Flexible presentations of graded monads

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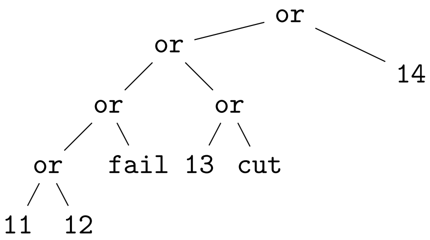
Example: nondeterminism with backtracking and cut

```
or(or(or(or(return11, return12), fail),
              or(return13, cut)), return14)
```



Terms should satisfy some equations:

$$\texttt{or}(M, N) \equiv M \qquad \text{whenever } M \text{ cuts}$$

$$\left(\begin{array}{l} \texttt{do x <- } N_1 \\ \quad\texttt{y <- } N_2 \\ \quad M \end{array}\right) \equiv \left(\begin{array}{l} \texttt{do y <- } N_2 \\ \quad\texttt{x <- } N_1 \\ \quad M \end{array}\right)$$

whenever M cuts
and N_1 cuts or `return`s something
and N_2 cuts or `return`s something

Models of effects from presentations

1. Effects can be modelled using monads [Moggi '89]
2. which often come from presentations [Plotkin and Power '02]
3. which induce algebraic operations [Plotkin and Power '03]

Example: (based on [Piróg and Staton '17])

1. Nondeterminism with can be modelled using a monad Cut
$$\mathrm{Cut}X = \mathrm{List}X \times \{\mathsf{cut}, \mathsf{nocut}\}$$
2. which comes from the presentation of monoids with a left zero:
$$\mathsf{or} : 2 \qquad \mathsf{fail} : 0 \qquad \mathsf{cut} : 0$$
$$\mathsf{or}(\mathsf{or}(x, y), z) \equiv \mathsf{or}(x, \mathsf{or}(y, z)) \qquad \mathsf{or}(\mathsf{fail}, x) \equiv x \equiv \mathsf{or}(x, \mathsf{fail}) \qquad \mathsf{or}(\mathsf{cut}, x) \equiv x$$
3. which induces algebraic operations
$$\mathrm{or}_X : \mathrm{Cut}\,X \times \mathrm{Cut}\,X \longrightarrow \mathrm{Cut}\,X$$
$$\mathrm{fail}_X : \mathbf{1} \longrightarrow \mathrm{Cut}\,X \qquad \mathrm{cut}_X : \mathbf{1} \longrightarrow \mathrm{Cut}\,X$$

Example: nondeterminism with backtracking and cut

$$\texttt{or}(M, N) \equiv M \qquad \text{whenever } M \text{ cuts}$$

$$\left(\begin{array}{l}\texttt{do x <- } N_1\\ \quad\;\texttt{y <- } N_2\\ \quad\; M\end{array}\right) \equiv \left(\begin{array}{l}\texttt{do y <- } N_2\\ \quad\;\texttt{x <- } N_1\\ \quad\; M\end{array}\right) \qquad \begin{array}{l}\text{whenever } M \text{ cuts}\\ \text{and } N_1 \text{ cuts or } \texttt{returns} \text{ something}\\ \text{and } N_2 \text{ cuts or } \texttt{returns} \text{ something}\end{array}$$

Grading

Grade computations by elements of an ordered monoid:

$$(\mathbb{E}, \leq, 1, \cdot)$$

so that they form a graded monad

For example:

- lists graded by $(\mathbb{N}, =, 1, \cdot)$

$$\mathrm{Vec}\, X\, e = \text{lists over } X \text{ of length exactly } e$$

- lists graded by $(\mathbb{N}, \leq, 1, \cdot)$

$$\mathrm{BVec}\, X\, e = \text{lists over } X \text{ of length at most } e$$

Example: grading nondeterminism with backtracking and cut

$$\texttt{or}(M,N) \equiv M \qquad \text{whenever } M \text{ has grade } \bot$$

Assign grades $e \in \{\bot, 1, \top\}$ to computations:

$\top$ don't know anything

$\vee|$

1 definitely cuts or `returns` something

$\vee|$

$\bot$ definitely cuts

Graded monad Cut:

$$\mathrm{Cut} X e = \{(\mathrm{xs}, c) \in \mathrm{List} X \times \{\mathsf{cut}, \mathsf{nocut}\} \mid (e = \bot \Rightarrow c = \mathsf{cut}) \wedge (e = 1 \Rightarrow c = \mathsf{cut} \vee \mathrm{xs} \neq [\,])\}$$

Kleisli extension:

$$\frac{f : X \to \mathrm{Cut} Y e}{f_d^\dagger : \mathrm{Cut} X d \to \mathrm{Cut} Y (d \cdot e)} \quad \text{where} \quad \begin{array}{rcl} \top \cdot e & = & \top \\ 1 \cdot e & = & e \\ \bot \cdot e & = & \bot \end{array}$$

Example: grading nondeterminism with backtracking and cut

1. Nondeterminism with cut can be modelled using a graded monad Cut

$$\begin{aligned} \mathrm{Cut} X e = \{(\mathrm{xs}, c) \in \mathrm{List} X \times \{\mathsf{cut}, \mathsf{nocut}\} \\ \mid (e = \bot \Rightarrow c = \mathsf{cut}) \\ \wedge (e = 1 \Rightarrow c = \mathsf{cut} \vee \mathrm{xs} \neq [\,])\} \end{aligned}$$

2. which comes from a graded presentation of monoids with a left zero?
3. which induces graded algebraic operations?

$$\mathrm{or}_{d_1, d_2, X} : \mathrm{Cut}\, X\, d_1 \times \mathrm{Cut}\, X\, d_2 \to \mathrm{Cut}\, X\, (d_1 \sqcap d_2) \qquad (d_1, d_2 \in \{\bot, 1, \top\})$$

$$\mathrm{fail}_X : \mathbf{1} \to \mathrm{Cut}\, X\, \top \qquad \mathrm{cut}_X : \mathbf{1} \to \mathrm{Cut}\, X\, \bot$$

The existing notions of graded presentation are not general enough

[Smirnov '08, Milius et al. '15, Dorsch et al. '19, Kura '20]

This work

Develop a notion of flexibly graded presentation

- Every flexibly graded presentation (Σ, E) induces
  - a canonical graded monad $\mathsf{T}_{(\Sigma,E)}$
  - along with a flexibly graded algebraic operation for each operation of the presentation
- Examples like Cut have computationally natural flexibly graded presentations
- The constructions are mathematically justified by locally graded categories, and a notion of flexibly graded abstract clone

Flexibly graded presentations

Given an ordered monoid $(\mathbb{E}, \leq, 1, \cdot)$ of grades,

a flexibly graded presentation (Σ, E) consists of

- a signature Σ: sets
$$\Sigma(d_1', \ldots, d_n'; d)$$
of operations
$$\frac{e \in \mathbb{E} \quad \Gamma \vdash t_1 : d_1' \cdot e \quad \cdots \quad \Gamma \vdash t_n : d_n' \cdot e}{\Gamma \vdash \mathrm{op}(e; t_1, \ldots, t_n) : d \cdot e}$$
- a collection of axioms E: sets
$$E(d_1', \ldots, d_n'; d)$$
of equations
$$x_1 : d_1', \ldots, x_n : d_n' \vdash t \equiv u : d$$

Part of the presentation of nondeterminism with cut:

$$\text{grades } \mathbb{E} = \{\bot \leq 1 \leq \top\}$$

$$\frac{\Gamma \vdash t_1 : d_1' \cdot e \qquad \Gamma \vdash t_2 : d_2' \cdot e}{\Gamma \vdash \mathrm{or}_{d_1', d_2'}(e; t_1, t_2) : (d_1' \sqcap d_2') \cdot e}$$

$$\mathrm{or}_{\bot, e}(1; x, y) \equiv x$$

Semantics

For every flexibly graded presentation (Σ, E), there is:

[Wood '76]

- a notion of (Σ, E)-algebra, forming a locally graded category $\mathbf{Alg}(\Sigma, E)$

A (Σ, E)-algebra $(A, [\![-]\!])$ is:

- a graded set $A : \mathbb{E} \longrightarrow \mathbf{Set}$
- with an interpretation

$$[\![\mathrm{op}]\!]_e : \prod_i A(d'_i \cdot e) \longrightarrow A(d \cdot e) \qquad \text{natural in } e$$

  of each $\mathrm{op} \in \Sigma(d'_1, \ldots, d'_n; d)$
- satisfying each axiom $t \equiv u$ of E:

$$[\![t]\!]_e = [\![u]\!]_e \qquad \text{for every } e$$

Semantics

For every flexibly graded presentation (Σ, E), there is: [Wood '76]

- a notion of (Σ, E)-algebra, forming a locally graded category $\mathbf{Alg}(\Sigma, E)$
- a sound and complete equational logic

$\Gamma \vdash t \equiv u : d$ generated by

$$\frac{(t, u) \in E(d'_1, \ldots, d'_n;\, d) \qquad \Gamma \vdash s_1 : d'_1 \cdot e \qquad \cdots \qquad \Gamma \vdash s_n : d'_n \cdot e}{\Gamma \vdash t\{e; x_1 \mapsto s_1, \ldots, x_n \mapsto s_n\} \equiv u\{e; x_1 \mapsto s_1, \ldots, x_n \mapsto s_n\} : d \cdot e}$$

and some other rules

Soundness and completeness:

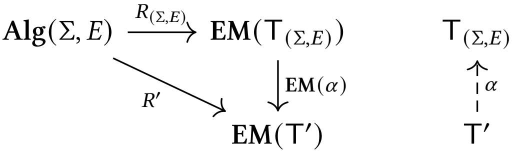
$$[\![t]\!] = [\![u]\!] \text{ in every } (\Sigma, E)\text{-algebra} \quad \Leftrightarrow \quad \Gamma \vdash t \equiv u : d \text{ is derivable}$$

Semantics

For every flexibly graded presentation (Σ, E), there is: [Wood '76]

- a notion of (Σ, E)-algebra, forming a locally graded category $\mathbf{Alg}(\Sigma, E)$
- a sound and complete equational logic
- a graded monad $\mathsf{T}_{(\Sigma,E)}$ on $\mathbf{Set}$ and concrete functor $R_{(\Sigma,E)} : \mathbf{Alg}(\Sigma, E) \to \mathbf{EM}(\mathsf{T}_{(\Sigma,E)})$, satisfying a universal property

For every graded monad T' and concrete functor $R' : \mathbf{Alg}(\Sigma, E) \to \mathbf{EM}(\mathsf{T}')$:



(But $R_{(\Sigma,E)}$ is usually not an isomorphism)

Semantics

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- a notion of (Σ, E)-algebra, forming a locally graded category $\mathbf{Alg}(\Sigma, E)$
- a sound and complete equational logic
- a graded monad $\mathsf{T}_{(\Sigma,E)}$ on $\mathbf{Set}$ and concrete functor $R_{(\Sigma,E)} : \mathbf{Alg}(\Sigma, E) \to \mathbf{EM}(\mathsf{T}_{(\Sigma,E)})$, satisfying a universal property
- for every op in Σ, a flexibly graded algebraic operation for $\mathsf{T}_{(\Sigma,E)}$

For op $\in \Sigma(d'_1, \ldots, d'_n; d)$:

$$\alpha_{\mathsf{op},X,e} : \textstyle\prod_i T_{(\Sigma,E)}X(d'_i \cdot e) \to T_{(\Sigma,E)}X(d \cdot e)$$

natural in e, and compatible with Kleisli extension

(Because each free $\mathsf{T}_{(\Sigma,E)}$-algebra $T_{(\Sigma,E)}X$ forms a (Σ, E)-algebra)

Semantics

For every flexibly graded presentation (Σ, E), there is: [Wood '76]

- a notion of (Σ, E)-algebra, forming a locally graded category $\mathbf{Alg}(\Sigma, E)$
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- a graded monad $\mathsf{T}_{(\Sigma,E)}$ on $\mathbf{Set}$ and concrete functor $R_{(\Sigma,E)} : \mathbf{Alg}(\Sigma, E) \to \mathbf{EM}(\mathsf{T}_{(\Sigma,E)})$, satisfying a universal property
- for every op in Σ, a flexibly graded algebraic operation for $\mathsf{T}_{(\Sigma,E)}$

A large class of graded monads have flexibly graded presentations:

- exactly the finitary graded monads on $\mathbf{Set}$
- correspondence goes via flexibly graded clones

Graded monads we care about have natural flexibly graded presentations