Canonical gradings of monads

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Example

The writer monad Wr for lists over a set C has:

object mapping	$\mathrm{Wr} : \mathbf{Set} \longrightarrow \mathbf{Set}$	$\mathrm{Wr}\, X = \mathrm{List}\, C \times X$
unit functions	$\eta_X : X \longrightarrow \mathrm{Wr}\, X$	$\eta_X x = ([], x)$
multiplication functions	$\mu_X : \mathrm{Wr}\,(\mathrm{Wr}\, X) \longrightarrow \mathrm{Wr}\, X$	$\mu_X(s_1, (s_2, x)) = (s_1 \mathbin{+\!\!+} s_2, x)$

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We can grade this by

- natural numbers $e \in \mathbb{N}$, to get a graded monad WrL:

$$\mathrm{WrL}\, e\, X = \mathrm{List}_{\leq e} C \times X \qquad \eta : X \to \mathrm{WrL}\, 0\, X \qquad \mu : \mathrm{WrL}\, e_1\, (\mathrm{WrL}\, e_2\, X) \to \mathrm{WrL}(e_1 + e_2)\, X$$

- subsets $e \subseteq C$, to get a graded monad WrS:

$$\mathrm{WrS}\, e\, X = \mathrm{List}\, e \times X \qquad \eta : X \to \mathrm{WrS}\, \emptyset\, X \qquad \mu : \mathrm{WrS}\, e_1\, (\mathrm{WrS}\, e_2\, X) \to \mathrm{WrS}(e_1 \cup e_2)\, X$$

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multiplication functions	$\mu_X : \mathrm{Wr}\,(\mathrm{Wr}\, X) \longrightarrow \mathrm{Wr}\, X$	$\mu_X(s_1, (s_2, x)) = (s_1 +\!\!+\, s_2, x)$

We can grade this by

- subsets $\Sigma \subseteq \mathrm{List}\, C$, to get a graded monad WrC:

$$\mathrm{WrC}\, \Sigma\, X = \Sigma \times X \qquad \eta : X \longrightarrow \mathrm{WrC}\, \mathsf{J}\, X \qquad \mu : \mathrm{WrC}\, \Sigma_1\, (\mathrm{WrC}\, \Sigma_2\, X) \longrightarrow \mathrm{WrC}(\Sigma_1 \boxdot \Sigma_2)\, X$$

where

$$\mathsf{J} = \{[]\} \qquad \Sigma_1 \boxdot \Sigma_2 = \{s_1 +\!\!+\, s_2 \mid s_1 \in \Sigma_1, s_2 \in \Sigma_2\}$$

Example

WrC is the canonical grading of Wr:

- WrL is

$$\mathbb{N} \xrightarrow{F} \mathcal{P}(\mathrm{List}\, C) \xrightarrow{\mathrm{WrC}} [\mathbf{Set}, \mathbf{Set}]$$

  where

$$Fe = \mathrm{List}_{\leq e} C \quad \subseteq \quad \mathrm{List} C$$

- WrS is

$$\mathcal{P} C \xrightarrow{F} \mathcal{P}(\mathrm{List}\, C) \xrightarrow{\mathrm{WrC}} [\mathbf{Set}, \mathbf{Set}]$$

  where

$$Fe = \mathrm{List} e \quad \subseteq \quad \mathrm{List} C$$

This work

More generally: given

- a (skew) monoidal category $\mathbf{D}$ (e.g. $[\mathbf{Set}, \mathbf{Set}]$)
- a class $\mathcal{M}$ of $\mathbf{D}$-morphisms (e.g. componentwise injective natural transformations)
- a monoid T in $\mathbf{D}$ (e.g. any monad on $\mathbf{Set}$)

satisfying some reasonable conditions, we have

- a notion of $\mathcal{M}$-grading of T
- T has a canonical $\mathcal{M}$-grading
- every other $\mathcal{M}$-grading factors through the canonical one

Grading objects

Given

- a category $\mathbf{D}$
- a class of $\mathbf{D}$-morphisms $\mathcal{M}$
- an object $T \in \mathbf{D}$

an $\mathcal{M}$-grading $(\mathbf{G}, G, g)$ of T is:

- a category $\mathbf{G}$ of grades
- with a functor $G : \mathbf{G} \to \mathbf{D}$
- and a natural transformation

$$g_e : Ge \rightarrowtail T$$

whose components are in $\mathcal{M}$

Example: writer monad

$\mathbf{D} = [\mathbf{Set}, \mathbf{Set}]$

$\mathcal{M}$ = componentwise injective nat. transformations

$T = \mathrm{List}\, C \times (-)$

$\mathbf{G} = (\mathbb{N}, \leq)$

$Ge = \mathrm{List}_{\leq e} C \times (-)$

$g_{e,X} = \lambda(s, x).\,(s, x)$

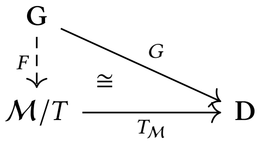
Canonical gradings of objects

The canonical $\mathcal{M}$-grading of $T \in \mathbf{D}$ has:

- category of grades $\mathcal{M}/T$: a grade is a pair (S, s), where $s : S \rightarrowtail T$ is in $\mathcal{M}$
- functor $T_{\mathcal{M}} : (S, s) \mapsto S : \mathcal{M}/T \to \mathbf{D}$
- natural transformation $g_{(S,s)} = s : G(S, s) \rightarrowtail T$

Universal property:

for every other grading $(\mathbf{G}, G, g)$ of T, there is an essentially unique functor $F : \mathbf{G} \to \mathcal{M}/T$ with isomorphisms $Ge \cong \hat{T}(Fe)$ for all e, commuting with the natural transformations



In other words: $\mathcal{M}/T$ is pseudoterminal in a 2-category of $\mathcal{M}$-gradings of T

Some examples

When $\mathbf{D} = [\mathbf{Set}, \mathbf{Set}]$, $\mathcal{M}$ = componentwise injective nat. transformations, canonical grades are subfunctors $S \hookrightarrow T$

- For $T = \mathrm{Id}$:

$$\mathcal{M}/T \simeq \{\bot \leq \top\} \qquad T_{\mathcal{M}}\bot = \emptyset \quad T_{\mathcal{M}}\top = \mathrm{Id}$$

- For $T = \mathrm{List}C \times (-)$:

$$\mathcal{M}/T \simeq (\mathcal{P}(\mathrm{List}C), \subseteq) \qquad T_{\mathcal{M}}(\Sigma \subseteq \mathrm{List}C) = \Sigma \times (-)$$

Some examples

If $T = V \Rightarrow (-)$ (for a set V), subfunctors $S \hookrightarrow T$ are equivalently upwards-closed sets

$$\Sigma \subseteq \mathrm{Equiv}_V$$

$R \in \Sigma \Rightarrow R' \in \Sigma$ whenever $R \subseteq R'$

of equivalence relations of V, via

$$\Sigma = \{R \in \mathrm{Equiv}_V \mid [-]_R \in S(V/R)\}$$

and these give a canonical grading

$$T_{\mathcal{M}}\,\Sigma\,X = \{f : V \to X \mid \exists R \in \Sigma.\, \forall v, v'.\, v\,R\,v' \Rightarrow f v = f v'\}$$

Graded monads [Smirnov '08, Melliès '12, Katsumata '14]

- A monad on $\mathbf{C}$ is a monoid in $([\mathbf{C},\mathbf{C}], \mathrm{Id}, \circ)$
- A $(\mathbf{G}, I, \odot)$-graded monad on $\mathbf{C}$ is a lax monoidal functor

$$\mathsf{T} = (T, \eta, \mu) : (\mathbf{G}, I, \odot) \to ([\mathbf{C},\mathbf{C}], \mathrm{Id}, \circ)$$

Explicitly:

$$T : \mathbf{G} \to [\mathbf{C},\mathbf{C}] \qquad \eta_X : X \to TXI \qquad \mu_{e_1,e_2,X} : Te_1(Te_2X) \to T(e_1 \odot e_2)X$$

Example:

$$\mathrm{WrL}\, e\, X = \mathrm{List}_{\leq e} C \times X \qquad \eta : X \to \mathrm{WrL}\, 0\, X \qquad \mu : \mathrm{WrL}\, e_1\, (\mathrm{WrL}\, e_2\, X) \to \mathrm{WrL}(e_1 + e_2)\, X$$

Grading monoids

Given

- a monoidal category $(\mathbf{D}, \mathsf{I}, \otimes)$
- a class of $\mathbf{D}$-morphisms $\mathcal{M}$
- a monoid T in $\mathbf{D}$

an $\mathcal{M}$-grading $(\mathbf{G}, \mathsf{G}, g)$ of T is:

- a monoidal category $\mathbf{G}$ of grades
- with a lax monoidal functor $G : \mathbf{G} \to \mathbf{D}$
- and a monoidal nat. trans.

$$g_e : Ge \rightarrowtail T$$

whose components are in $\mathcal{M}$

Example: writer monad

$(\mathbf{D}, \mathsf{I}, \otimes) = ([\mathbf{Set}, \mathbf{Set}], \mathrm{Id}, \circ)$

$\mathcal{M}$ = componentwise injective nat. trans.

$\mathsf{T} = \mathrm{List}\, C \times (-)$ (a writer monad)

$\mathbf{G} = (\mathbb{N}, \leq)$, with addition

$Ge = \mathrm{List}_{\leq e} C \times (-)$ (a graded writer monad)

$g_{e,X} = \lambda(s, x).\, (s, x)$

Canonical gradings of monoids

If

- T is a monoid in $(\mathbf{D}, \mathsf{I}, \otimes)$

then $\mathbf{D}/T$ forms a monoidal category:

$$\mathsf{J} \;=\; (I, \eta) \qquad\qquad (S, s) \boxdot (S', s') \;=\; (S \otimes S', (s \otimes s') \circ \mu_T)$$

$$\mathsf{I} \xrightarrow{\quad\eta\quad} T \qquad\qquad S \otimes S' \xrightarrow{s \otimes s'} T \otimes T \xrightarrow{\mu} T$$

and

$$(S, s) \mapsto S : \mathbf{D}/T \to \mathbf{D}$$

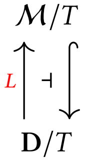
forms a monoidal functor

Canonical gradings of monoids

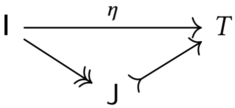
If

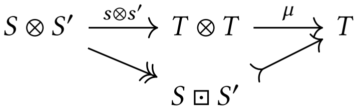
- T is a monoid in $(\mathbf{D}, \mathsf{I}, \otimes)$
- $\mathcal{M}$ forms a factorization system $(\mathcal{E}, \mathcal{M})$
- $\mathcal{E}$ is closed under $\otimes$ on both sides

then $\mathcal{M}/T$ forms a monoidal category:



$$\mathsf{J} = L(I, \eta) \qquad (S, s) \boxdot (S', s') = L(S \otimes S', (s \otimes s') \circ \mu_T)$$





and

$$(S, s) \mapsto S : \mathcal{M}/T \to \mathbf{D}$$

forms a lax monoidal functor

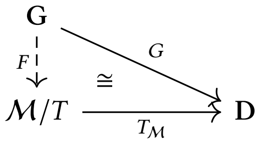
Canonical gradings of monoids

The canonical $\mathcal{M}$-grading of a monoid T in $(\mathbf{D}, \mathsf{I}, \otimes)$ has:

- monoidal category of grades $\mathcal{M}/T$
- lax monoidal functor $T_{\mathcal{M}} : (S, s) \mapsto S : \mathcal{M}/T \to \mathbf{D}$
- monoidal natural transformation $g_{(S,s)} = s : G(S, s) \rightarrowtail T$

Universal property:

for every other grading $(\mathbf{G}, G, g)$ of the monoid T, there is an essentially unique lax monoidal $F : \mathbf{G} \to \mathcal{M}/T$ with isomorphisms $Ge \cong \hat{T}(Fe)$ for all e, commuting with the natural transformations



In other words: $\mathcal{M}/T$ is pseudoterminal in a 2-category of $\mathcal{M}$-gradings of T

Example: writer

Take

- $\mathbf{D} = [\mathbf{Set}, \mathbf{Set}]$, with endofunctor composition
- $(\mathcal{E}, \mathcal{M}) = (\text{componentwise surjective}, \text{componentwise injective})$
- T is a writer monad
$$T = \mathrm{List}C \times (-)$$

Then:

- Subfunctors $S \hookrightarrow \mathrm{Wr}$ are equivalently subsets $\Sigma \subseteq \mathrm{List}C$ via

$$\Sigma = \{s \in \mathrm{List}C \mid (s, \star) \in S1\} \qquad SX = \{(s, x) \in \mathrm{List}C \times X \mid s \in \Sigma\}$$

So the canonical grading is $\mathcal{P}(\mathrm{List}C)$ with

$$\mathsf{T}_{\mathcal{M}} : (\mathcal{P}(\mathrm{List}C), \subseteq) \to [\mathbf{Set}, \mathbf{Set}] \qquad \mathsf{T}_{\mathcal{M}}\Sigma = \Sigma \times (-)$$
$$\mathsf{J} = \{[]\} \qquad \Sigma_1 \boxdot \Sigma_2 = \{s_1 \mathbin{+\!\!+} s_2 \mid s_1 \in \Sigma_1, s_2 \in \Sigma_2\}$$

Grading by sets of shapes

For $\mathbf{D} = [\mathbf{Set}, \mathbf{Set}]$, there is also a factorization system

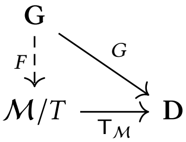
$$\mathcal{E} = \text{natural transformations } \alpha \text{ such that } \alpha_1 \text{ is surjective}$$
$$\mathcal{M} = \text{cartesian natural transformations } \alpha \text{ such that } \alpha_1 \text{ is injective}$$

satisfying

$$\mathcal{M}/T \simeq (\mathcal{P}(T1), \subseteq)$$

Summary

Given a suitable class $\mathcal{M}$ of morphisms, every monoid T has a canonical $\mathcal{M}$-grading $\mathsf{T}_{\mathcal{M}} : \mathcal{M}/\mathsf{T} \to \mathbf{D}$



In particular, we can canonically grade monads (and algebraic operations for them)