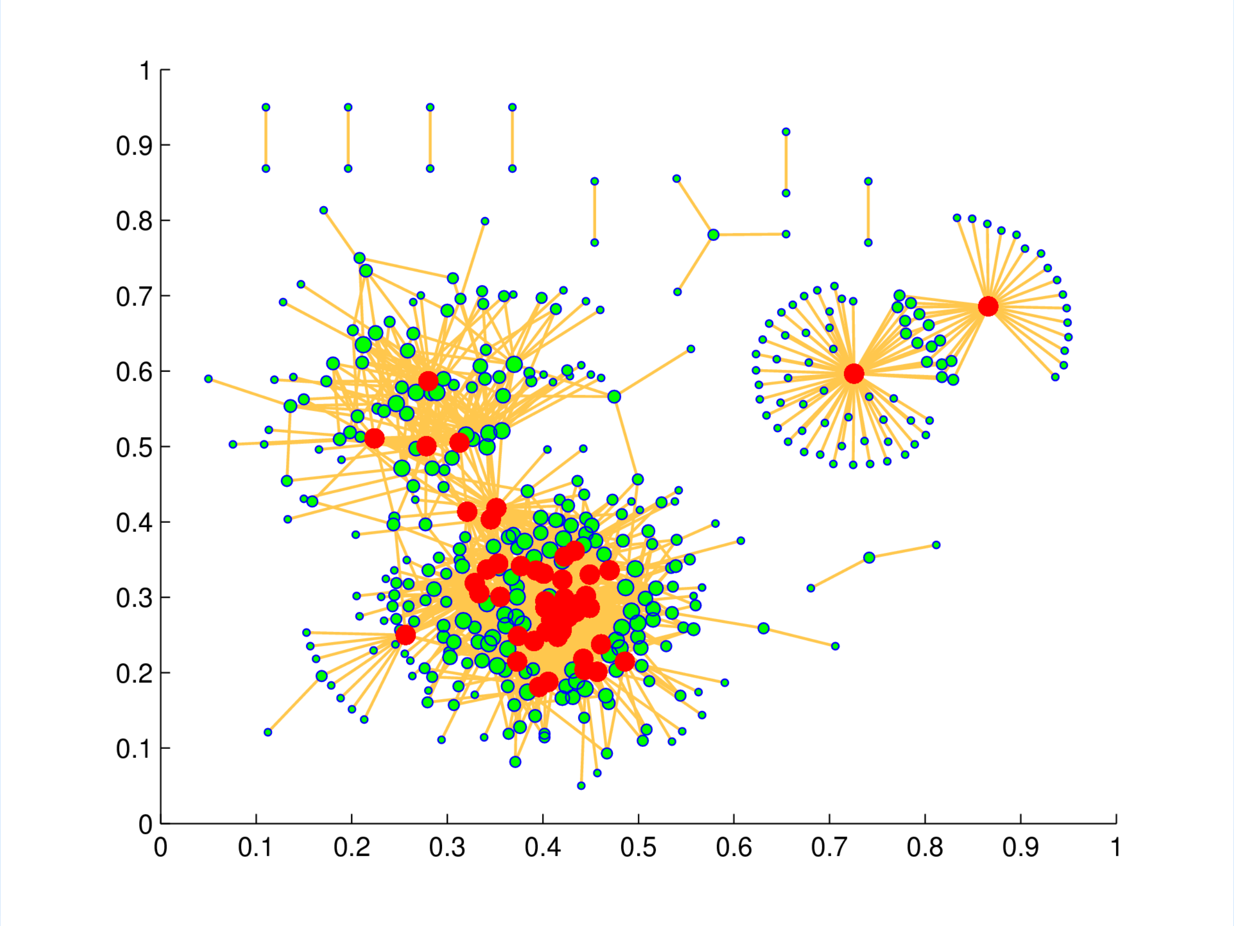
- Packet capture and deep packet inspection.
- 5 mins intervals.
- 10 Applications are tracked. Each forming a subgraph of interactions on the network (i.e. routers interacting with the same application).
- Aim: Given a graph can we estimate the application?.

*Exploiting Dynamicity in Graph-based Traffic Analysis: Techniques and Applications, Marios Iliofotou, M. Faloutsos, and M. Mitzenmacher, In ACM CoNEXT 2009, Dec.

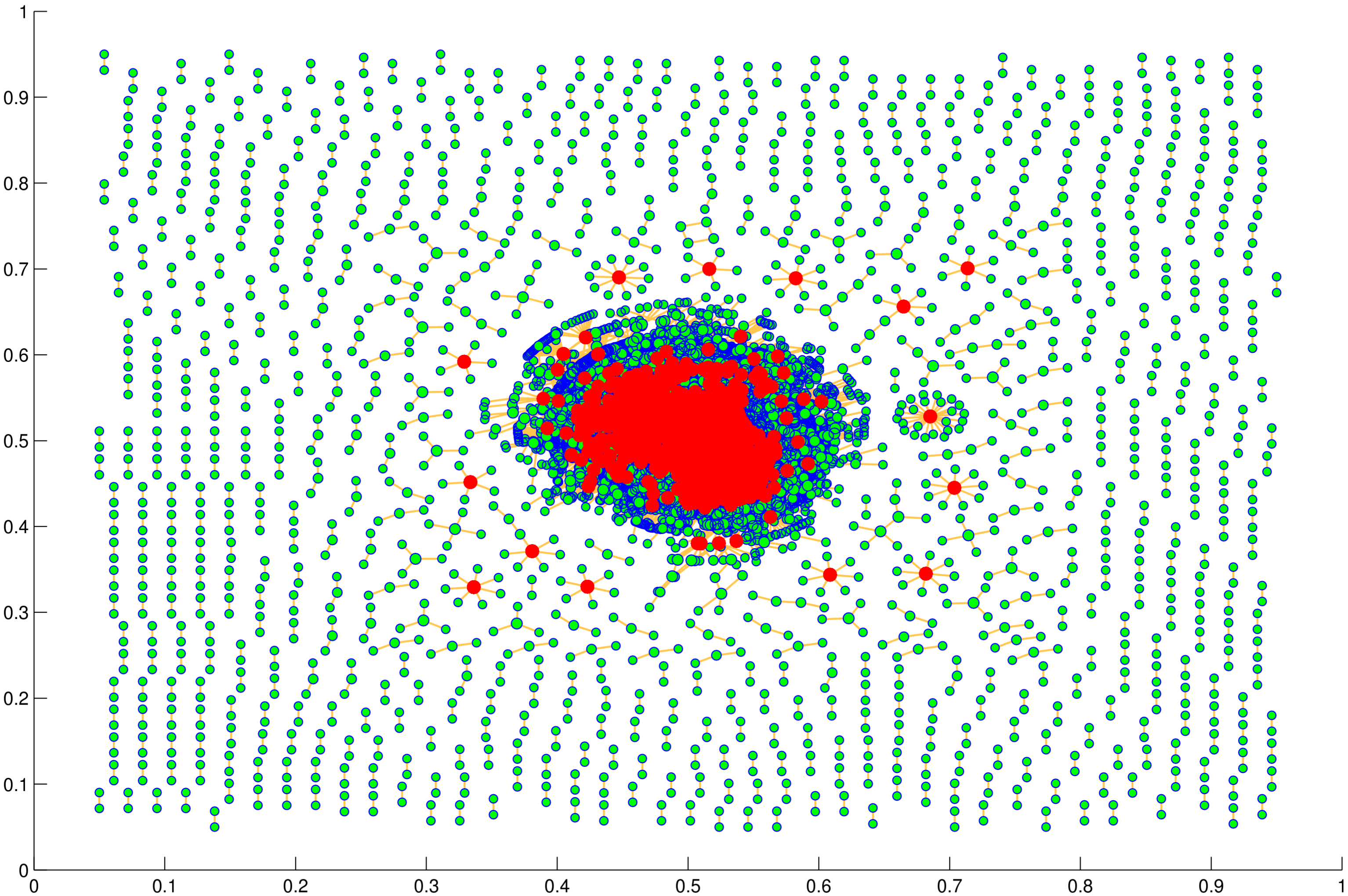
An application graph. (E-donkey)



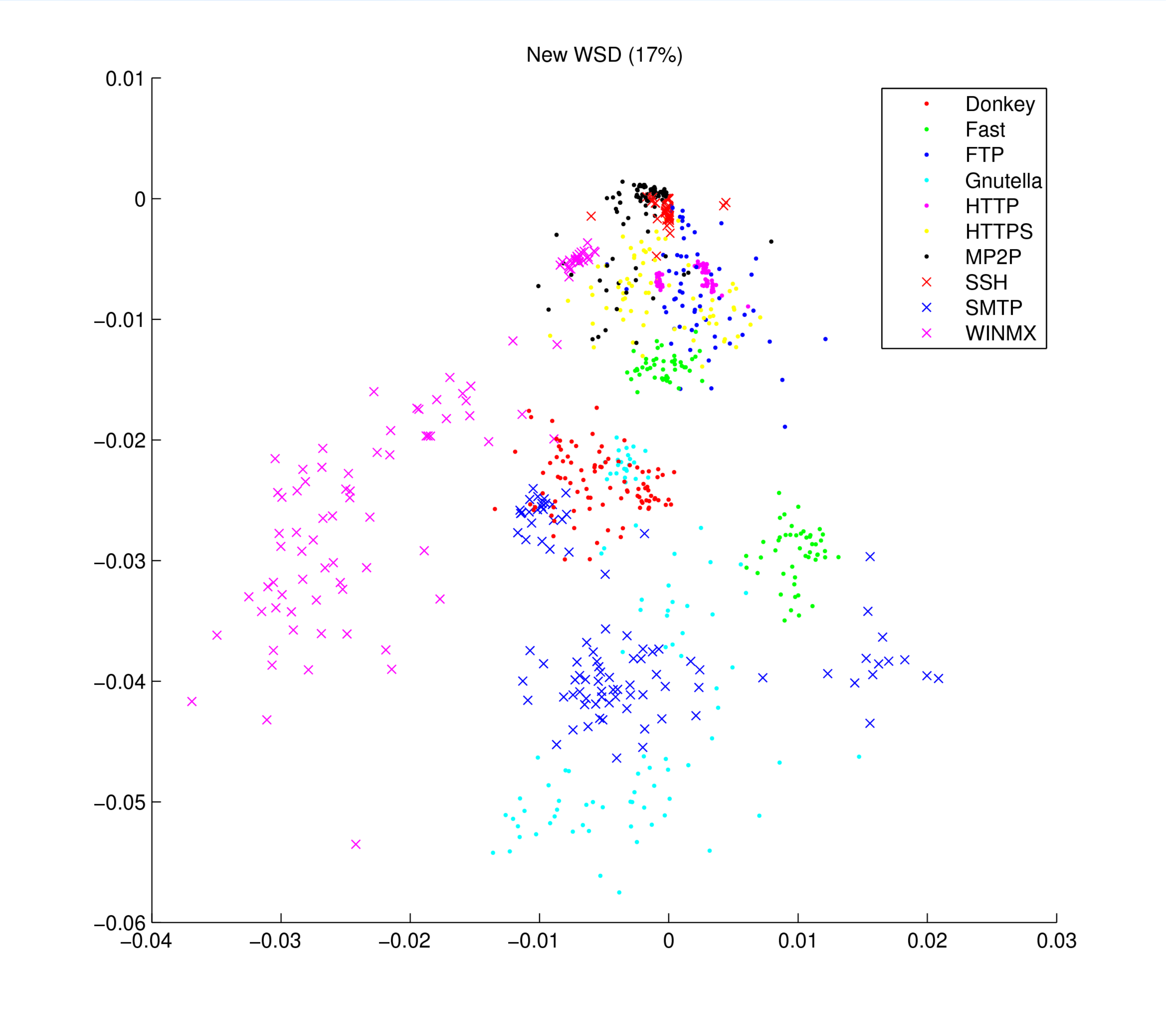
Example: MP2P



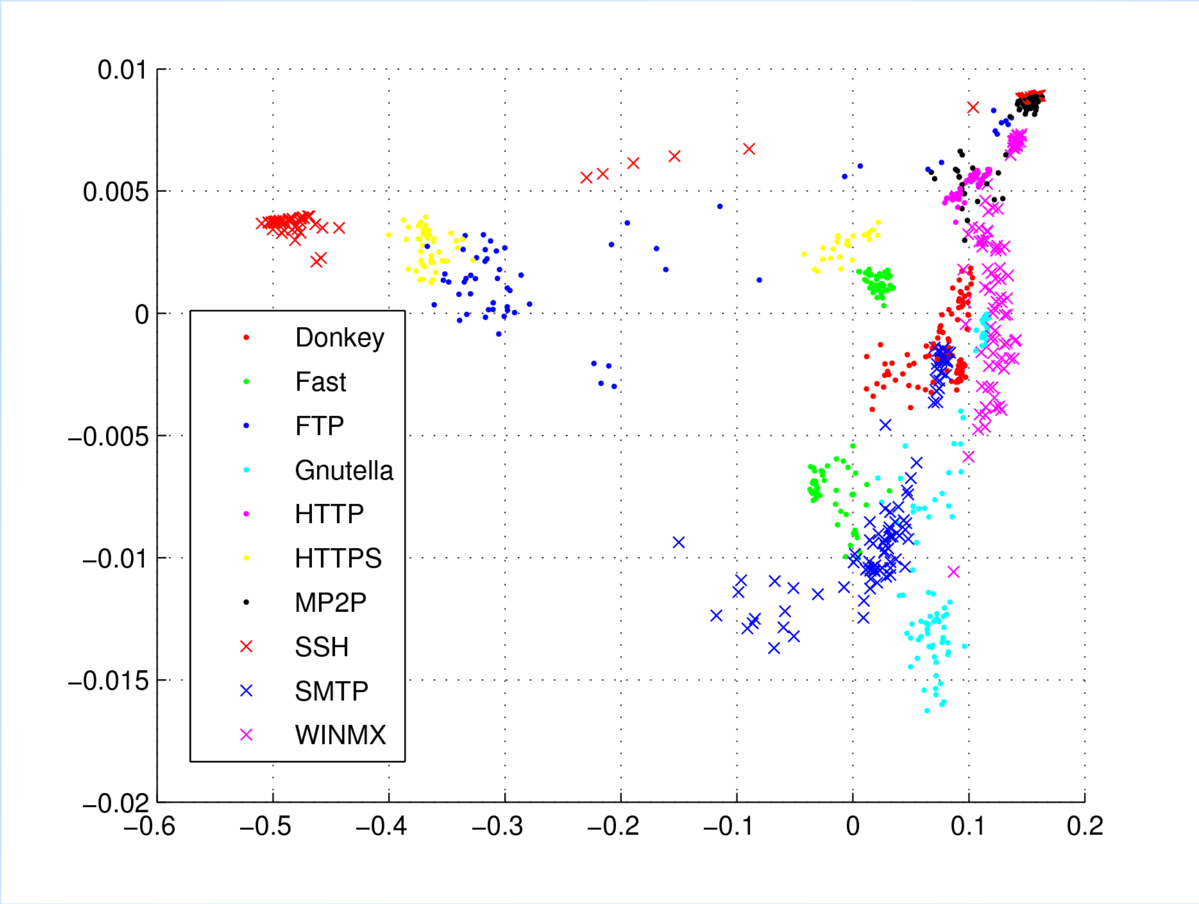
Example: SMTP



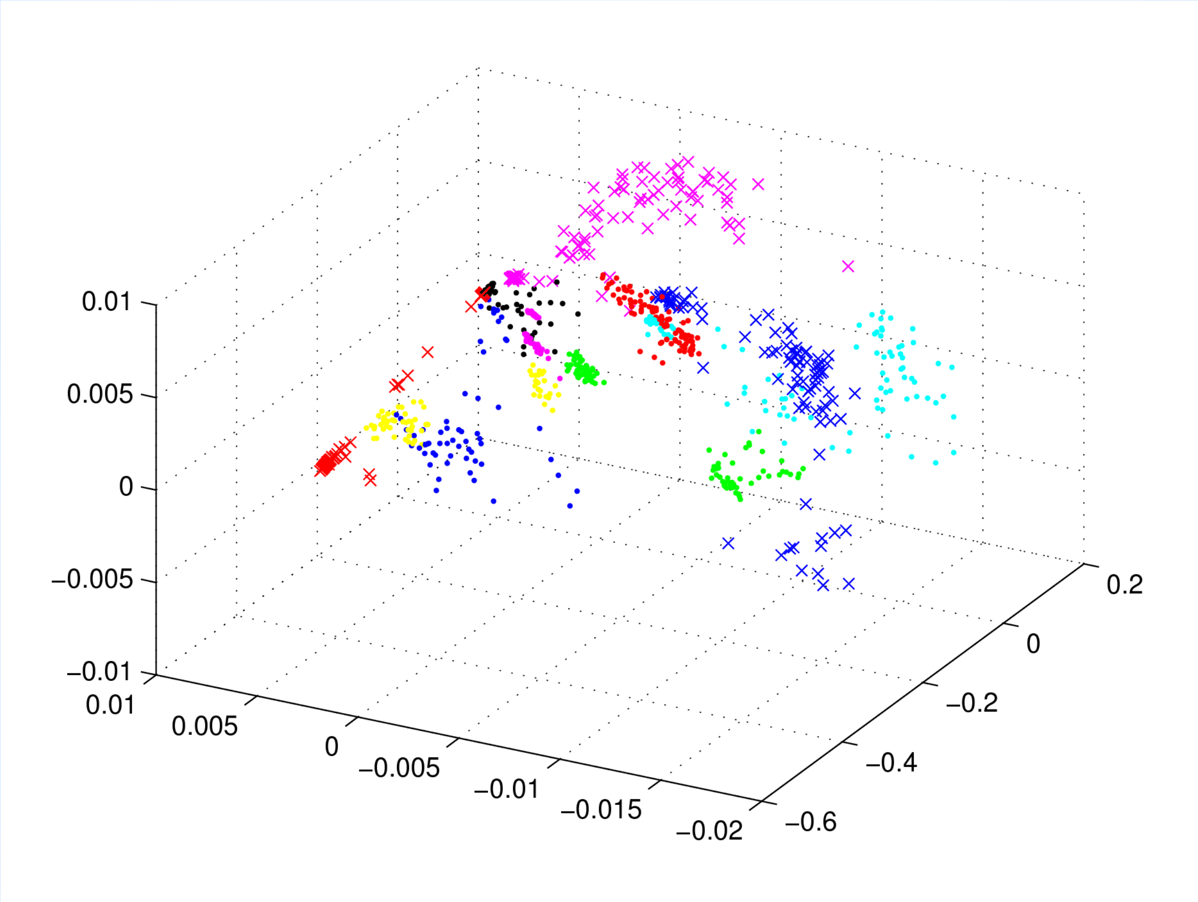
Result using random projection.



Result using MDS.



From a different angle.



Clustering results.

TABLE 1
Confusion matrix for Internet applications.

	Donkey	Fast	FTP	Gnutella	HTTP	HTTPS	MP2P	SSH	SMTP	WINMX
Donkey	**0.96**	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.04	0.00
Fast	0.00	**1.00**	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
FTP	0.00	0.00	**0.67**	0.00	0.00	**0.20**	0.13	0.00	0.00	0.00
Gnutella	**0.22**	0.02	0.00	**0.74**	0.00	0.00	0.00	0.00	0.02	0.00
HTTP	0.00	0.00	0.06	0.00	**0.91**	0.03	0.00	0.00	0.00	0.00
HTTPS	0.00	0.03	0.03	0.00	0.00	**0.94**	0.00	0.00	0.00	0.00
MP2P	0.00	0.00	0.04	0.00	0.07	0.00	**0.89**	0.00	0.00	0.00
SSH	0.00	0.00	0.03	0.00	0.00	0.06	**0.31**	**0.60**	0.00	0.00
SMTP	0.17	0.04	0.00	0.00	0.00	0.00	0.00	0.00	**0.78**	0.00
WINMX	0.00	0.00	0.00	0.00	0.00	0.00	**0.22**	0.00	0.04	**0.74**

Application 5: The Orbis topology generator.

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Orbis is a topology generator based on the *configuration model*.

- Paper: Sigcomm '06.
- Given a particular degree distribution we form link stubs:

The Orbis topology generator.

•

Connect Stubs at random until all connections are complete.

- This is known as the **1k** model and is defined solely by a degree distribution.
- A **0K** model is defined solely by the average degree.

The Orbis topology generator.

•

Connect Stubs at random until all connections are complete.

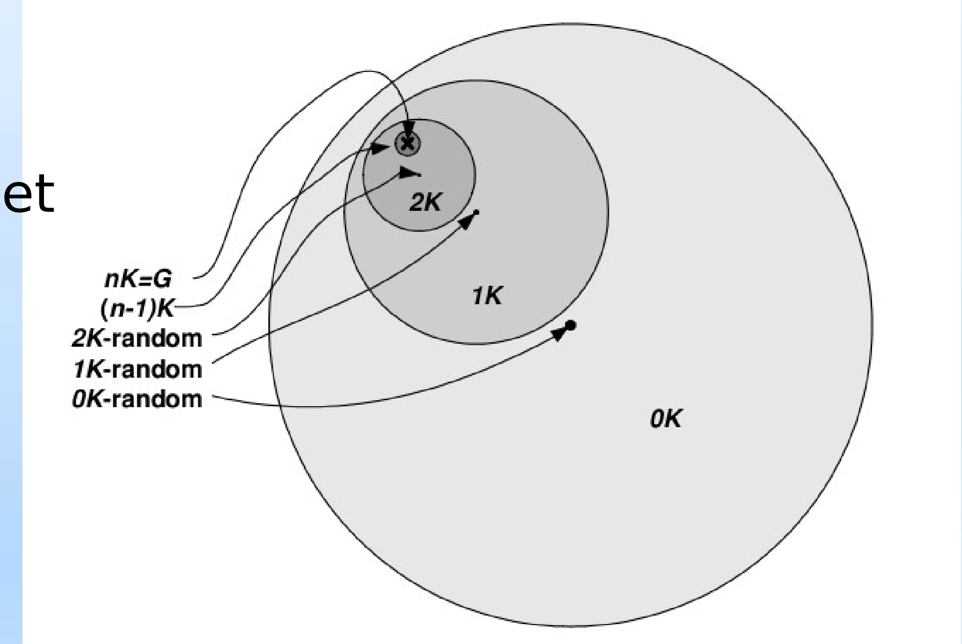
- A **2k** model and is defined by the joint-degree distribution and connects stubs based on the samples from this distribution:
- i.e. a node with degree 3 may connect to a node with degree 2 and so on.

The dK series.

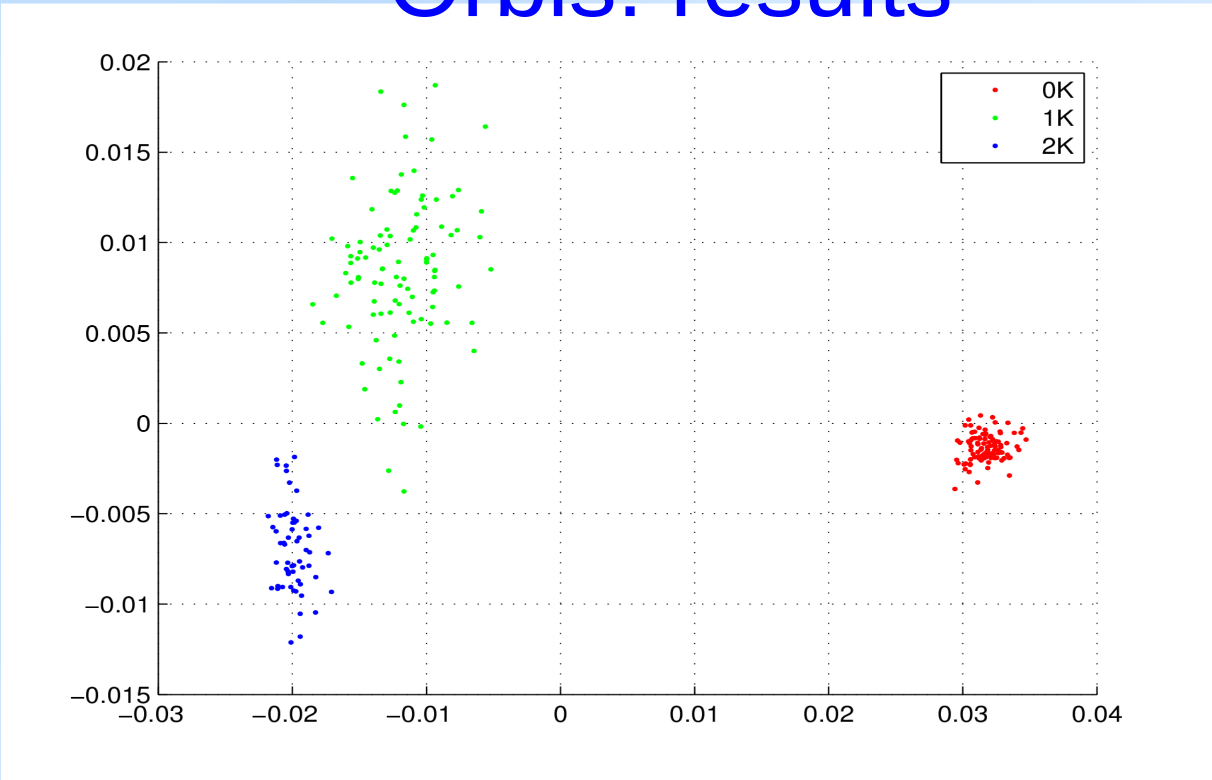
- The **0k, 1k, 2k** form a series called the **dK** series which specifies a subset of random graphs in which

  nk ⊂ ...⊂ **2k** ⊂ **1k** ⊂ **0k**

- This was shown diagrammatically in the Sigcomm '06 paper.
- **Aim**: To validate this subdivision of graphs using WSD and MDS.

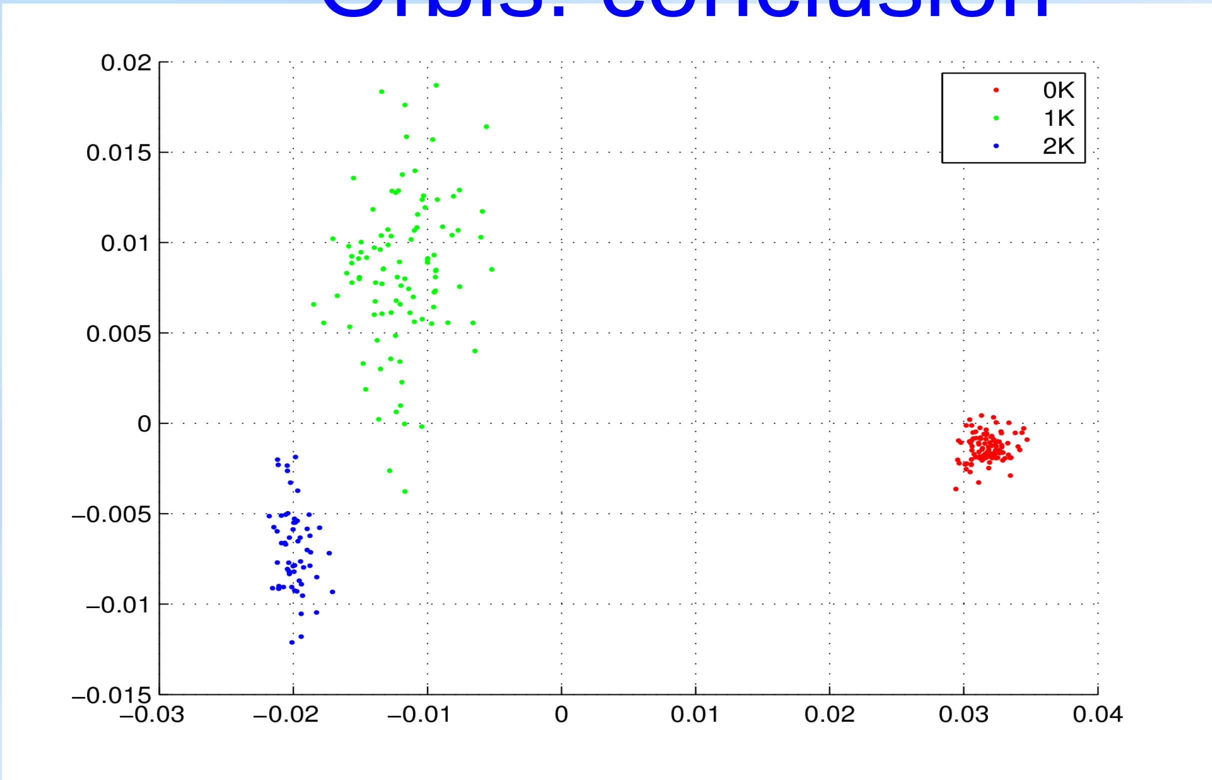


Orbis: results



- An AB model with 5000 nodes and average degree 5.6 is used as the base model for extracting an average degree, a degree distribution and a joint degree distribution.
- 100 graphs from the following ensembles are sampled:
- **0k** ensemble (average degree 5.6)
- **1k** ensemble (power law distribution)
- **2k** ensemble (no name).

Orbis: conclusion



Result:

- **0k –** very narrow set of graphs generated (ER – narrow degree distribution)
- **1k** ensemble (larger set of graphs)
- **2k** ensemble – the joint degree distribution of an AB model is not even close to the average of a **1K** specified by a degree distribution.

Application 6: Facial expression recognition.

•

Yale face dataset.

- 15 subjects performing 11 facial expressions.
- Image:

→ resize 256x256

→ quadtree decomposition

→ graph

→ WSD

→ MDS

→ clustering.

Original image.

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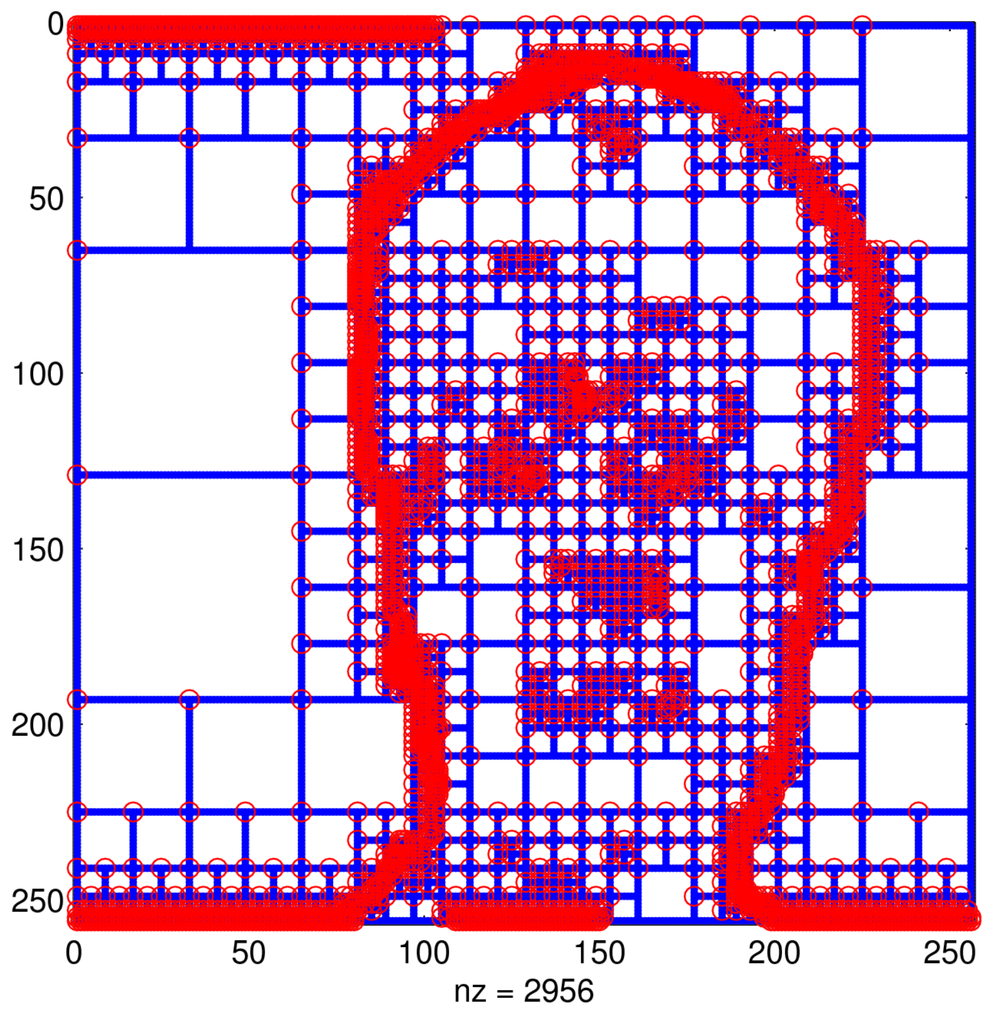
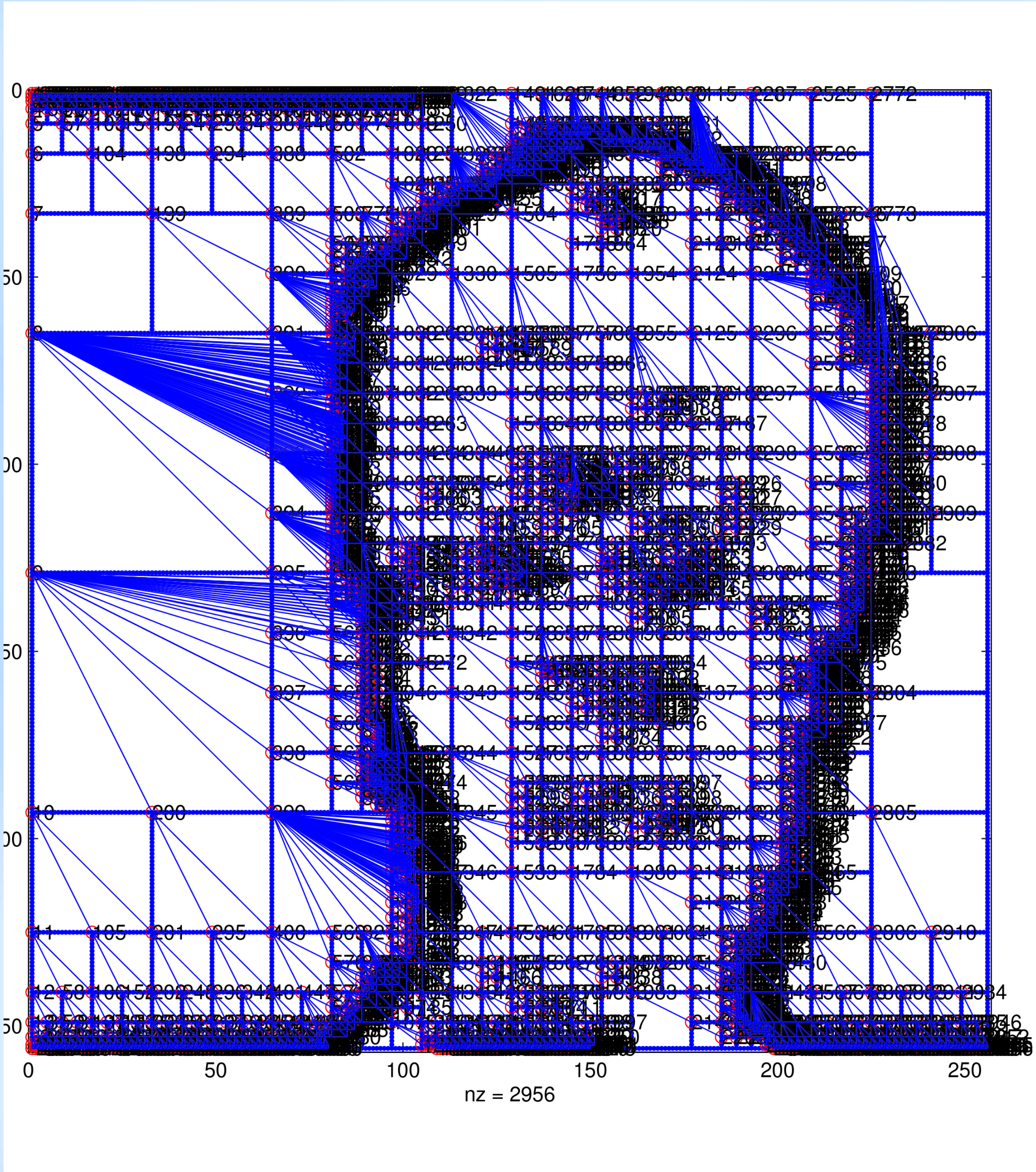
Resized image.

Quadtree decompostion.

•



Graph construction.



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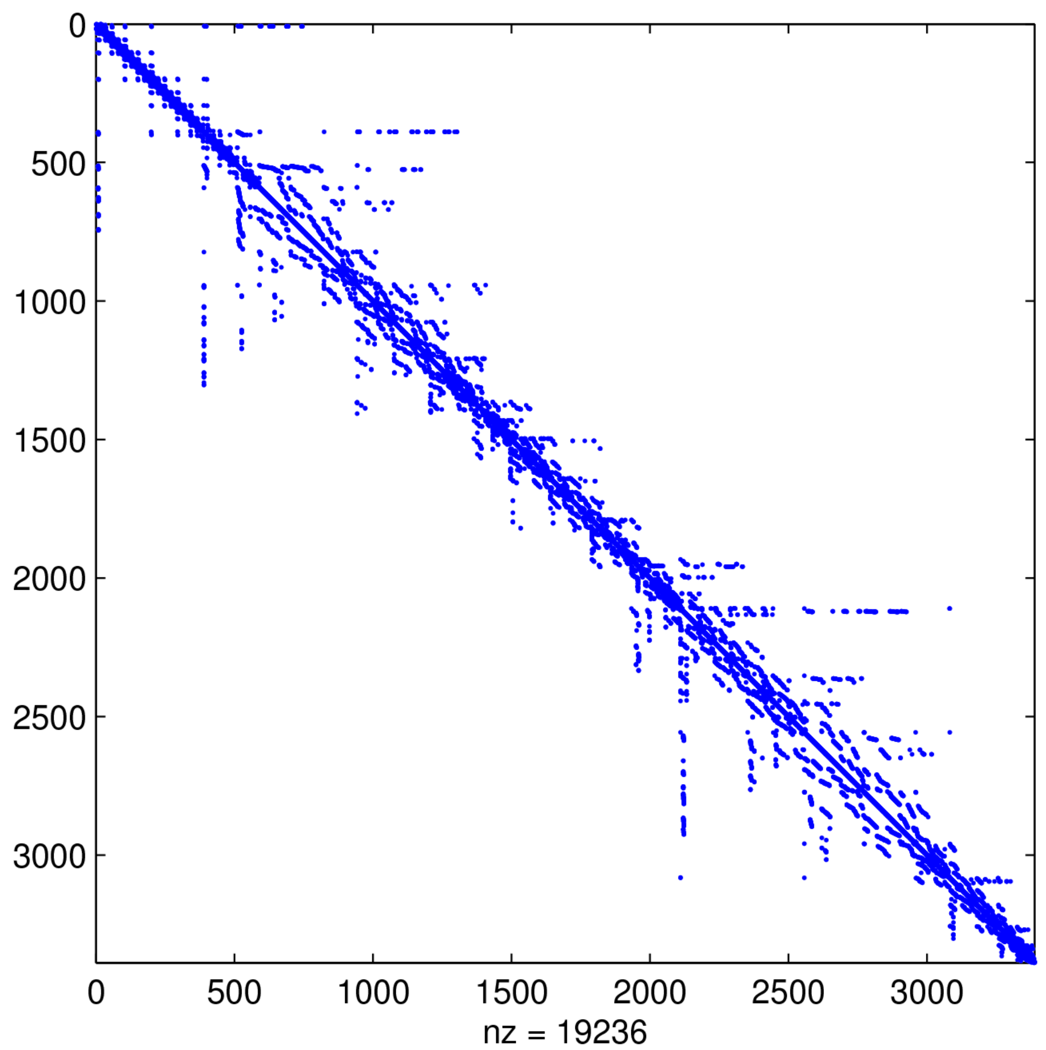
(weighted) Adjacency matrix.



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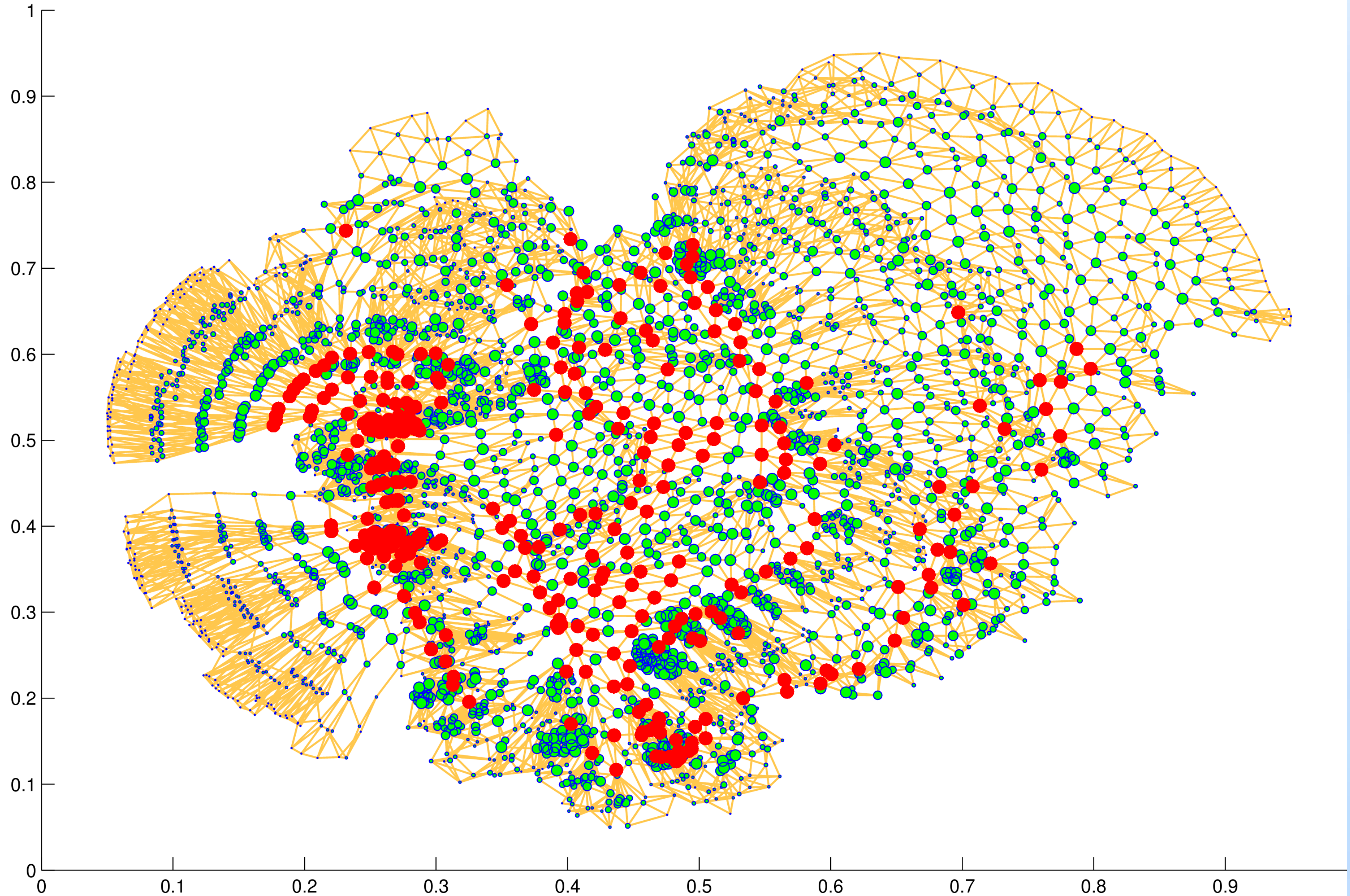
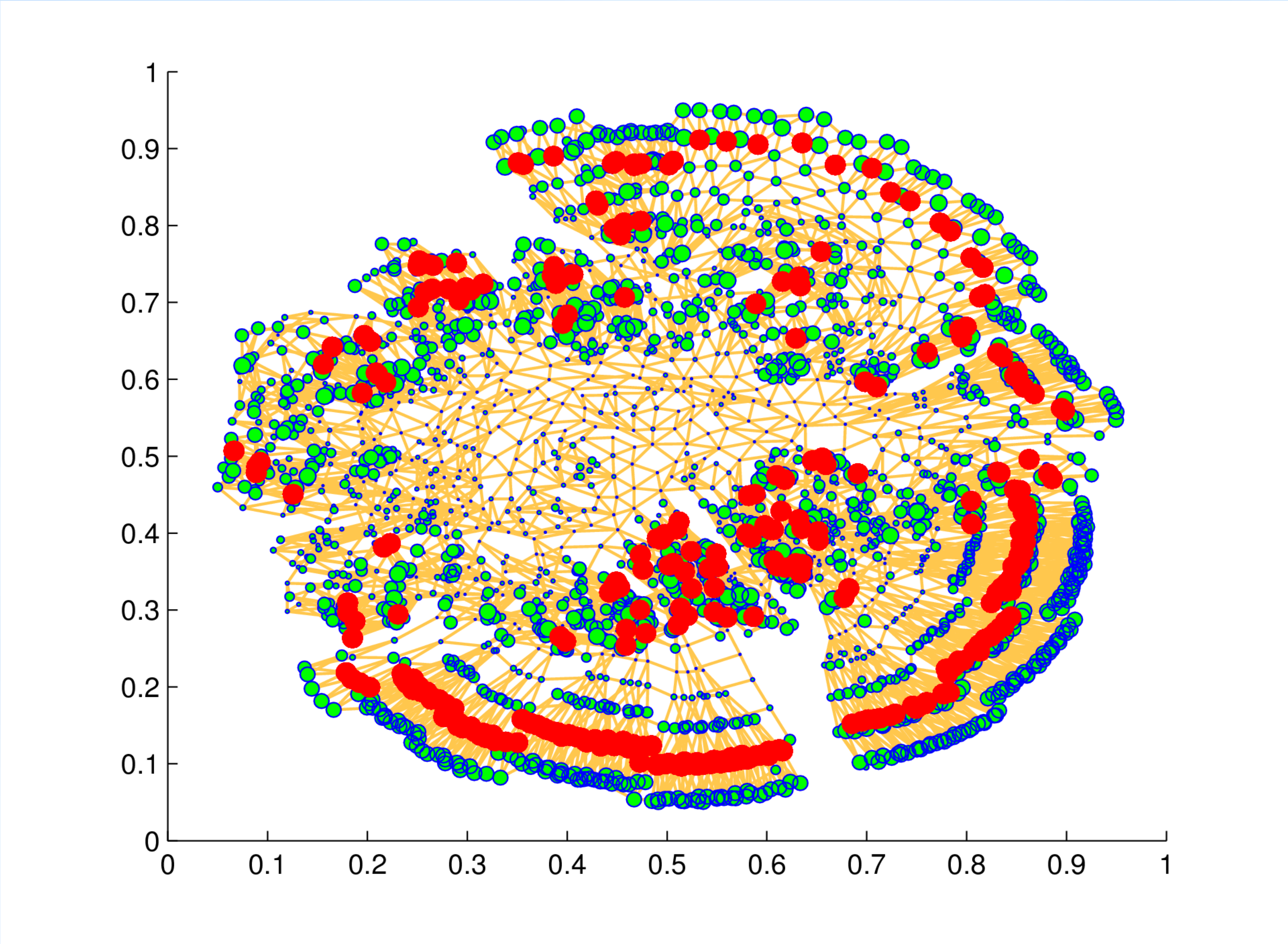
Weighted graph (face 1).



Weighted graph (face 10).

•



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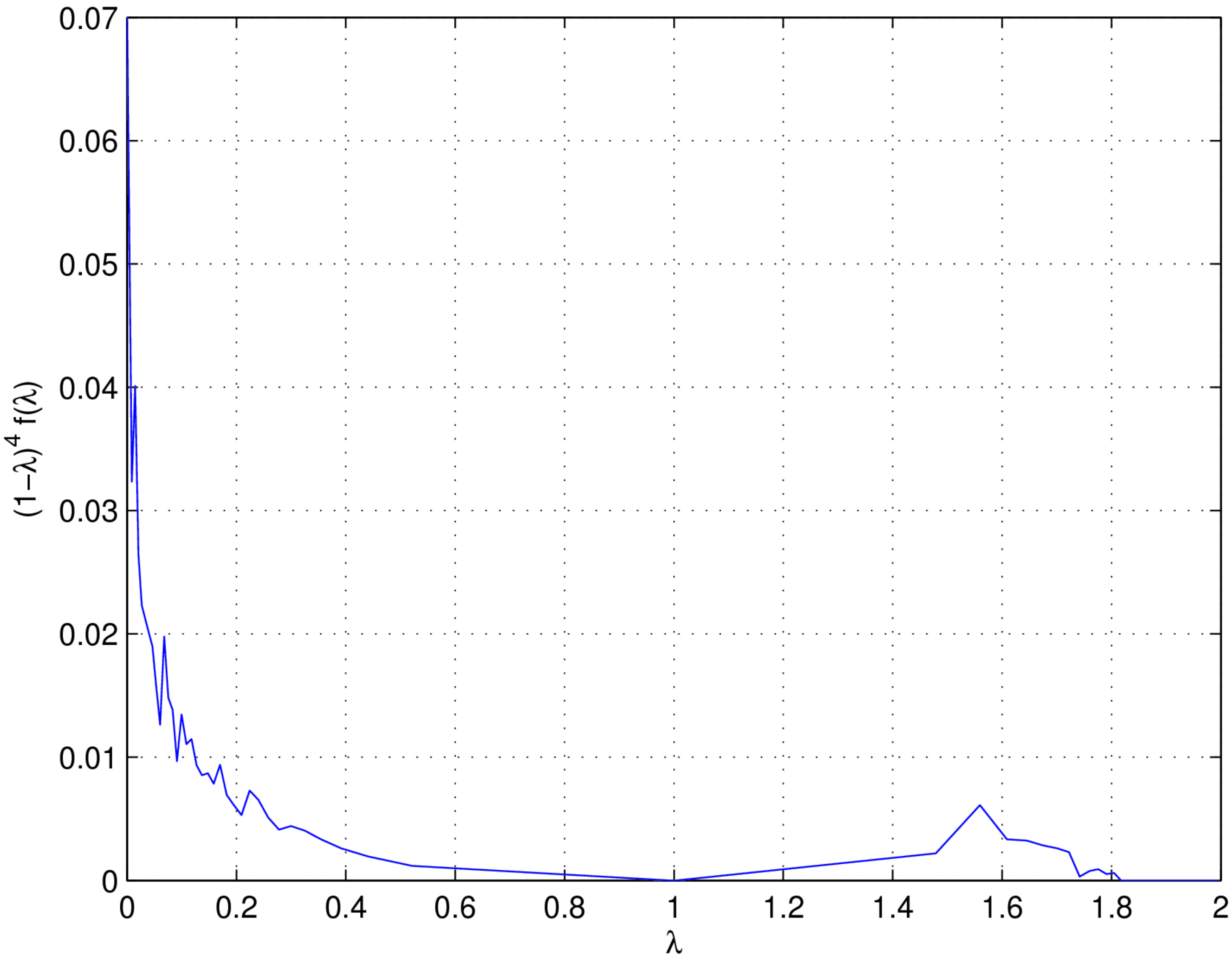
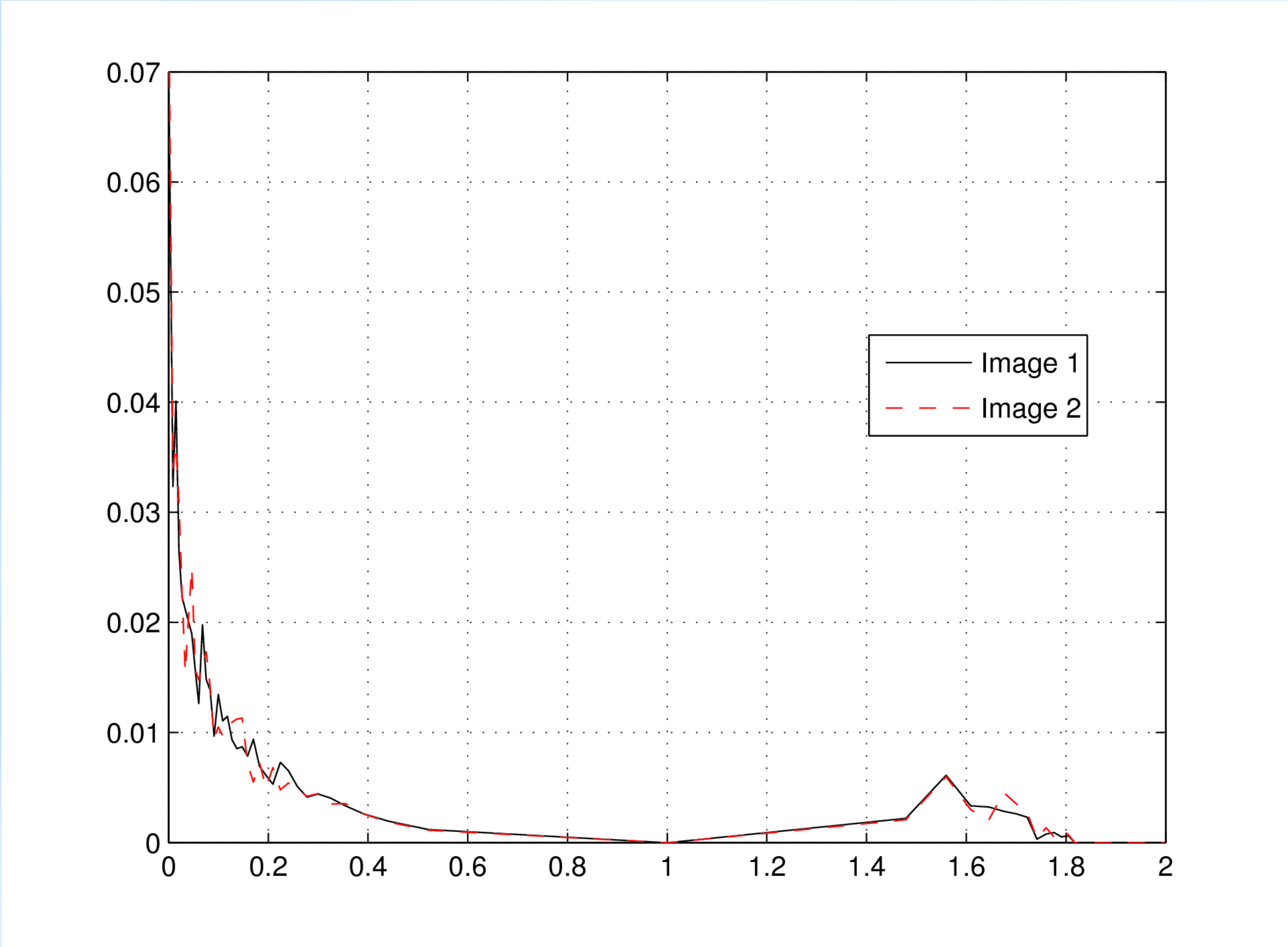
WSD.

•

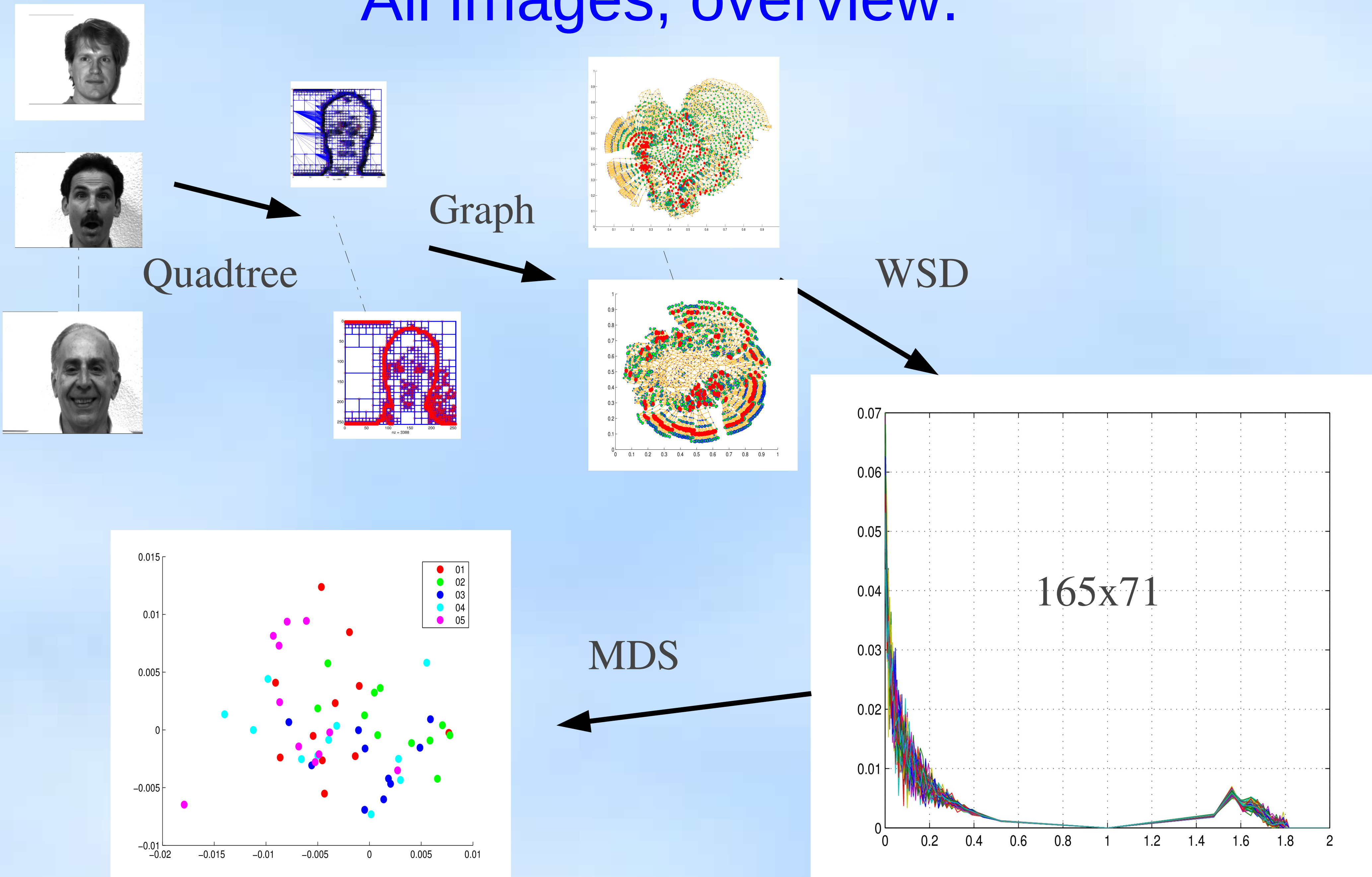


0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
$(1-\lambda)^4 f(\lambda)$
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
λ

WSD's of two images.



All images, overview.



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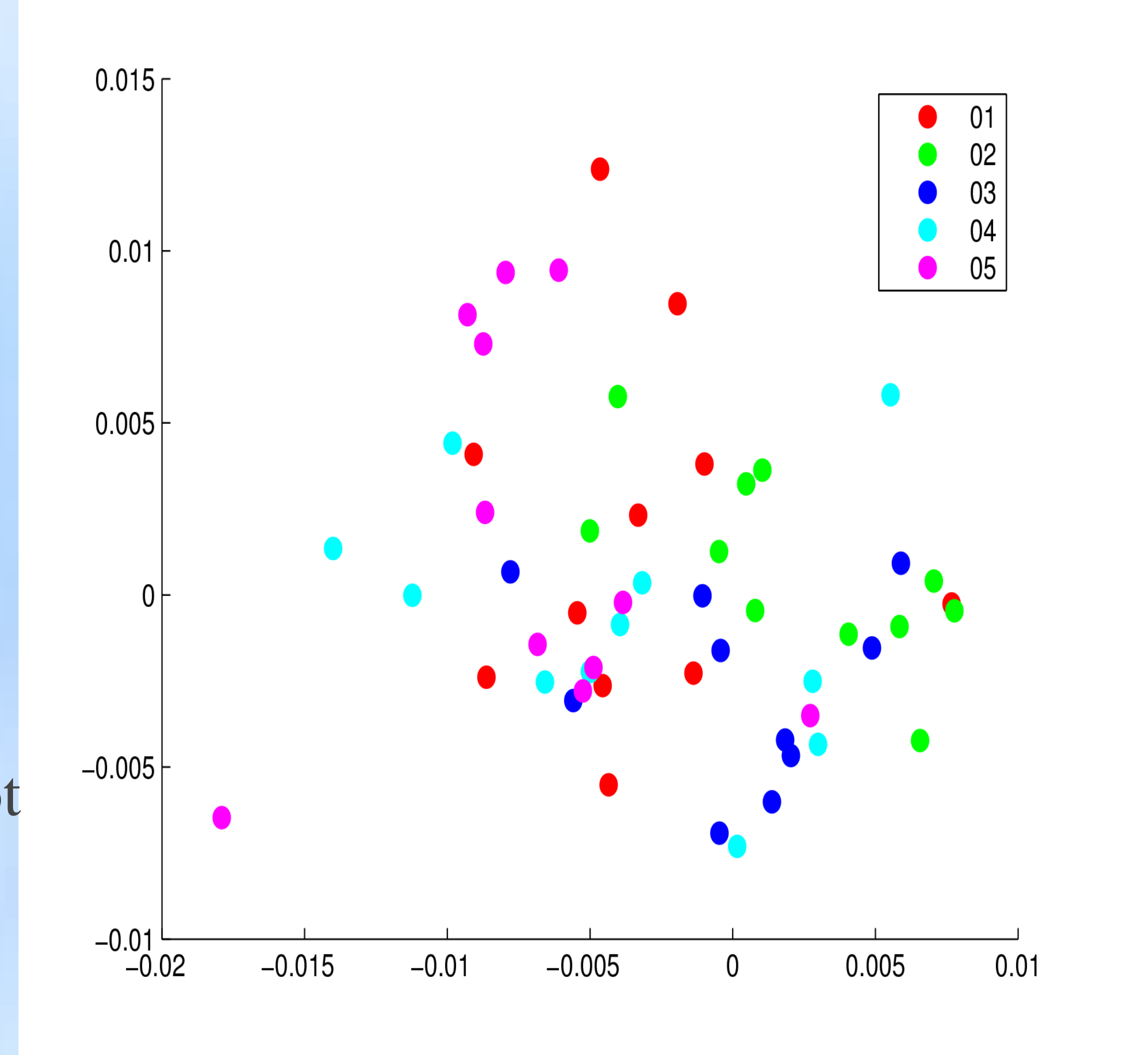
5 subjects all facial expressions.

- Only 5 subjects shown for clarity.
- The separation between subjects is not very good.
- Clusters do exist.
- Semantic information is missing.
- Shadow effects introduce a lot of noise.



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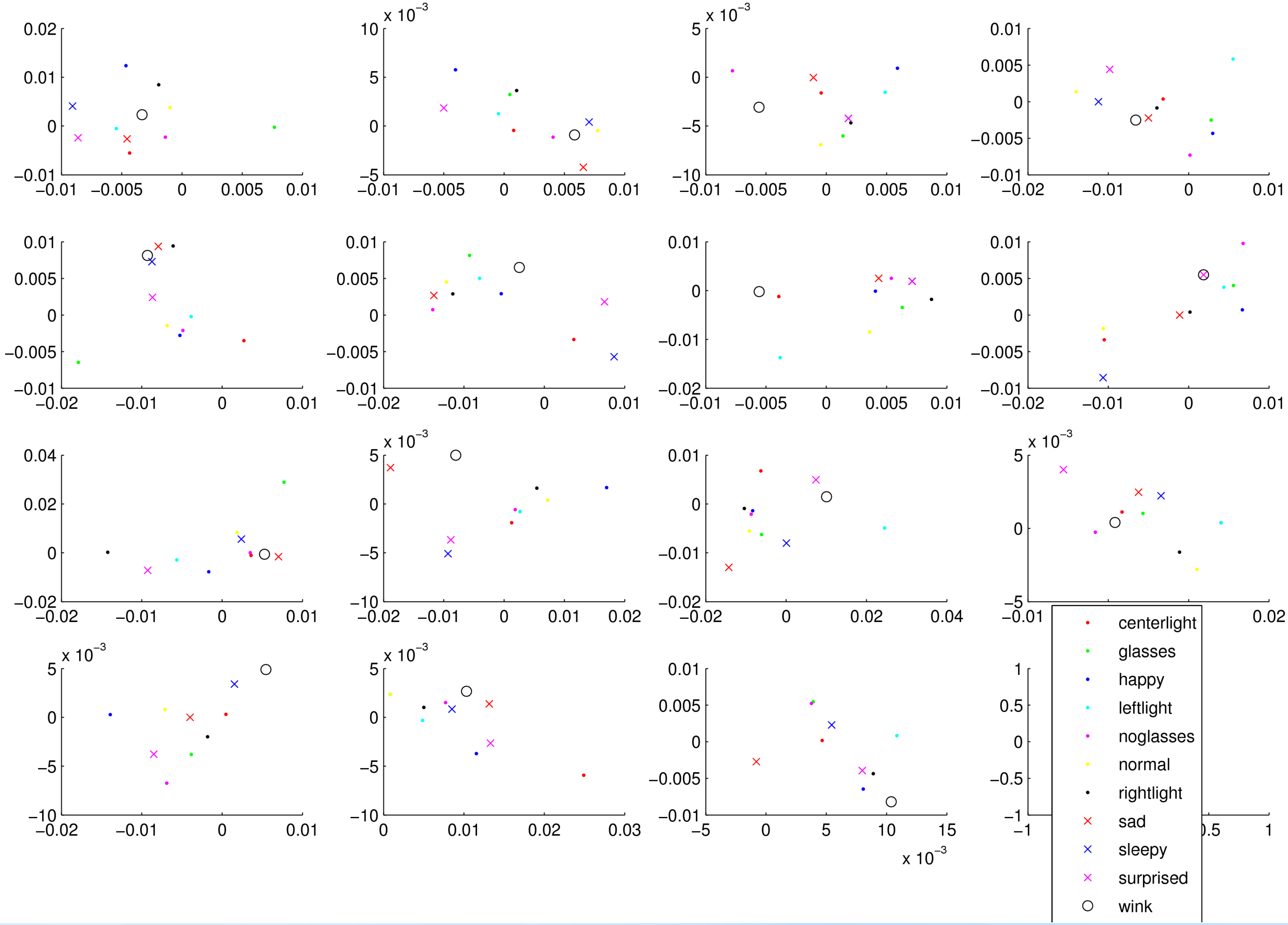
Breakdown by expression.



Summary.

- A graph metric, the WSD was introduced.
  - Based on N-cycles
  - Normalised Laplacian eigenvalue distribution.
  - Weighted distribution.
  - Quadratic norm between weighted distributions forms a graph metric.
- *Model* fitting applications:

  Optimum parameters for a topology generator.
  - Tracking evolution of the internet.
- Clustering applications:
  - Discriminating between topology generators.
  - Network application identification.
  - Orbis model analysis.
  - Face recognition*

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Questions ?

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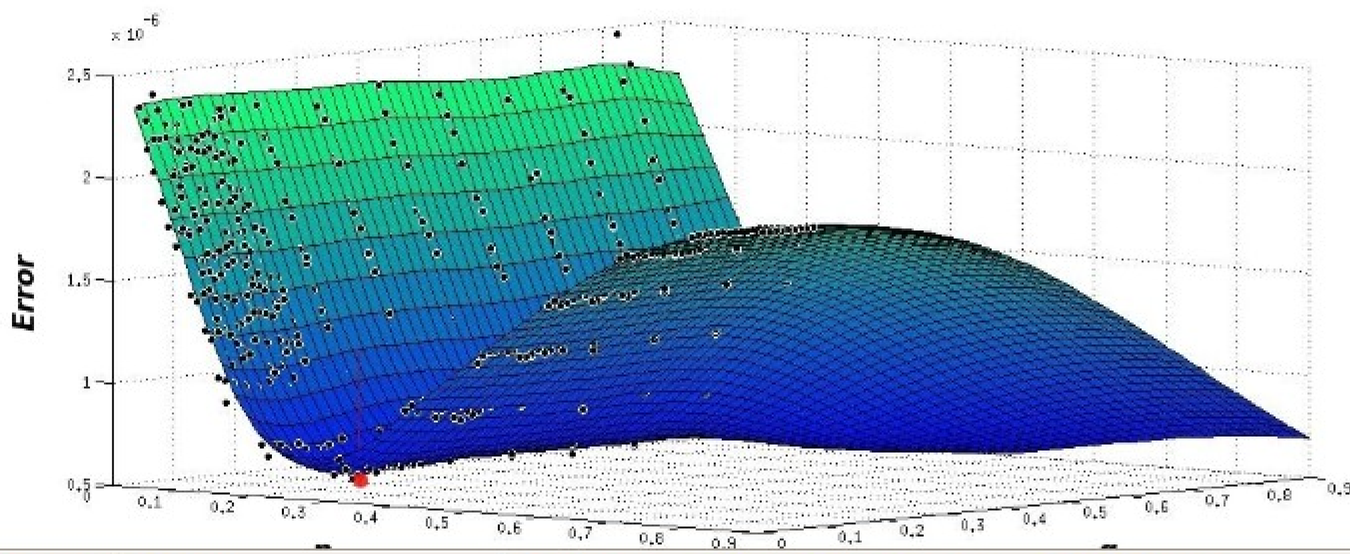
Computer Laboratory
Weighted Spectral Distribution Toolbox

Computer Laboratory > Research > Systems Research Group > NetOS > MASTS > Weighted Spectral Distribution Toolbox

Weighted Spectral Distribution Toolkit



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