Sweedler theory of monads

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Interaction laws [Katsumata, Rivas, Uustalu '20]

To run,

an effectful (effect-requesting) program behaving as
a computation

needs to **interact** with

a environment
that an effect-providing (coeffectful) machine behaves as

For example:

- a nondeterministic program needs a machine making choices
- a stateful program needs a machine coherently responding to fetch and store commands

Interaction laws [Katsumata, Rivas, Uustalu '20]

A monad–comonad interaction law (on **Set**) consists of

- A monad T (TX: computations, X: results)
- A comonad D (DY: environments, Y: states)

- A family of functions

$$\psi_{X,Y} : TX \times DY \to X \times Y$$

satisfying some laws.

(More generally: work in a locally presentable symmetric monoidal closed category)

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A monad–comonad interaction law (on $\mathbf{Set}$) consists of

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- A comonad D (DY: environments, Y: states)
- A monad R ($R(X \times Y)$: residual computations)
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satisfying some laws.

Example:

$TX = V \Rightarrow (V \times X)$ (state) $DY = V \times (V \Rightarrow Y)$ (costate) $RZ = Z$

$$\begin{array}{rcl} \psi_{X,Y} : \quad (V \Rightarrow (V \times X)) \times (V \times (V \Rightarrow Y)) & \longrightarrow & X \times Y \\ (t, (v, f)) & \mapsto & \text{let } (v', x) = t\, v \text{ in } (x, f\, v') \end{array}$$

Interaction laws [Katsumata, Rivas, Uustalu '20]

Example:

- $TX = \mu Z.X + Z^2$ – inductively generated by

$$\frac{x \in X}{\mathtt{Lf}(x) \in TX} \qquad \frac{\ell \in TX \qquad r \in TX}{\mathtt{Br}(\ell, r) \in TX}$$

- $DY = \nu Z.Y \times (2 \times Z)$ – coinductively generated by

$$\frac{y \in Y \qquad d \in DY}{(y, \mathtt{Left}(d)) \in DY} \qquad \frac{y \in Y \qquad d \in DY}{(y, \mathtt{Right}(d)) \in DY}$$

- $RZ = Z$

$$\begin{aligned} \psi_{X,Y} : TX \times DY &\longrightarrow R(X \times Y) \\ (\mathtt{Lf}(x), (y, \_)) &\mapsto (x, y) \\ \\ (\mathtt{Br}(\ell, r), (y, \mathtt{Left}(d))) &\mapsto \psi(\ell, d) \\ (\mathtt{Br}(\ell, r), (y, \mathtt{Right}(d))) &\mapsto \psi(r, d) \end{aligned}$$

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Example:

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$$\frac{x \in X}{\mathtt{Lf}(x) \in TX} \qquad \frac{\ell \in TX \qquad r \in TX}{\mathtt{Br}(\ell, r) \in TX}$$

- $DY = \nu Z.Y \times (1 + 2 \times Z)$ – coinductively generated by

$$\frac{y \in Y}{(y, \mathtt{Stop}) \in DY} \qquad \frac{y \in Y \qquad d \in DY}{(y, \mathtt{Left}(d)) \in DY} \qquad \frac{y \in Y \qquad d \in DY}{(y, \mathtt{Right}(d)) \in DY}$$

- $RZ = 1 + Z$

$$\begin{array}{rcl} \psi_{X,Y} : TX \times DY & \rightarrow & R(X \times Y) \\ (\mathtt{Lf}(x), (y, \_)) & \mapsto & \mathtt{inr}(x, y) \\ (\mathtt{Br}(\ell, r), (y, \mathtt{Stop})) & \mapsto & \mathtt{inl} \star \\ (\mathtt{Br}(\ell, r), (y, \mathtt{Left}(d))) & \mapsto & \psi(\ell, d) \\ (\mathtt{Br}(\ell, r), (y, \mathtt{Right}(d))) & \mapsto & \psi(r, d) \end{array}$$

This work: Sweedler theory of monads

Given any two of T, D, R, is there a universal choice for the other (co)monad, forming an interaction law?

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Given any two of T, D, R, is there a universal choice for the other (co)monad, forming an interaction law?

Main theorem: if

- the two given (co)monads are accessible

then the universal choice for the third:

- exists
- has a characterization in terms of its (co)algebras
- is accessible

Sweedler power

Given D, R, the *Sweedler power* is the final interacting monad: a monad $D \mathrel{-\!\!\star} R$ with bijections

$$\frac{TX \times DY \longrightarrow R(X \times Y) \quad \text{iteraction law}}{T \longrightarrow D \mathrel{-\!\!\star} R \quad \text{monad morphism}}$$

$$\begin{array}{ccc} TX \times DY & & \\ \downarrow{\scriptstyle \psi^\sharp_X \times DY} & \searrow^{\psi_{X,Y}} & \\ (D \mathrel{-\!\!\star} R)X \times DY & \longrightarrow & R(X \times Y) \end{array} \qquad \begin{array}{c} T \\ \downarrow{\scriptstyle \psi^\sharp} \\ D \mathrel{-\!\!\star} R \end{array}$$

Sweedler power

Given D, R, the *Sweedler power* is the final interacting monad: a monad $D \multimap\!\!\star\, R$ with bijections

$$\frac{TX \times DY \to R(X \times Y) \quad \text{iteraction law}}{\overline{T \to D \multimap\!\!\star\, R \quad \text{monad morphism}}}$$

If D, R are accessible functors, then

$$\frac{TX \times DY \to R(X \times Y) \quad \text{natural in } X, Y}{\overline{TX \to (D \multimap\!\!\star\, R)X \quad \text{natural in } X}}$$

where

$$(D \multimap\!\!\star\, R)X = \int_Y DY \Rightarrow R(X \times Y) \qquad \left(\begin{matrix} Y\text{-natural families of functions} \\ f_Y : DY \to R(X \times Y) \end{matrix} \right)$$

For D, R accessible, the Sweedler power is $D \multimap\!\!\star\, R$

- Example: if $DY = V \times (V \Rightarrow Y)$ (costate) and $R = \mathrm{Id}$, then $(D \multimap\!\!\star\, R)X \cong V \Rightarrow (V \times X)$ (state)

Sweedler hom

Given T, R, the *Sweedler hom* is the final interacting comonad: a comonad $\mathcal{M}(T,R)$ with bijections

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$$\frac{TX \times DY \to R(X \times Y) \quad \text{iteraction law}}{D \to \mathcal{M}(T,R) \quad \text{comonad morphism}}$$

If T is a free monad F^* (F accessible), then

$$\begin{array}{ll} F^*X \times DY \to R(X \times Y) & \text{interaction law} \\ \hline\hline F^* \to D \multimap\!\!\star\, R & \text{monad morphism} \\ \hline\hline F \to D \multimap\!\!\star\, R & \text{natural transformation} \\ \hline\hline D \to F \multimap\!\!\star\, R & \text{natural transformation} \\ \hline\hline D \to (F \multimap\!\!\star\, R)^\dagger & \text{comonad morphism} \end{array}$$

so $\mathcal{M}(F^*,R) \cong (F \multimap\!\!\star\, R)^\dagger$

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If T is a free monad F^* (F accessible), then $\mathcal{M}(F^*, R) \cong (F \multimap\!\!\star\, R)^\dagger$

Example: if $TX = \mu Z.X + Z^2$, and $RZ = 1 + Z$ then

$$\begin{aligned} \mathcal{M}(T,R) &\cong ((-)^2 \multimap\!\!\star\, R)^\dagger \\ &\cong (1 + (2 \times -))^\dagger \\ &\cong \nu Z.\, Y \times (1 + 2 \times Z) \end{aligned}$$

So we have interaction laws:

$$\frac{TX \times (\mathcal{M}(T,R))Y \to R(X \times Y)}{\overline{\mathcal{M}(T,R) \to \mathcal{M}(T,R)}} \qquad \frac{TX \times (\nu Z.\, Y \times (2 \times Z)) \to R(X \times Y)}{\overline{(\nu Z.\, (-) \times (2 \times Z)) \to \mathcal{M}(T,R)}}$$

Sweedler hom, (co)algebraically

There is a (co)algebraic perspective on interaction laws [Uustalu and Voorneveld '20]:

$$\frac{\psi : TX \times DY \to R(X \times Y) \quad \text{interaction law}}{\Psi : \mathbf{Coalg}(D) \to [\mathbf{Alg}(R), \mathbf{Alg}(T)]^{\mathrm{op}} \quad \text{such that}}$$

$$\begin{array}{ccccc}
\mathbf{Coalg}(D) & & \xrightarrow{\quad\Psi\quad} & & [\mathbf{Alg}(R), \mathbf{Alg}(T)]^{\mathrm{op}} \\
{\scriptstyle U}\downarrow & & & & \downarrow{\scriptstyle [\mathbf{Alg}(R), U]^{\mathrm{op}}} \\
\mathbf{Set} & \xrightarrow[(Y \mapsto Y \Rightarrow -)^{\mathrm{op}}]{} & [\mathbf{Set}, \mathbf{Set}]^{\mathrm{op}} & \xrightarrow[{[U, \mathbf{Set}]^{\mathrm{op}}}]{} & [\mathbf{Alg}(R), \mathbf{Set}]^{\mathrm{op}}
\end{array}$$

$$\Psi : \quad \begin{array}{c} Y \\ \downarrow{\scriptstyle y} \\ DY \end{array} \quad \mapsto \quad \begin{array}{c} RZ \\ \downarrow{\scriptstyle z} \\ Z \end{array} \quad \mapsto \quad \begin{array}{c} T(Y \Rightarrow Z) \\ \downarrow{\scriptstyle \dots} \\ Y \Rightarrow Z \end{array}$$

Sweedler hom, (co)algebraically

$$\begin{array}{ccccc} \mathbf{SRun}_R(T) & \longrightarrow & & & [\mathbf{Alg}(R),\mathbf{Alg}(T)]^{\mathrm{op}} \\ {\scriptstyle U}\downarrow & \lrcorner & & & \downarrow{\scriptstyle [\mathbf{Alg}(R),U]^{\mathrm{op}}} \\ \mathbf{Set} & \xrightarrow[(Y\mapsto Y\Rightarrow -)^{\mathrm{op}}]{} & [\mathbf{Set},\mathbf{Set}]^{\mathrm{op}} & \xrightarrow[[U,\mathbf{Set}]^{\mathrm{op}}]{} & [\mathbf{Alg}(R),\mathbf{Set}] \end{array}$$

- $U : \mathbf{SRun}_R(T) \to \mathbf{Set}$ is comonadic, the comonad is $\mathcal{M}(T,R)$
- $\mathbf{SRun}_R(T)$: R-residual stateful runners of T

Stateful runners

An R-residual stateful runner of T is: [Uustalu and Voorneveld '20]

- a set Y (the carrier)
- with a natural family of functions $\theta_X : TX \times Y \to R(X \times Y)$
- satisfying a unit law and a multiplication law

$$
\begin{array}{ccc}
X \times Y & = & X \times Y \\
\downarrow \eta_{X \times Y} & & \downarrow \eta_{X \times Y} \\
TX \times Y & \xrightarrow[\theta_X]{} & R(X \times Y)
\end{array}
\qquad
\begin{array}{ccccc}
TTX \times Y & \xrightarrow{\theta_{TX}} & R(TX \times Y) & \xrightarrow{R\theta_X} & RR(X \times Y) \\
\downarrow \mu_{X \times Y} & & & & \downarrow \mu_{X \times Y} \\
TX \times Y & & \xrightarrow[\theta_{X \times Y}]{} & & R(X \times Y)
\end{array}
$$

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TX \times Y & & \xrightarrow[\theta_{X\times Y}]{} & & R(X \times Y)
\end{array}
$$

If R, T are accessible, then:

- $U : \mathbf{SRun}_R(T) \to \mathbf{Set}$ is comonadic
- the induced comonad is the Sweedler hom $\mathcal{M}(T, R)$

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Example: if $TX = V \Rightarrow X$ (reader) and $RZ = Z$ (identity), then

- stateful runners $(Y, \theta : (V \Rightarrow X) \times Y \to X \times Y)$ are equivalently pairs $(Y, Y \to V)$
- the Sweedler hom $\mathcal{M}(T, R)$ is $V \times (-)$ (coreader comonad)

$$(t, (v, y)) \mapsto (t\, v, y) : (V \Rightarrow X) \times (V \times Y) \to X \times Y$$

Interaction laws and Sweedler theory

If D, R are accessible, then

$$\frac{TX \times DY \longrightarrow R(X \times Y) \quad \text{interaction law}}{\overline{T \longrightarrow (D \multimap\!\!\star\, R) \qquad \text{monad morphism}}}$$

where

$$(D \multimap\!\!\star\, R)X = \int_Y DY \Rightarrow R(X \times Y)$$

Duoidal Sweedler theory [López Franco and Vasilakopoulou '20]

Consider a category $\mathbf{D}$ equipped with e.g. $\mathbf{D} = [\mathbf{Set}, \mathbf{Set}]_{\text{acc}}$

- a monoidal structure $(I, \diamond)$ e.g. composition
- a symmetric closed monoidal structure $(J, \star, -\!\!\star)$ e.g. Day convolution wrt $\otimes$
- some structural laws, satisfying equations

($\mathbf{D}$ is a symmetric duoidal category with $\star$ closed)

If T, R are $\diamond$-monoids, and D is a $\diamond$-comonoid, a *measuring map* is a monoid morphism

$$T \to D -\!\!\star R$$

- e.g. an interaction law

Duoidal Sweedler theory [López Franco and Vasilakopoulou '20]

If the appropriate adjoints exist, we have functors:

$$\star : \mathbf{Comon(D)} \times \mathbf{Comon(D)} \to \mathbf{Comon(D)} \qquad \text{(comonoid tensor)}$$

$$\mathcal{C} : \mathbf{Comon(D)}^{\mathrm{op}} \times \mathbf{Comon(D)} \to \mathbf{Comon(D)} \qquad \text{(comonoid int. hom)}$$

$$\multimap\!\!\star : \mathbf{Comon(D)}^{\mathrm{op}} \times \mathbf{Mon(D)} \to \mathbf{Mon(D)} \qquad \text{(Sweedler power)}$$

$$\triangleright : \mathbf{Comon(D)} \times \mathbf{Mon(D)} \to \mathbf{Mon(D)} \qquad \text{(Sweedler copower)}$$

$$\mathcal{M} : \mathbf{Mon(D)}^{\mathrm{op}} \times \mathbf{Mon(D)} \to \mathbf{Comon(D)}$$

(Sweedler hom/univ. measuring comonoid)

$$\frac{D_0 \to \mathcal{C}(D_1, D) \quad \text{in } \mathbf{Comon(D)}}{D_0 \star D_1 \to D \quad \text{in } \mathbf{Comon(D)}} \qquad \frac{D \to \mathcal{M}(T,R) \quad \text{in } \mathbf{Comon(D)}}{\dfrac{D \triangleright T \to R \quad \text{in } \mathbf{Mon(D)}}{T \to D \multimap\!\!\star R \quad \text{measuring}}}$$

so:

- $\mathbf{Comon(D)}$ forms a symmetric monoidal closed category
- $\mathbf{Mon(D)}$ enriches over $\mathbf{Comon(D)}$, and has powers and copowers

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These operations exist for $\mathbf{D} = [\mathbf{Set}, \mathbf{Set}]_{\mathrm{acc}}$

For accessible (co)monads on a locally presentable symmetric monoidal closed $\mathbf{C}$, the Sweedler operations

$$\begin{aligned} \multimap\!\star &: \mathbf{Comnd}_{\mathsf{acc}}(\mathbf{C})^{\mathrm{op}} \times \mathbf{Mnd}_{\mathsf{acc}}(\mathbf{C}) \to \mathbf{Mnd}_{\mathsf{acc}}(\mathbf{C}) && \text{(Sweedler power)} \\ \triangleright &: \mathbf{Comnd}_{\mathsf{acc}}(\mathbf{C}) \times \mathbf{Mnd}_{\mathsf{acc}}(\mathbf{C}) \to \mathbf{Mnd}_{\mathsf{acc}}(\mathbf{C}) && \text{(Sweedler copower)} \\ \mathcal{M} &: \mathbf{Mnd}_{\mathsf{acc}}(\mathbf{C})^{\mathrm{op}} \times \mathbf{Mnd}_{\mathsf{acc}}(\mathbf{C}) \to \mathbf{Comnd}_{\mathsf{acc}}(\mathbf{C}) && \text{(Sweedler hom)} \end{aligned}$$

exist, provide universal interaction laws

$$TX \otimes DY \longrightarrow R(X \otimes Y)$$

and we can characterize their (co)algebras

- There is also a generalization to enriched (co)monads