What Makes a Strong Monad?

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Strong monads are important for several applications, in particular, in the denotational semantics of effectful languages, where strength is needed to sequence computations that have free variables. Strength is non-trivial: it can be difficult to determine whether a monad has any strength at all, and monads can be strong in multiple ways. We therefore review some of the most important known facts about strength and prove some new ones. In particular, we present a number of equivalent characterizations of strong functor and strong monad, and give some conditions that guarantee existence or uniqueness of strengths. We look at strength from three different perspectives: actions of a monoidal category **V**, enrichment over **V**, and powering over **V**. We are primarily motivated by semantics of effects, but the results are also useful in other contexts.

1 Introduction

Following Moggi [23], effectful computations are often modelled using strong monads. Strength also appears in other applications; for example, strength is crucial for the notion of commutative monad [12] used in the construction of tensor products on categories of algebras [21, 13], and in measure theory [16]; strong functors are also important in the study of abstract syntax [5]. It can be difficult in these contexts to determine whether a given functor or monad admits a strength, and various facts have been proved about strength to help with this. Some appear in published work (often as a small lemma in a paper not primarily about strength) [24, 28, 20], while others are folklore. These have some overlap, and levels of generality vary.

We collect together a number of important results about strength. There are two groups of results in particular that we focus on. One is the equivalence of various definitions of strong functor and strong monad. These are useful in particular for reasoning about strong functors and monads, and are also useful for constructing strengths for ordinary functors and monads. The other is results concerning existence and uniqueness of strengths for functors and monads. Several of these results are known, but a good number are, to the best of our knowledge, new.

The difference between monads and strong monads is best seen by looking at the *Kleisli extension* operator. If T is the underlying endofunctor of a monad, then every morphism $f : X \to TY$ induces a morphism $f^{\dagger} : TX \to TY$, as on the left below. In the Cartesian case, if T forms a *strong* monad, then the Kleisli extension has the more general form on the right.

$$\frac{f : X \to TY}{f^{\dagger} : TX \to TY} \qquad \frac{f : \Gamma \times X \to TY}{f^{\dagger} : \Gamma \times TX \to TY}$$

Our main interest is the semantics of effects (though the results we give here can be applied more widely). Strength in this case enables interpretation of terms with free variables. Consider the following typing rule:

$$\frac{\Gamma \vdash t : A \qquad \Gamma, x : A \vdash t' : B}{\Gamma \vdash \mathsf{let}\ x = t\ \mathsf{in}\ t' : B}$$

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In a monadic model of a call-by-value language, the terms t and t' would be interpreted as morphisms $[\![t]\!] : [\![\Gamma]\!] \to T[\![A]\!]$ and $[\![t']\!] : [\![\Gamma]\!] \times [\![A]\!] \to T[\![B]\!]$, where TX is the object of (possibly effectful) computations that return values in X. Using the strong Kleisli extension of $[\![t']\!]$ we can interpret the let as

$$[\![\mathsf{let}\ x = t\ \mathsf{in}\ t']\!] : [\![\Gamma]\!] \xrightarrow{\langle \mathrm{id}_{[\![\Gamma]\!]}, [\![t]\!]\rangle} [\![\Gamma]\!] \times T[\![A]\!] \xrightarrow{[\![t']\!]^{\dagger}} T[\![B]\!]$$

The Kleisli extension of an ordinary monad suffices when Γ is empty (because then $[\![\Gamma]\!] = \mathbf{1}$), but we need the strong version in general.

Instead of assuming products, we work in the more general setting of an *action* of a monoidal category on another category. Strengths with respect to an action appear for example in [2, 4, 22, 9, 30]. Working with actions instead of a Cartesian, symmetric monoidal or general monoidal structure does not add much complexity, but is useful for some of the results we give. We also approach strength from two other perspectives. The *enriched* perspective is well-known for categories enriched over themselves and goes back to Kock [15]; by generalizing to actions, we remove the self-enrichment restriction. The third perspective, which we call *powering*, is less well-known, but was also first considered by Kock [14]. The same three-perspective approach can be found in the nLab article on strong monads [29], but for the most part still only for the self-enriched case.

We discuss actions, strong functors, and strong monads in Sections 2 to 4, looking especially at uniqueness and existence of strengths for functors. Our novel contributions are sufficient criteria for unique existence (based on our notion of *functional completeness*), and for non-unique existence (based on our notion of *weak functional completeness*). We also provide a number of examples. We consider enrichment in Section 5 and powering in Section 6. In Appendix A, we discuss biactions, bistrong functors and commutative monads.

2 Monoidal categories and actions

We begin by recalling the notions of *monoidal category* and *action*, and give our primary examples of these.

Definition 2.1. A *monoidal category* $(\mathbf{V}, I, \otimes)$ consists of a category $\mathbf{V}$, an object $I \in \mathbf{V}$ called the *unit*, and a functor $\otimes : \mathbf{V} \times \mathbf{V} \to \mathbf{V}$ called the *tensor*, equipped with three natural isomorphisms

$$\lambda_\Gamma : I \otimes \Gamma \to \Gamma \qquad \rho_\Gamma : \Gamma \to \Gamma \otimes I \qquad \alpha_{\Gamma_1,\Gamma_2,\Gamma_3} : (\Gamma_1 \otimes \Gamma_2) \otimes \Gamma_3 \to \Gamma_1 \otimes (\Gamma_2 \otimes \Gamma_3)$$

satisfying the following coherence conditions:

$$\begin{array}{ccc} \Gamma \otimes \Delta & = & \Gamma \otimes \Delta \\ \downarrow{\scriptstyle \rho_\Gamma \otimes \Delta} & & \uparrow{\scriptstyle \Gamma \otimes \lambda_\Delta} \\ (\Gamma \otimes I) \otimes \Delta & \xrightarrow[\alpha_{\Gamma,I,\Delta}]{} & \Gamma \otimes (I \otimes \Delta) \end{array}$$

$$((\Gamma_1 \otimes \Gamma_2) \otimes \Gamma_3) \otimes \Gamma_4 \xrightarrow{\alpha_{\Gamma_1 \otimes \Gamma_2, \Gamma_3, \Gamma_4}} (\Gamma_1 \otimes \Gamma_2) \otimes (\Gamma_3 \otimes \Gamma_4) \xrightarrow{\alpha_{\Gamma_1,\Gamma_2,\Gamma_3 \otimes \Gamma_4}} \Gamma_1 \otimes (\Gamma_2 \otimes (\Gamma_3 \otimes \Gamma_4))$$

$$((\Gamma_1 \otimes \Gamma_2) \otimes \Gamma_3) \otimes \Gamma_4 \xrightarrow{\alpha_{\Gamma_1,\Gamma_2,\Gamma_3} \otimes \Gamma_4} (\Gamma_1 \otimes (\Gamma_2 \otimes \Gamma_3)) \otimes \Gamma_4 \xrightarrow{\alpha_{\Gamma_1,\Gamma_2 \otimes \Gamma_3,\Gamma_4}} \Gamma_1 \otimes ((\Gamma_2 \otimes \Gamma_3) \otimes \Gamma_4) \xrightarrow{\Gamma_1 \otimes \alpha_{\Gamma_2,\Gamma_3,\Gamma_4}} \Gamma_1 \otimes (\Gamma_2 \otimes (\Gamma_3 \otimes \Gamma_4))$$

Example 2.2. Every category $\mathbf{V}$ with finite products forms a *Cartesian monoidal category* $(\mathbf{V}, \mathbf{1}, \times)$, in which the unit is the terminal object $\mathbf{1}$, and the tensor of X and Y is the binary product $X \times Y$.

Example 2.3. Let $\mathbf{Set}_\star$ be the category of pointed sets and point-preserving functions. Objects of $\mathbf{Set}_\star$ are sets X with a distinguished element $\star \in X$; morphisms are functions $f : X \to Y$ such that $f\star = \star$. We consider two monoidal structures on $\mathbf{Set}_\star$. The first is the Cartesian monoidal structure, which is inherited from $\mathbf{Set}$ (the product $X \times Y$ is the product of sets, with distinguished element $(\star,\star)$). The second is the *smash product* $X \otimes Y = \{(x,y) \in X \times Y \mid x = \star \Leftrightarrow y = \star\}$, which has the two-element pointed set $\{\star,1\}$ as the unit. On morphisms, $\otimes$ is given by $(f \otimes g)(x,y) = (\star,\star)$ if $fx = \star$ or $gy = \star$, and by $(f \otimes g)(x,y) = (fx, gy)$ otherwise.

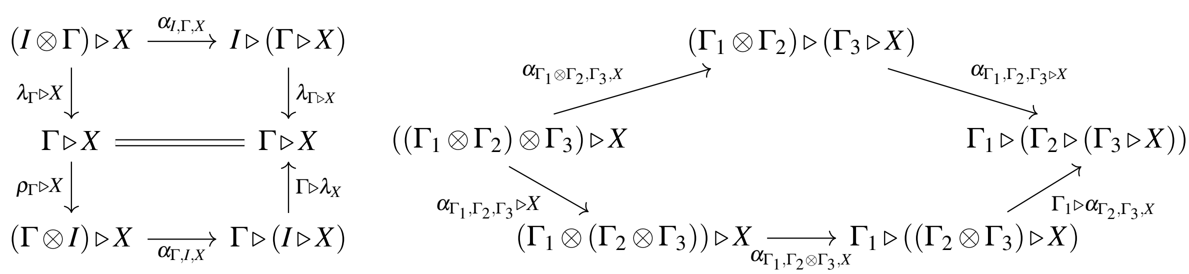
Example 2.4. Let $\mathsf{M} = (M,1,\cdot)$ be a (set-theoretic) monoid. The category $\mathbf{Act}\,\mathsf{M}$ of right M-actions has as objects sets X equipped with a function $(*) : X \times M \to X$ such that $x * 1 = x$ and $x * (m \cdot m') = (x * m) * m'$ for all $x \in X$ and $m,m' \in M$. Morphisms $f : X \to Y$ in $\mathbf{Act}\,\mathsf{M}$ are functions that preserve the action, i.e. $f(x*m) = (fx)*m$ for all $x \in X$ and $m \in M$. The category $\mathbf{Act}\,\mathsf{M}$ is Cartesian monoidal; the terminal object $\mathbf{1}$ is the one-element set equipped with the unique $*$, and the product $X \times Y$ is the product of sets with $(x,y)*m = (x*m, y*m)$.

When M is natural numbers with addition, $\mathbf{Act}\,\mathsf{M}$ is isomorphic to the category of sets X equipped with an endofunction $e : X \to X$; morphisms $f : X \to Y$ are functions such that $f \circ e = e \circ f$. The action on an object X is $x * n = e^n x$. This is isomorphic to the category $[\mathbb{N},\mathbf{Set}]$ where $\mathbb{N}$ is the one-object category with natural numbers as morphisms and addition as composition.

Definition 2.5. A *(left) action*[1] of a monoidal category $(\mathbf{V}, I, \otimes)$ on a category $\mathbf{C}$ is a functor $\triangleright : \mathbf{V} \times \mathbf{C} \to \mathbf{C}$ equipped with two natural isomorphisms

$$\lambda_X : I \triangleright X \to X \qquad \alpha_{\Gamma',\Gamma,X} : (\Gamma' \otimes \Gamma) \triangleright X \to \Gamma' \triangleright (\Gamma \triangleright X)$$

satisfying the following coherence conditions:



A left action of $\mathbf{V}$ on $\mathbf{C}$ is the same as a monoidal functor from $\mathbf{V}$ to $[\mathbf{C},\mathbf{C}]$, where we equip $[\mathbf{C},\mathbf{C}]$ with the composition monoidal structure.

Example 2.6. The tensor of any monoidal category $\mathbf{V}$ (in particular, the examples above) forms an action of $\mathbf{V}$ on itself, with $X \triangleright Y = X \otimes Y$.

Example 2.7. Consider $\mathbf{V} = \mathbf{Set}$ with the Cartesian monoidal structure. A category $\mathbf{C}$ *has copowers* over $\mathbf{Set}$ when for all sets Γ and objects $X \in \mathbf{C}$, the coproduct $\Gamma \bullet X = \coprod_{\gamma \in \Gamma} X \in \mathbf{V}$ exists. The object $\Gamma \bullet X$ is the *copower* of Γ and X; its universal property is that morphisms $f : \Gamma \bullet X \to Y$ are in natural bijection with tuples $(f_\gamma : X \to Y)_{\gamma \in \Gamma}$ of morphisms, by taking $f_\gamma = f \circ \mathrm{in}_\gamma$. If $\mathbf{C}$ has copowers over $\mathbf{Set}$, then they form an action $\Gamma \triangleright X = \Gamma \bullet X$ of $(\mathbf{Set}, \mathbf{1}, \times)$ on $\mathbf{C}$. For $\mathbf{C} = \mathbf{Set}$, the copower $X \bullet Y$ is just the Cartesian product $X \times Y$.

In the relationship between strength and enrichment explained below in Section 5, the action $\triangleright$ forms the *copowers* (or *tensors*) of the enriched category in a more general sense of 'copower'. Any locally small category $\mathbf{C}$ is uniquely $\mathbf{Set}$-enriched. Its copowers, if they exist, are given by small coproducts as described above.

[1] A category $\mathbf{C}$ with a left action of a monoidal category $\mathbf{V}$ is also called a (left) $\mathbf{V}$-*actegory*.

3 Strong functors

Throughout this section, we suppose a monoidal category $\mathbf{V}$, whose objects Γ we view as *contexts* (because of their role in the introduction as interpretations of typing contexts). We then consider strong functors $F : \mathbf{C} \to \mathbf{D}$, where $\mathbf{C}$ and $\mathbf{D}$ are categories equipped with actions $\triangleright_{\mathbf{C}} : \mathbf{V} \times \mathbf{C} \to \mathbf{C}$ and $\triangleright_{\mathbf{D}} : \mathbf{V} \times \mathbf{D} \to \mathbf{D}$. We have no need to assume that $\mathbf{V}$ is a *symmetric* monoidal category (but this is the case for all of our examples). A $\mathbf{C}$-morphism $\Gamma \triangleright_{\mathbf{C}} X \to Y$ can be thought of as a morphism from X to Y *in context* Γ, and similarly for $\mathbf{D}$.

There are several equivalent definitions of strong functor. The following is not the standard one, but matches closely the intuition that the context Γ should be preserved, enabling the interpretation of terms with free variables.

Definition 3.1. A *(left) strong functor* $F : \mathbf{C} \to \mathbf{D}$ consists of an object $FX \in \mathbf{D}$ for each object $X \in \mathbf{C}$ and a $\mathbf{D}$-morphism $F^{(\Gamma)} f : \Gamma \triangleright_{\mathbf{D}} FX \to FY$ for each $\mathbf{C}$-morphism $f : \Gamma \triangleright_{\mathbf{C}} X \to Y$, such that $F^{(\Gamma)}$ is natural in $\Gamma \in \mathbf{V}$, and

$$
\begin{array}{rll}
F^{(I)} \lambda_X = \lambda_{FX} & & \text{for } X \in \mathbf{C} \\
F^{(\Gamma' \otimes \Gamma)}(g \circ (\Gamma' \triangleright_{\mathbf{C}} f) \circ \alpha_{\Gamma',\Gamma,X}) = F^{(\Gamma')} g \circ (\Gamma' \triangleright_{\mathbf{D}} F^{(\Gamma)} f) \circ \alpha_{\Gamma',\Gamma,FX} & & \text{for } f : \Gamma \triangleright_{\mathbf{C}} X \to Y,\ g : \Gamma' \triangleright_{\mathbf{C}} Y \to Z
\end{array}
$$

If $F, G : \mathbf{C} \to \mathbf{D}$ are strong functors, then a *strong natural transformation* $\tau : F \Rightarrow G$ consists of a morphism $\tau_X : FX \to GX$ for each $X \in \mathbf{C}$, such that $\tau_Y \circ F^{(\Gamma)} f = G^{(\Gamma)} f \circ (\Gamma \triangleright_{\mathbf{D}} \tau_X)$ for each $f : \Gamma \triangleright_{\mathbf{C}} X \to Y$.

Every strong functor F has an underlying ordinary functor $\underline{F} : \mathbf{C} \to \mathbf{D}$, given on objects by $\underline{F} X = FX$ and on morphisms $f : X \to Y$ by $\underline{F} f = F^{(I)}(f \circ \lambda_X) \circ \lambda_{FX}^{-1} : FX \to FY$. Every strong natural transformation $\alpha : F \Rightarrow G$ is a natural transformation $\alpha : \underline{F} \Rightarrow \underline{G}$. There is an identity strong functor Id and each pair of strong functors $F : \mathbf{C} \to \mathbf{D}$ and $G : \mathbf{D} \to \mathbf{E}$ has a composition $G \cdot F : \mathbf{C} \to \mathbf{E}$. These are given on objects X and morphisms $f : \Gamma \triangleright_{\mathbf{C}} X \to Y$ by

$$
\mathrm{Id} X = X \quad \mathrm{Id}^{(\Gamma)} f = f \qquad\qquad (G \cdot F) X = G(FX) \quad (G \cdot F)^{(\Gamma)} f = G^{(\Gamma)}(F^{(\Gamma)} f)
$$

Example 3.2. Let $\triangleright_{\mathbf{C}}$ and $\triangleright_{\mathbf{D}}$ both be the action of $\mathbf{Set}$ on itself given by the Cartesian monoidal structure. The strong functor $\mathrm{List} : \mathbf{Set} \to \mathbf{Set}$ maps each set X to the set of lists over X; on functions $f : \Gamma \times X \to Y$ it is given by $\mathrm{List}^{(\Gamma)} f\,(\gamma, [x_1, \ldots, x_n]) = [f(\gamma, x_1), \ldots, f(\gamma, x_n)]$. The ordinary functor $\underline{\mathrm{List}} : \mathbf{Set} \to \mathbf{Set}$ is then just the usual list functor, given on functions by $\underline{\mathrm{List}}\, f\,[x_1, \ldots, x_n] = [f x_1, \ldots, f x_n]$.

An alternative definition is that a strong functor is an ordinary functor equipped with a *strength*. This is the more common definition, and we make some use of it below.

Definition 3.3. A *(left) strength* for an ordinary functor $F : \mathbf{C} \to \mathbf{D}$ is a family of $\mathbf{D}$-morphisms $\mathrm{str}_{\Gamma,X} : \Gamma \triangleright_{\mathbf{D}} FX \to F(\Gamma \triangleright_{\mathbf{C}} X)$, natural in $\Gamma \in \mathbf{V}$ and $X \in \mathbf{C}$ and such that

$$
\begin{array}{ccc}
I \triangleright_{\mathbf{D}} FX & & \\
{\scriptstyle \mathrm{str}_{I,X}} \downarrow & \searrow^{\lambda_{FX}} & \\
F(I \triangleright_{\mathbf{C}} X) & \xrightarrow[F\lambda_X]{} & FX
\end{array}
\qquad
\begin{array}{ccccc}
(\Gamma' \otimes \Gamma) \triangleright_{\mathbf{D}} FX & \xrightarrow{\alpha_{\Gamma',\Gamma,FX}} & \Gamma' \triangleright_{\mathbf{D}} (\Gamma \triangleright_{\mathbf{D}} FX) & \xrightarrow{\Gamma' \triangleright_{\mathbf{D}} \mathrm{str}_{\Gamma,X}} & \Gamma' \triangleright_{\mathbf{D}} F(\Gamma \triangleright_{\mathbf{C}} X) \\
{\scriptstyle \mathrm{str}_{\Gamma' \otimes \Gamma,X}} \downarrow & & & & \downarrow {\scriptstyle \mathrm{str}_{\Gamma',\Gamma \triangleright_{\mathbf{C}} X}} \\
F((\Gamma' \otimes \Gamma) \triangleright_{\mathbf{C}} X) & & \xrightarrow[F\alpha_{\Gamma',\Gamma,X}]{} & & F(\Gamma' \triangleright_{\mathbf{C}} (\Gamma \triangleright_{\mathbf{C}} X))
\end{array}
$$

We show the equivalence between these two definitions of strong functor and the corresponding fact for strong natural transformations.

Proposition 3.4. *If $F : \mathbf{C} \to \mathbf{D}$ is an ordinary functor, then there is a bijection between (1) strong functors $\hat{F}$ such that $\underline{\hat{F}} = F$, and (2) strengths* str *for F. If $F, G : \mathbf{C} \to \mathbf{D}$ are functors equipped with the equivalent data of this bijection, then a natural transformation $\tau : F \Rightarrow G$ is a strong natural transformation $\tau : \hat{F} \Rightarrow \hat{G}$ exactly when the following commutes:*

$$\begin{array}{ccc} \Gamma \triangleright_{\mathbf{D}} FX & \xrightarrow{\Gamma \triangleright_{\mathbf{D}} \tau_X} & \Gamma \triangleright_{\mathbf{D}} GX \\ {\scriptstyle \mathrm{str}_{\Gamma,X}}\downarrow & & \downarrow{\scriptstyle \mathrm{str}_{\Gamma,X}} \\ F(\Gamma \triangleright_{\mathbf{C}} X) & \xrightarrow[\tau_{\Gamma \triangleright_{\mathbf{C}} X}]{} & G(\Gamma \triangleright_{\mathbf{C}} X) \end{array}$$

Proof. By the Yoneda lemma, families of functions $\hat{F}^{(\Gamma)} : \mathbf{C}(\Gamma \triangleright_{\mathbf{C}} X, Y) \to \mathbf{D}(\Gamma \triangleright_{\mathbf{D}} FX, FY)$ natural in $Y \in \mathbf{C}$ are in bijection with morphisms $\mathrm{str}_{\Gamma,X} : \Gamma \triangleright_{\mathbf{D}} FX \to F(\Gamma \triangleright_{\mathbf{C}} X)$. If $\hat{F}$ is a strong functor with $\underline{\hat{F}} = F$, then $\hat{F}^{(\Gamma)}$ is natural in Y. Moreover, such a natural family forms a strong functor $\hat{F}$ with $\underline{\hat{F}} = F$ exactly when $\mathrm{str}_{\Gamma,X} = \hat{F}^{(\Gamma)} \mathrm{id}_{\Gamma \triangleright_{\mathbf{C}} X}$ is a strength for F. The fact about natural transformations follows immediately from the fact that given a strength str, the corresponding strong functor is given by $\hat{F}^{(\Gamma)} f = F f \circ \mathrm{str}_{\Gamma,X}$. □

3.1 Uniqueness and existence of strengths

It is well-known that every endofunctor F on **Set** has a unique strength with respect to the Cartesian monoidal structure. Various other results about uniqueness of strengths have been proved (e.g. [23, 28, 20]). Uniqueness results are useful for determining whether a given functor admits a strength at all when for some reason there is only one candidate to check. Conversely, existence results make it easier to construct strengths for functors. We supply uniqueness and existence results for strengths in this section.

We first define a notion of *functional completeness* for an action, which guarantees both uniqueness and existence of strengths.[2] We call the elements $\gamma \in \mathbf{V}(I, \Gamma)$ the *points* of Γ. For each morphism $f : \Gamma \triangleright_{\mathbf{D}} X \to Y$ in $\mathbf{D}$, we have a function $(\!| f |\!) : \mathbf{V}(I, \Gamma) \to \mathbf{D}(X, Y)$, *applying* f to points, by defining $(\!| f |\!) \gamma = f \circ (\gamma \triangleright_{\mathbf{D}} X) \circ \lambda_X^{-1}$.

Definition 3.5. We say that $\triangleright_{\mathbf{D}}$ is *functionally complete* if, for every function $\zeta : \mathbf{V}(I, \Gamma) \to \mathbf{D}(X, Y)$, there is a unique $\mathbf{D}$-morphism $f : \Gamma \triangleright_{\mathbf{D}} X \to Y$ such that $(\!| f |\!) = \zeta$.

Writing $\Phi_\Gamma \zeta$ for the unique f from the definition, we get a family of functions Φ that are the inverses of the functions $(\!| - |\!)$. This family Φ is natural in Γ, X and Y. Moreover, it satisfies

$$\begin{array}{rl} \Phi_I \zeta = \zeta \, \mathrm{id}_I \circ \lambda_X & \text{for } \zeta : \mathbf{V}(I, I) \to \mathbf{D}(X, Y) \\ \Phi_{\Gamma' \otimes \Gamma} \zeta = \Phi_{\Gamma'}(\lambda \gamma'. \Phi_\Gamma(\lambda \gamma.\ \zeta((\gamma' \otimes \gamma) \circ \rho_I))) \circ \alpha_{\Gamma', \Gamma, X} & \text{for } \zeta : \mathbf{V}(I, \Gamma' \otimes \Gamma) \to \mathbf{D}(X, Y) \end{array}$$

The key consequence is the following (which is a corollary of Propositions 3.10 and 3.13 below):

Proposition 3.6. *If $\triangleright_{\mathbf{D}}$ is functionally complete, then every functor $F : \mathbf{C} \to \mathbf{D}$ has a unique strength, and every natural transformation $\tau : F \Rightarrow G$ of functors $\mathbf{C} \to \mathbf{D}$ is strong.*

Example 3.7. The category **Set** with the Cartesian product is functionally complete; this is why endofunctors on **Set** have unique strengths. More generally, if **D** has copowers over **Set**, then the action $\bullet$ of

[2] This notion is similar in spirit to functional completeness in categorical logic [17, 25], but not the same. In categorical logic, $\mathbf{V}$ would be functionally complete if, for any $\mathbf{V}[\Gamma]$-morphism $z : X \to Y$, there were a unique $\mathbf{V}$-morphism $f : \Gamma \otimes X \to Y$ such that $J f \circ (\gamma \otimes X) \circ \lambda_X^{-1} = z$. Here by $\mathbf{V}[\Gamma]$ we mean the monoidal category obtained by freely extending $\mathbf{V}$ with a morphism $\gamma : I \to \Gamma$—an "indeterminate" point of Γ—and by J we mean the inclusion of $\mathbf{V}$ in $\mathbf{V}[\Gamma]$. (For a Cartesian monoidal $\mathbf{V}$, one would extend to a Cartesian monoidal category $\mathbf{V}[\Gamma]$ instead of just a monoidal category.) We are looking for a more distinctive name for our notion.

$(\mathbf{Set}, \mathbf{1}, \times)$ on $\mathbf{D}$ is functionally complete. In this case, points $\gamma \in \mathbf{Set}(\mathbf{1}, \Gamma)$ are just elements of the set Γ. Functional completeness thus says equivalently that, for every function $\zeta : \Gamma \to \mathbf{D}(X, Y)$, there is a unique $f : \Gamma \bullet X \to Y$ such that $f \circ \mathrm{in}_\gamma = \zeta\gamma$ for all $\gamma \in \Gamma$. This is exactly the universal property of $\Gamma \bullet X$.

Example 3.8. In contrast, the Cartesian product of pointed sets is not functionally complete as an action of $(\mathbf{Set}_\star, \mathbf{1}, \times)$ on itself: every Γ has only one point, so the morphisms $f : \Gamma \times X \to Y$ fail to be unique. The Cartesian product of posets also fails to be functionally complete as an action of $(\mathbf{Poset}, \mathbf{1}, \times)$ on itself: the morphisms $f : \Gamma \times X \to Y$ in this case are necessarily given by $f(\gamma, x) = \zeta(\gamma)(x)$, so are unique if they exist, but this f may fail to be monotone.

There are few examples of functionally complete actions. We break down the notion of functional completeness into *well-pointedness*, which guarantees uniqueness of strength, and the existence of a *weak functional completeness structure*, which guarantees existence of a canonical strength for each functor. Both of these have more examples.

Definition 3.9. The action $\triangleright_\mathbf{D}$ is said to be *well-pointed* (or *have enough points*) when $(\!|-|\!)$ is injective, i.e. when $(\!|f|\!) = (\!|g|\!)$ implies $f = g$ for all $f, g : \Gamma \triangleright_\mathbf{D} X \to Y$.

If the action is a monoidal category acting on itself, well-pointedness in our sense is equivalent to Abramsky and Heunen's *(monoidal) well-pointedness* [1]. According to their definition, a monoidal category $\mathbf{V}$ is well-pointed if two morphisms $f, g : X \otimes X' \to Y$ are equal whenever $f \circ (\xi \otimes \xi') \circ \rho_I = g \circ (\xi \otimes \xi') \circ \rho_I$ for all $\xi : I \to X$, $\xi' : I \to X'$. For a Cartesian monoidal category acting on itself, our notion of well-pointedness agrees with the usual notion defined with respect to a terminal object (two morphisms $f, g : X \to Y$ are equal if $f \circ \xi = g \circ \xi$ for all $\xi : \mathbf{1} \to X$). A similar simplification is possible when each $- \triangleright_\mathbf{D} X$ has a right adjoint (i.e. there is a corresponding *enrichment* in the sense of Section 5, for example when $\triangleright_\mathbf{D}$ is the action of a monoidal closed category on itself). When these right adjoints exist, if two $\mathbf{V}$-morphisms $f, g : \Gamma \to \Delta$ are equal whenever $f \circ \gamma = g \circ \gamma$ for all $\gamma : I \to \Gamma$, then $\triangleright_\mathbf{D}$ is well-pointed.

Functional completeness is a strictly stronger property than well-pointedness; the latter implies that strengths are unique if they exist, but does not guarantee existence.

Proposition 3.10. *Suppose that $\triangleright_\mathbf{D}$ is well-pointed. A functor $F : \mathbf{C} \to \mathbf{D}$ has a strength exactly when, for every Γ, X, there is a morphism $\mathrm{str}_{\Gamma,X} : \Gamma \triangleright_\mathbf{D} FX \to F(\Gamma \triangleright_\mathbf{C} X)$ such that*

$$(\!|\mathrm{str}_{\Gamma,X}|\!)\gamma = F((\!|\mathrm{id}_{\Gamma \triangleright_\mathbf{C} X}|\!)\gamma) \quad \textit{for } \gamma \in \mathbf{V}(I, \Gamma) \tag{1}$$

When the (necessarily unique) morphisms $\mathrm{str}_{\Gamma,X}$ exist, str *is the only strength for F. Moreover, if $G, H : \mathbf{C} \to \mathbf{D}$ are strong functors, then every natural transformation $\tau : G \Rightarrow H$ is strong.*

Proof. If a family of morphisms str is a strength, then

$$(\!|\mathrm{str}_{\Gamma,X}|\!)\gamma = F(\gamma \triangleright_\mathbf{C} X) \circ \mathrm{str}_{I,X} \circ \lambda_{FX}^{-1} = F(\gamma \triangleright_\mathbf{C} X) \circ F\lambda_X^{-1} = F((\!|\mathrm{id}_{\Gamma \triangleright_\mathbf{C} X}|\!)\gamma)$$

using naturality in Γ and the strength axiom for λ. Well-pointedness therefore implies uniqueness. To show that every family of morphisms str satisfying the condition is a strength, it suffices by well-pointedness to consider the image of each axiom under $(\!|-|\!)$ and then calculate. A similar proof shows that every τ is strong. □

Example 3.11. The smash product $\otimes$ of pointed sets is a well-pointed action of $(\mathbf{Set}_\star, I, \otimes)$ on $\mathbf{Set}_\star$. In this case, the unit I is the two-element pointed set $\{\star, 1\}$; since morphisms $I \to \Gamma$ send $\star$ to $\star$, points of

Γ are in bijection with elements of Γ, and so, for any ζ, there is at most one f with $(\!|f|\!) = \zeta$. In contrast, the Cartesian product of pointed sets is not well-pointed, because every Γ has only one point.

The Cartesian product of posets is a well-pointed action of $(\mathbf{Poset}, \mathbf{1}, \times)$ on itself, again because points of Γ are in bijection with elements of Γ. If F is an endofunctor on **Poset**, then any strength for F would necessarily be given by $\mathrm{str}_{\Gamma,X}(\gamma,t) = F(\lambda x.(\gamma,x))t$; this forms a strength for F exactly when it is monotone. This property enables us to show that some functors have no strength. For example, the functor $|-| : \mathbf{Poset} \to \mathbf{Poset}$ that sends $(X,\leq)$ to the discrete poset $(X,=)$ has no strength because $\mathrm{str}_{\Gamma,X}$ is not a monotone function $\Gamma \times |X| \to |\Gamma \times X|$.

For existence of strengths, we introduce the following.

Definition 3.12. A *weak functional completeness structure* Φ for $\triangleright_{\mathbf{D}}$ is an assignment of a **D**-morphism $\Phi_\Gamma \zeta : \Gamma \triangleright X \to Y$ satisfying $(\!|\Phi_\Gamma \zeta|\!) = \zeta$ to each function $\zeta : \mathbf{V}(I,\Gamma) \to \mathbf{D}(X,Y)$. We require this assignment to be natural in Γ, X, Y, and to satisfy

$$\Phi_{\Gamma' \otimes \Gamma} \zeta = \Phi_{\Gamma'}(\lambda \gamma'. \Phi_\Gamma(\lambda \gamma.\, \zeta((\gamma' \otimes \gamma) \circ \rho_I))) \circ \alpha_{\Gamma',\Gamma,X} \qquad \text{for } \zeta : \mathbf{V}(I, \Gamma' \otimes \Gamma) \to \mathbf{D}(X,Y)$$

If $\triangleright_{\mathbf{D}}$ is well-pointed, then there can be at most one weak functional completeness structure Φ for $\triangleright_{\mathbf{D}}$, because morphisms f such that $(\!|f|\!) = \zeta$ are unique if they exist; such a Φ exists exactly when $\triangleright_{\mathbf{D}}$ is functionally complete. If $\triangleright_{\mathbf{D}}$ is not well-pointed, then in general there can be several weak functional completeness structures for $\triangleright_{\mathbf{D}}$.

Proposition 3.13. *Let Φ be a weak functional completeness structure for the action $\triangleright_{\mathbf{D}}$. For every functor $F : \mathbf{C} \to \mathbf{D}$, there is a strong functor $\hat{F}$ such that $\underline{\hat{F}} = F$; this is defined on morphisms $f : \Gamma \triangleright_{\mathbf{C}} X \to Y$ by $\hat{F}^{(\Gamma)} f = \Phi_\Gamma(\lambda \gamma. F((\!|f|\!)\gamma))$. Moreover, every natural transformation $\alpha : F \Rightarrow G$ is a strong natural transformation $\hat{F} \Rightarrow \hat{G}$, for $\hat{F}$ and $\hat{G}$ thus constructed.*

Weak functional completeness does not necessarily deliver the canonical strengths for the identity functor and the composition of functors with strength.

Example 3.14. We construct a weak functional completeness structure for $(\mathbf{Set}_\star, \mathbf{1}, \times)$ acting on itself. Every Γ has exactly one point $\star$, so to give a function $\zeta : \mathbf{Set}_\star(\mathbf{1},\Gamma) \to \mathbf{Set}_\star(X,Y)$ is just to choose a morphism $\zeta(\star) : X \to Y$ of pointed sets. We can therefore define the morphism $\Phi_\Gamma \zeta : \Gamma \times X \to Y$ by $\Phi_\Gamma \zeta(\gamma,x) = \zeta(\star)(x)$. This is in fact the only possible Φ in this case, even though the action fails to be well-pointed. By Proposition 3.13, every endofunctor F on pointed sets forms a strong functor with $\hat{F}^{(\Gamma)} f(\gamma,t) = F(\lambda x. f(\star,x))t$; this corresponds to the strength $\mathrm{str}_{\Gamma,X}(\gamma,t) = F(\lambda x.(\star,x))t$. There may in general be other strengths for F. For example, the identity functor on $\mathbf{Set}_\star$ also has the canonical strength $\mathrm{id}_{\Gamma \times X} : \Gamma \times X \to \Gamma \times X$.

Example 3.15. We give an example of an action that has multiple weak functional completeness structures. Fix a set E, and let $\mathbf{D}$ be the Kleisli category of the monad $-+E$ on **Set**: objects are sets, and morphisms $f \in \mathbf{D}(X,Y)$ are functions $f : X \to Y + E$; the identities are the left coprojections inl, and the composition of $f : X \to Y + E$ with $g : Y \to Z + E$ is $[g, \mathrm{inr}] \circ f : X \to Z + E$. This category is coCartesian, as is the Kleisli category of any monad on any coCartesian category. It therefore forms a monoidal category $(\mathbf{D}, \mathbf{0}, +)$, which acts on itself. Every Γ has exactly one point $[]$ because $\mathbf{0}$ is initial, so a function $\zeta : \mathbf{D}(\mathbf{0},\Gamma) \to \mathbf{D}(X,Y)$ just chooses a single function $\zeta([]) : X \to Y + E$. For each $e \in E$, we therefore have a morphism $\Phi^e_\Gamma \zeta = [\mathrm{inr} \circ e \circ \langle\rangle, \zeta([])] \in \mathbf{D}(\Gamma + X, Y)$, and Φ^e is a weak functional completeness structure. Hence in general, there is more than one Φ.

We note that, if $\mathbf{D}$ has copowers over **Set**, then well-pointedness of $\mathbf{D}$ amounts to the canonical morphisms $\mathbf{V}(I,\Gamma) \bullet X \to \Gamma \triangleright X$ being epimorphisms, while weak functional completeness amounts to the monoidal natural transformation $\mathbf{V}(I,-) \bullet (=) \to (-) \triangleright (=)$ being a split monomorphism (in the category of lax monoidal functors $\mathbf{V} \to [\mathbf{D},\mathbf{D}]$). Functional completeness is equivalent to the latter being an isomorphism.

4 Strong monads

We now turn to strong monads. There is a richer collection of equivalent definitions of strong monad than there is of strong functor. Because of our focus on semantics, the primary definition we use asks for a strong Kleisli extension operator $(-)^\dagger$, as in the introduction. Again we work in the action-based setting, so we suppose a monoidal category $\mathbf{V}$ that acts on a category $\mathbf{C}$. We drop the subscript on the action, writing $\triangleright$ instead of $\triangleright_{\mathbf{C}}$.

Definition 4.1. A *strong monad* $\mathsf{T} = (T, \eta, (-)^\dagger)$ consists of an object $TX \in \mathbf{C}$ and morphism $\eta_X : X \to TX$ for each $X \in \mathbf{C}$, and a morphism $f^\dagger : \Gamma \triangleright TX \to TY$ for each $f : \Gamma \triangleright X \to TY$, such that $(-)^\dagger$ is natural in $\Gamma \in \mathbf{V}$ and

$$\begin{aligned} (\eta_X \circ \lambda_X)^\dagger &= \lambda_{TX} && \text{for } X \in \mathbf{C} \\ f^\dagger \circ (\Gamma \triangleright \eta_X) &= f && \text{for } f : \Gamma \triangleright X \to TY \\ g^\dagger \circ (\Gamma' \triangleright f^\dagger) \circ \alpha_{\Gamma',\Gamma,TX} &= (g^\dagger \circ (\Gamma' \triangleright f) \circ \alpha_{\Gamma',\Gamma,X})^\dagger && \text{for } f : \Gamma \triangleright X \to TY,\ g : \Gamma' \triangleright Y \to TZ \end{aligned}$$

If S and T are strong monads, then a *strong monad morphism* $\tau : \mathsf{S} \to \mathsf{T}$ consists of a morphism $\tau_X : SX \to TX$ for each $X \in \mathbf{C}$, such that $\tau_X \circ \eta_X = \eta_X$ for each $X \in \mathbf{C}$ and $\tau_Y \circ f^\dagger = (\tau_Y \circ f)^\dagger \circ (\Gamma \triangleright \tau_X)$ for each $f : X \to SY$.

The morphisms η are collectively called the unit of T, and $(-)^\dagger$ is the Kleisli extension. If $\mathsf{T} = (T, \eta, (-)^\dagger)$ is a strong monad, then the assignment on objects T extends to a strong functor with $T^{(\Gamma)} f = (\eta_Y \circ f)^\dagger : \Gamma \triangleright TX \to TY$ for $f : \Gamma \triangleright X \to Y$. The unit η is then a strong natural transformation $\eta : \mathrm{Id} \Rightarrow T$, as is the *multiplication* $\mu : T \cdot T \Rightarrow T$, given by $\mu_X = \lambda^\dagger_{TX} \circ \lambda^{-1}_{TTX}$. Every strong monad morphism $\tau : \mathsf{S} \to \mathsf{T}$ is a strong natural transformation $\tau : S \Rightarrow T$.

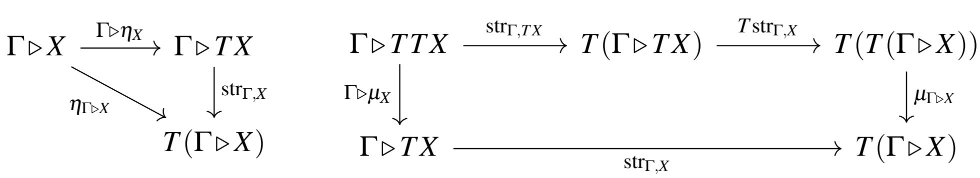
Example 4.2. Consider **Set** with the Cartesian monoidal structure, acting on itself. The strong monad List on **Set** maps each set X to the set List$X$ of lists over X. The unit η is given by the singleton lists $\eta_X x = [x]$. The Kleisli extension $f^\dagger : \Gamma \times \mathrm{List}X \to \mathrm{List}Y$ of $f : \Gamma \times X \to \mathrm{List}Y$ is defined by $f^\dagger(\gamma, [x_1, \ldots, x_n]) = f(\gamma, x_1) +\!\!+ \cdots +\!\!+ f(\gamma, x_n)$ where $+\!\!+$ is concatenation of lists. The strong functor List is the strong functor defined in Example 3.2 (and the ordinary functor $\underline{\mathrm{List}}$ is then the usual list endofunctor on **Set**).

If $\mathsf{T} = (T, \eta, (-)^\dagger)$ is a strong monad on $\mathbf{C}$, then the underlying functor $\underline{T}$ forms an ordinary monad $\underline{\mathsf{T}} = (\underline{T}, \eta, \mu)$ on $\mathbf{C}$ (where the multiplication μ is defined as above). Every strong monad morphism $\tau : \mathsf{S} \to \mathsf{T}$ is a monad morphism $\tau : \underline{\mathsf{S}} \to \underline{\mathsf{T}}$.

We now give several equivalent characterizations of strong monads. In addition to the definition above, strong monads can be defined in terms of strong functors, in terms of strengths, and also by lifting the action of $\mathbf{V}$ to the Kleisli category.

Proposition 4.3. *For each monad $\mathsf{T} = (T, \eta, \mu)$ on $\mathbf{C}$ there are bijections between*

1. *strong monads $\hat{\mathsf{T}}$ such that $\underline{\hat{\mathsf{T}}} = \mathsf{T}$;*
2. *strong functors $\hat{T}$ such that $\underline{\hat{T}} = T$ and such that η and μ are strong natural transformations $\mathrm{Id} \Rightarrow \hat{T}$ and $\hat{T} \cdot \hat{T} \Rightarrow \hat{T}$;*
3. *strengths* str *for the functor T, such that the following diagrams commute:*



4. *liftings of $\triangleright$ to the Kleisli category of T, i.e. actions $\triangleright_{\mathsf{T}}$ of $\mathbf{V}$ on $\mathbf{Kl}\,\mathsf{T}$ such that the following diagram commutes (up to equality, where K_{T} is the Kleisli inclusion):*

$$\begin{array}{ccc} \mathbf{V}\times\mathbf{C} & \xrightarrow{\ \triangleright\ } & \mathbf{C} \\ {\scriptstyle \mathbf{V}\times K_{\mathsf{T}}}\downarrow & & \downarrow{\scriptstyle K_{\mathsf{T}}} \\ \mathbf{V}\times\mathbf{Kl}\,\mathsf{T} & \xrightarrow[\triangleright_{\mathsf{T}}]{} & \mathbf{Kl}\,\mathsf{T} \end{array}$$

If S,T are monads on $\mathbf{C}$ equipped with the equivalent data from this bijection, then the following conditions on monad morphisms $\tau : \mathsf{S} \to \mathsf{T}$ are equivalent: (1) τ is a strong monad morphism $\hat{\mathsf{S}} \to \hat{\mathsf{T}}$; (2) τ is a strong natural transformation $\hat{S} \to \hat{T}$; (3) τ makes the diagram on the left below commute; (4) τ makes the diagram on the right below commute.

$$\begin{array}{ccc} \Gamma\triangleright SX & \xrightarrow{\Gamma\triangleright\tau_X} & \Gamma\triangleright TX \\ {\scriptstyle \mathrm{str}_{\Gamma,X}}\downarrow & & \downarrow{\scriptstyle \mathrm{str}_{\Gamma,X}} \\ S(\Gamma\triangleright X) & \xrightarrow[\tau_{\Gamma\triangleright X}]{} & T(\Gamma\triangleright X) \end{array} \qquad \begin{array}{ccc} \mathbf{V}\times\mathbf{Kl}\,\mathsf{S} & \xrightarrow{\triangleright_{\mathsf{S}}} & \mathbf{Kl}\,\mathsf{S} \\ {\scriptstyle \mathbf{V}\times\mathbf{Kl}\,\tau}\downarrow & & \downarrow{\scriptstyle \mathbf{Kl}\,\tau} \\ \mathbf{V}\times\mathbf{Kl}\,\mathsf{T} & \xrightarrow[\triangleright_{\mathsf{T}}]{} & \mathbf{Kl}\,\mathsf{T} \end{array}$$

Proof. For the bijection between (1) and (2), strong monads induce strong functors as above. If $\hat{T}$ is a strong functor with $\underline{\hat{T}} = T$, then the unit of $\hat{\mathsf{T}}$ is η and the Kleisli extension is given by $f^\dagger = \mu_Y \circ \hat{T}^{(\Gamma)} f$ for $f : \Gamma\triangleright X \to TY$. The bijection between (2) and (3) is a special case of Proposition 3.4, in particular, the two diagrams in (3) correspond to η and μ being strong. To go from (3) to (4), define the action $\triangleright_{\mathsf{T}}$ on objects by $\Gamma\triangleright_{\mathsf{T}} X = \Gamma\triangleright X$, on $\mathbf{V}$-morphisms by $\sigma\triangleright_{\mathsf{T}} X = \sigma\triangleright X$, and on morphisms $f \in \mathbf{Kl}\,\mathsf{T}(X,Y) = \mathbf{C}(X,TY)$ by $\Gamma\triangleright_{\mathsf{T}} f = \mathrm{str}_{\Gamma,Y}\circ(\Gamma\triangleright f) \in \mathbf{Kl}\,\mathsf{T}(\Gamma\triangleright X,\Gamma\triangleright Y)$. To go from (4) to (3), use $\mathrm{id}_{TX} \in \mathbf{Kl}\,\mathsf{T}(TX,X)$ to define $\mathrm{str}_{\Gamma,X} = \Gamma\triangleright_{\mathsf{T}} \mathrm{id}_{TX} \in \mathbf{Kl}\,\mathsf{T}(\Gamma\triangleright TX,\Gamma\triangleright X)$. The equivalence of the conditions on monad morphisms follows from the definition of each bijection. □

The fourth characterization of strong monad is important because of its connection with the semantics of call-by-value languages in *Freyd categories* [27]. Indeed, one possible definition of Freyd category explicitly requires such an action $\triangleright_{\mathsf{T}}$ [18]. The Kleisli inclusion K_{T} forms a strong functor with $\triangleright_{\mathsf{T}}$ as the action of $\mathbf{V}$ on $\mathbf{Kl}\,\mathsf{T}$, as does its right adjoint. If $\tau : \mathsf{S} \to \mathsf{T}$ is a strong monad morphism, then $\mathbf{Kl}\,\tau : \mathbf{Kl}\,\mathsf{S} \to \mathbf{Kl}\,\mathsf{T}$ also forms a strong functor.

We again emphasize that strength is additional structure a monad can be equipped with, not merely a property. Some monads admit multiple strengths and some admit no strength at all.

Example 4.4. Suppose a monoid $\mathsf{M} = (M,1,\cdot)$ in $\mathbf{Set}$, and consider the product of right M-actions as an action of the Cartesian monoidal category $\mathbf{Act}\,\mathsf{M}$ on itself. Equipping M with the discrete action $m * m' = m$ makes M into a monoid in $\mathbf{Act}\,\mathsf{M}$. The M-writer monad Wr_{M} on $\mathbf{Act}\,\mathsf{M}$ is the functor $\mathrm{Wr}_M = - \times M$ equipped with unit $\eta_X x = (x,1)$ and multiplication $\mu_X((x,m'),m) = (x, m\cdot m')$. If M is commutative, then Wr_{M} forms a strong monad in at least two ways. As for every writer monad on a monoidal category, the inverse of the associator is a strength $\mathrm{str}_{\Gamma,X}(\gamma,(x,m)) = ((\gamma,x),m)$; the bijections above induce a strong monad in which the Kleisli extension of $f : \Gamma\times X \to \mathrm{Wr}_M Y$ is given by $f^\dagger(\gamma,(x,m)) = (y, m\cdot m')$ where $(y,m') = f(\gamma,x)$. Using commutativity, there is also a second strength $\mathrm{str}'_{\Gamma,X}(\gamma,(x,m)) = ((\gamma * m, x), m)$; this induces a strong monad with Kleisli extension $f^\dagger(\gamma,(x,m)) = (y, m\cdot m')$ where $(y,m') = f(\gamma * m, x)$.

This example can also be adjusted for the product of sets as an action of $(\mathbf{Act}\,\mathsf{M},\mathbf{1},\times)$ on $\mathbf{Set}$. In this case, commutativity of M is not needed.

4.1 Free monads on strong endofunctors

It is frequently useful to be able to construct the free monad T on an endofunctor F. In general, a strength for F will not induce a strength for T; we give a sufficient condition for this to be the case below. First we note that, for many applications (even without strength), T being free (as in *free object*) is not enough. One often wants the monad T to be *algebraically free* [10], meaning there is an isomorphism $\mathbf{Alg}\,\mathsf{T} \cong \mathbf{alg}\,F$ that commutes with the forgetful functors. (We write $\mathbf{Alg}\,\mathsf{T}$ for the Eilenberg-Moore category of the monad T, and $\mathbf{alg}\,F$ for the category of algebras of the functor F.) Algebraic freeness, thus defined, is not a universal property, but it still identifies a monad up to a unique isomorphism. Algebraically free implies free; the converse holds if $\mathbf{C}$ is complete [10, Proposition 22.4].

In general, even the algebraically free monad will not be strong when F has a strength. To obtain a strength for the monad, we need to refine the notion of free algebra. Several versions of the following notion have appeared in the literature before (for example [2, 26, 6]).

Definition 4.5. If F is a strong endofunctor on $\mathbf{C}$, an $\underline{F}$-algebra (A,a) equipped with a morphism $f : X \to A$ is called the *strongly free F-algebra* on $X \in \mathbf{C}$ if, for all $\Gamma \in \mathbf{V}$, $(B,b) \in \mathbf{alg}\,\underline{F}$ and $g : \Gamma \triangleright X \to B$, there is a unique morphism $h : \Gamma \triangleright A \to B$ such that the following diagram commutes.

$$\begin{array}{ccccc}
\Gamma \triangleright X & \xrightarrow{\Gamma \triangleright f} & \Gamma \triangleright A & \xleftarrow{\Gamma \triangleright a} & \Gamma \triangleright FA \\
& \searrow^{g} & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle F^{(\Gamma)}h} \\
& & B & \xleftarrow{b} & FB
\end{array}$$

If (A,f,a) is the strongly free F-algebra on X, then it is also the free $\underline{F}$-algebra on X.

Proposition 4.6. *Suppose a strong functor F. If the strongly free F-algebra (TX,η_X,σ_X) exists for each object X, then T forms a strong monad T in which the unit is η and the Kleisli extension $g^\dagger$ of $g : \Gamma \triangleright X \to TY$ is the unique morphism $h : \Gamma \triangleright TX \to TY$ such that*

$$\begin{array}{ccccc}
\Gamma \triangleright X & \xrightarrow{\Gamma \triangleright \eta_X} & \Gamma \triangleright TX & \xleftarrow{\Gamma \triangleright \sigma_X} & \Gamma \triangleright F(TX) \\
& \searrow^{g} & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle F^{(\Gamma)}h} \\
& & TY & \xleftarrow{\sigma_Y} & F(TY)
\end{array}$$

The monad $\underline{\mathsf{T}}$ is algebraically free on $\underline{F}$.

It is well-known that, in the presence of right adjoints to $\Gamma \triangleright -$ (in particular, when $\mathbf{V}$ is right closed, in the case of $\mathbf{V}$ acting on itself), ordinary free algebras suffice to construct a strength (see for example [4, Theorem 5]); we explain this result in the context of *powering* in Section 6.1. Free algebras are also strongly free when they can be constructed as colimits that are preserved by $\Gamma \triangleright - : \mathbf{C} \to \mathbf{C}$ for each Γ. (See e.g. Kelly [10] for the construction of free algebras as colimits.)

4.2 Uniqueness and existence of strengths for monads

The situation for uniqueness of strengths carries over immediately from functors (Section 3.1) to monads.

Proposition 4.7. *Suppose that $\mathsf{T} = (T,\eta,\mu)$ is a monad on $\mathbf{C}$ and that the action $\triangleright$ is well-pointed. If the functor T forms a strong functor $\hat{T}$ with $T = \underline{\hat{T}}$ (necessarily uniquely), then T forms a strong monad $\hat{\mathsf{T}}$ with $\underline{\hat{\mathsf{T}}} = \mathsf{T}$ (again uniquely); moreover, every monad morphism between strong monads is a strong monad morphism. In particular, if $\triangleright$ is functionally complete, then every monad T on $\mathbf{C}$ forms a strong monad in exactly one way.*

Existence of strengths for monads is more problematic. The strengths assigned to the functor T by a weak functional completeness structure Φ will not in general make T into a strong monad. For example, consider the Cartesian monoidal category $\mathbf{Set}_\star$ acting on itself. This has a single weak functional completeness structure Φ that assigns to the identity functor on $\mathbf{Set}_\star$ the strength $\mathrm{str}_{\Gamma,X}(\gamma,x) = (*,x)$. The unit of the identity monad is not a strong natural transformation with respect to this strength (its domain is the identity functor with the canonical strength!), so Φ does not make the identity monad into a strong monad. In fact, if the strength assigned to the identity endofunctor on $\mathbf{C}$ by a weak functional completeness structure Φ for an arbitrary action of $\mathbf{V}$ on $\mathbf{C}$ makes the identity monad into a strong monad, then it follows that the action is functionally complete. To see this, note that strong naturality of η implies the identity monad forms a strong monad in only one way: the underlying strong functor has $\mathrm{Id}^{(\Gamma)} f = f$. If Φ makes the identity monad into a strong monad, we therefore have $f = \mathrm{Id}^{(\Gamma)} f = \Phi_\Gamma(\lambda\gamma.\, \mathrm{Id}((\!|f|\!)\gamma)) = \Phi_\Gamma (\!|f|\!)$, so $(\!|-|\!)$ is a bijection, which implies functional completeness.

5 Enrichment

So far, we have considered strength only from the perspective of actions of the monoidal category $\mathbf{V}$. A well-known result of Kock [15] is that, in a certain situation, strong functors are the same as enriched functors. More precisely, Kock shows that if $\mathbf{V}$ is a monoidal category that is (left) closed in the sense that each $- \otimes X : \mathbf{V} \to \mathbf{V}$ has a right adjoint $X \multimap - : \mathbf{V} \to \mathbf{V}$, then strengths $\Gamma \otimes FX \to F(\Gamma \otimes X)$ for an endofunctor $F : \mathbf{V} \to \mathbf{V}$ are in bijection with suitable natural transformations $X \multimap Y \to FX \multimap FY$. The latter make F into an enriched functor $\mathbf{V} \to \mathbf{V}$ (in the sense of enriched category theory [11]), where $\mathbf{V}$ enriches over itself using the closed structure.

It is less well-known that this connection between enrichment and strength holds more generally. If $\mathbf{C}$ and $\mathbf{D}$ are any categories that enrich over $\mathbf{V}$, and suitable adjoints exist, then enriched functors $\mathbf{C} \to \mathbf{D}$ are the same as strong functors $\mathbf{C} \to \mathbf{D}$. There are similar bijections for natural transformations, and for monads. Strength and enrichment are therefore just two perspectives on the same structure. In particular, facts from enriched category theory can be transferred along these bijections to become facts about strength.

We give the precise connection between strength and enrichment in this section, again working with an general monoidal category $\mathbf{V}$. Again, for what we are interested in, we do not need symmetry.

Definition 5.1. An *enrichment* of a category $\mathbf{C}$ over a monoidal category $(\mathbf{V}, I, \otimes)$ is a functor $\rightarrowtriangle : \mathbf{C}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{V}$ equipped with natural transformations

$$j_X : I \to X \rightarrowtriangle X \qquad M_{X,Y,Z} : (Y \rightarrowtriangle Z) \otimes (X \rightarrowtriangle Y) \to X \rightarrowtriangle Z$$

such that the functions $\hat{\jmath}_{X,Y} : \mathbf{C}(X,Y) \to \mathbf{V}(I, X \rightarrowtriangle Y)$ given by $\hat{\jmath}_{X,Y} f = (X \rightarrowtriangle f) \circ j_X$ are bijections and such that the following coherence conditions are satisfied:

$$\begin{array}{ccc}
I \otimes (X \rightarrowtriangle Y) & \xrightarrow{j_Y \otimes (X \rightarrowtriangle Y)} & (Y \rightarrowtriangle Y) \otimes (X \rightarrowtriangle Y) \\
{\scriptstyle \lambda_{X \rightarrowtriangle Y}} \downarrow & & \downarrow {\scriptstyle M_{X,Y,Y}} \\
X \rightarrowtriangle Y & = & X \rightarrowtriangle Y \\
{\scriptstyle \rho_{X \rightarrowtriangle Y}} \downarrow & & \uparrow {\scriptstyle M_{X,X,Y}} \\
(X \rightarrowtriangle Y) \otimes I & \xrightarrow[(X \rightarrowtriangle Y) \otimes j_X]{} & (X \rightarrowtriangle Y) \otimes (X \rightarrowtriangle X)
\end{array}$$

$$\begin{array}{ccc}
((Y \rightarrowtriangle Z) \otimes (X \rightarrowtriangle Y)) \otimes (W \rightarrowtriangle X) & \xrightarrow{M_{X,Y,Z} \otimes (W \rightarrowtriangle X)} & (X \rightarrowtriangle Z) \otimes (W \rightarrowtriangle X) \\
{\scriptstyle \alpha_{Y \rightarrowtriangle Z, X \rightarrowtriangle Y, W \rightarrowtriangle X}} \downarrow & & \downarrow {\scriptstyle M_{W,X,Z}} \\
(Y \rightarrowtriangle Z) \otimes ((X \rightarrowtriangle Y) \otimes (W \rightarrowtriangle X)) & & W \rightarrowtriangle Z \\
{\scriptstyle (Y \rightarrowtriangle Z) \otimes M_{W,X,Y}} \searrow & & \nearrow {\scriptstyle M_{W,Y,Z}} \\
& (Y \rightarrowtriangle Z) \otimes (W \rightarrowtriangle Y) &
\end{array}$$

The objects $X \mathrel{-\!\triangleright} Y$ are the hom-objects of the enrichment; the natural transformation j gives the identities and M is composition. Since we do not assume that $\mathbf{V}$ is symmetric, the order of composition is very important. The bijection condition in the definition means that the enriched category has $\mathbf{C}$ as the underlying ordinary category.

Example 5.2. Recall from Example 2.7 that every monoidal category $\mathbf{V}$ acts on itself with $\Gamma \triangleright X = \Gamma \otimes X$. If each $- \otimes X$ has a right adjoint $X \multimap -$, then $\multimap$ forms an enrichment of $\mathbf{V}$ over itself. (This fact is an instance of Proposition 5.4 below.) This includes for example the category of actions $\mathbf{Act}\,\mathsf{M}$ of any set-theoretic monoid M, with the Cartesian monoidal structure (because $\mathbf{Act}\,\mathsf{M}$ has exponentials). Another example is the smash product $\otimes$ of pointed sets, for which $X \multimap Y$ is $\mathbf{Set}_\star(X,Y)$, with distinguished element $\lambda x.\star$.

For $\mathbf{V} = \mathbf{Set}$ with the Cartesian monoidal structure, every (locally small) $\mathbf{C}$ has a unique enrichment. The object $X \mathrel{-\!\triangleright} Y \in \mathbf{Set}$ is the hom-set $\mathbf{C}(X,Y)$; the structural laws $j_X : 1 \to \mathbf{C}(X,X)$ and $M_{X,Y,Z} : \mathbf{C}(Y,Z) \times \mathbf{C}(X,Y) \to \mathbf{C}(X,Z)$ are the identities and composition. If $\mathbf{C}$ has copowers over $\mathbf{Set}$ (which form an action of $\mathbf{Set}$ on $\mathbf{C}$ as in Example 2.7), then $- \bullet X \dashv \mathbf{C}(X,-)$.

Definition 5.3. If $\mathbf{C}$ and $\mathbf{D}$ are enriched categories, then an *enriched functor* $F : \mathbf{C} \to \mathbf{D}$ consists of an object $FX \in \mathbf{D}$ for each object $X \in \mathbf{C}$ and a morphism $\mathrm{fmap}_{X,Y} : X \mathrel{-\!\triangleright}_{\mathbf{C}} Y \to FX \mathrel{-\!\triangleright}_{\mathbf{D}} FY$ for each $X, Y \in \mathbf{C}$, such that the following diagrams commute:

$$\begin{array}{ccc} I & \xrightarrow{j_X} & X \mathrel{-\!\triangleright}_{\mathbf{C}} X \\ & \searrow^{j_{FX}} & \downarrow{\scriptstyle \mathrm{fmap}_{X,X}} \\ & & FX \mathrel{-\!\triangleright}_{\mathbf{D}} FX \end{array} \qquad \begin{array}{ccc} (Y \mathrel{-\!\triangleright}_{\mathbf{C}} Z) \otimes (X \mathrel{-\!\triangleright}_{\mathbf{C}} Y) & \xrightarrow{M_{X,Y,Z}} & X \mathrel{-\!\triangleright}_{\mathbf{C}} Z \\ {\scriptstyle \mathrm{fmap}_{Y,Z} \otimes \mathrm{fmap}_{X,Y}}\downarrow & & \downarrow{\scriptstyle \mathrm{fmap}_{X,Z}} \\ (FY \mathrel{-\!\triangleright}_{\mathbf{D}} FZ) \otimes (FX \mathrel{-\!\triangleright}_{\mathbf{D}} FY) & \xrightarrow[M_{FX,FY,FZ}]{} & FX \mathrel{-\!\triangleright}_{\mathbf{D}} FZ \end{array}$$

If $F, G : \mathbf{C} \to \mathbf{D}$ are enriched functors, an *enriched natural transformation* $\tau : F \Rightarrow G$ consists of a $\mathbf{D}$-morphism $\tau_x : FX \to GX$ for each $X \in \mathbf{C}$, such that the following diagram commutes:

$$\begin{array}{ccc} X \mathrel{-\!\triangleright}_{\mathbf{C}} Y & \xrightarrow{\mathrm{fmap}_{X,Y}} & FX \mathrel{-\!\triangleright}_{\mathbf{D}} FY \\ {\scriptstyle \mathrm{fmap}_{X,Y}}\downarrow & & \downarrow{\scriptstyle FX \mathrel{-\!\triangleright} \tau_Y} \\ GX \mathrel{-\!\triangleright}_{\mathbf{D}} GY & \xrightarrow[\tau_X \mathrel{-\!\triangleright} GY]{} & FX \mathrel{-\!\triangleright}_{\mathbf{D}} GY \end{array}$$

In the case of $\mathbf{V} = \mathbf{Set}$ with the Cartesian monoidal structure, enriched functors and natural transformations are just the same as ordinary functors and natural transformations. This is a counterpart to the fact that ordinary functors and natural transformations are uniquely strong with respect to $\bullet$ (if $\mathbf{D}$ has copowers).

The connection between enrichment and strength is the following.

Proposition 5.4. *Suppose, for each $X \in \mathbf{C}$, a functor $- \triangleright X : \mathbf{V} \to \mathbf{C}$ with a right adjoint $X \mathrel{-\!\triangleright} - : \mathbf{C} \to \mathbf{V}$. Also suppose that $- \otimes \Gamma$ has a right adjoint $\Gamma \multimap - : \mathbf{V} \to \mathbf{V}$ for each $\Gamma \in \mathbf{V}$.*[3] *Then there is a bijection between (1) the additional data required for $\triangleright$ to form an action of $\mathbf{V}$ on $\mathbf{C}$ and (2) the additional data required for $\mathrel{-\!\triangleright}$ to form an enrichment of $\mathbf{C}$ over $\mathbf{V}$ such that the morphisms $(\Gamma \triangleright X) \mathrel{-\!\triangleright} Y \to \Gamma \multimap (X \mathrel{-\!\triangleright} Y)$, obtained from M by transposition, are isomorphisms. If both $\mathbf{C}$ and $\mathbf{D}$ are equipped with an action and an enrichment related by this bijection, then strong functors $\mathbf{C} \to \mathbf{D}$ are in bijection with enriched functors $\mathbf{C} \to \mathbf{D}$. Moreover, if F, G are strong, then natural transformations $\tau : \underline{F} \Rightarrow \underline{G}$ are strong if and only if they are enriched.*

This proposition is not new. A proof is given for the more general case of enrichment in a bicategory by Gordon and Power [7]. Janelidze and Kelly [8] also give a proof of the first part of this proposition for enrichment in a monoidal category; they describe the construction of the enrichment as "often-rediscovered folklore". We sketch the proof here.

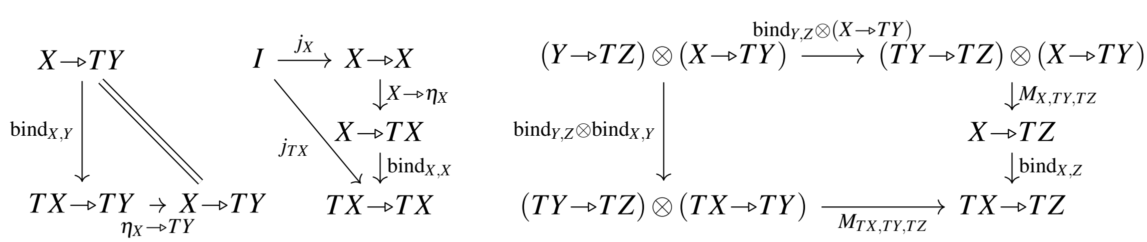
Proof. It is a standard fact about adjunctions that making $\triangleright$ into a bifunctor is equivalent to making $\mathrel{-\!\triangleright}$ into a bifunctor. By transposition, morphisms $\lambda_X : I \triangleright X \to X$ are in bijection with morphisms $j_X : I \to X \mathrel{-\!\triangleright} X$, and λ_X is an isomorphism exactly when $\hat{\jmath}_{X,Y} : \mathbf{C}(X,Y) \to \mathbf{V}(I, X \mathrel{-\!\triangleright} Y)$ is a bijection for all Y. Families of morphisms $\alpha_{\Gamma',\Gamma,X} : (\Gamma' \otimes \Gamma) \triangleright X \to \Gamma' \triangleright (\Gamma \triangleright X)$ natural in Γ, Γ' are in bijection with families of morphisms $M_{X,Y,Z} : (Y \mathrel{-\!\triangleright} Z) \otimes (X \mathrel{-\!\triangleright} Y) \to X \mathrel{-\!\triangleright} Z$ natural in Y, Z by the Yoneda lemma and transposition, and $\alpha_{\Gamma,\Gamma',X}$ is invertible for all Γ, Γ' exactly when the morphisms $(\Gamma \triangleright X) \mathrel{-\!\triangleright} Y \to \Gamma \multimap (X \mathrel{-\!\triangleright} Y)$ induced by M are invertible for all Γ, Y. Each of the coherence laws of an action corresponds to one of the laws of an enrichment.

Each strong functor $F : \mathbf{C} \to \mathbf{D}$ comes with functions $\mathbf{C}(\Gamma \triangleright_{\mathbf{C}} X, Y) \to \mathbf{D}(\Gamma \triangleright_{\mathbf{D}} FX, FY)$ natural in Γ. By transposition, natural transformations of this type are in bijection with Γ-natural transformations $\mathbf{V}(\Gamma, X \mathrel{-\!\triangleright}_{\mathbf{C}} Y) \to \mathbf{V}(\Gamma, FX \mathrel{-\!\triangleright}_{\mathbf{D}} FY)$, hence, by the Yoneda lemma, with morphisms $X \mathrel{-\!\triangleright}_{\mathbf{C}} Y \to FX \mathrel{-\!\triangleright}_{\mathbf{D}} FY$. The axioms of enriched and strong functors transfer along this bijection, as do strong and enriched naturality. □

Remark 5.5. There is a more conceptual (and more technical) proof, which we outline. Wood [31] shows that the 2-category of categories enriched over **V** embeds fully faithfully into that of what he calls *large* **V**-*categories*.[4] Modulo size issues, these are categories enriched over $[\mathbf{V}^{\mathrm{op}}, \mathbf{Set}]$ with the convolution monoidal structure. There is a similar embedding of categories equipped with actions of **V** in large **V**-categories. By characterizing the images of these embeddings, it is possible to transfer data between the action perspective and enrichment perspective under the assumptions of Proposition 5.4. This also works for the powered categories of Section 6. Large **V**-categories then provide a perspective on strength that strictly subsumes all of the three perspectives we consider here. The *locally indexed categories* used by Levy [18] and Egger et al. [3] for strength with respect to Cartesian products are similar (but not quite identical) to large **V**-categories; the $[\mathbf{V}^{\mathrm{op}}, \mathbf{Set}]$ perspective is used by Melliès [22].

One of the advantages of considering enrichment is that the concept of *enriched monad* (corresponding to strong monad) admits a particularly lightweight definition.

Definition 5.6. If **C** is an enriched category, then an *enriched monad* on **C** consists of an object $TX \in \mathbf{C}$ and morphism $\eta_X : X \to TX$ for each $X \in \mathbf{C}$, and a morphism $\mathrm{bind}_{X,Y} : X \mathrel{-\!\triangleright} TY \to TX \mathrel{-\!\triangleright} TY$ for each $X, Y \in \mathbf{C}$, such that the following diagrams commute:



[3] We do not claim that it is *necessary* for $\multimap$ to exist in order to connect strength and enrichment, but the statement of this proposition is complicated without $\multimap$.

[4] Following Levy [19], we prefer to call them *locally* **V**-*graded categories*.

An *enriched monad morphism* $\tau : \mathsf{S} \to \mathsf{T}$ consists of a morphism $\tau_X : SX \to TX$ for each $X \in \mathbf{C}$, such that $\tau_X \circ \eta_X = \eta_X$, and such that the following diagram commutes:

$$\begin{array}{ccccc} X \multimap SY & \xrightarrow{\mathrm{bind}_{X,Y}} & SX \multimap SY & \xrightarrow{SX \multimap \tau_Y} & \\ \downarrow{\scriptstyle X \multimap \tau_Y} & & & & SX \multimap TY \\ X \multimap TY & \xrightarrow[\mathrm{bind}_{X,Y}]{} & TX \multimap TY & \xrightarrow{\tau_X \multimap TY} & \end{array}$$

Remark 5.7. In Haskell (and similar languages), the `Monad` type class asks for a polymorphic function `(>>=) :: m a -> (a -> m b) -> m b`. This corresponds to the natural transformation bind above (with arguments reversed). Instances of `Monad` are actually enriched monads (and by the following proposition, strong monads), not ordinary monads, which is why there is no need to provide a strength in Haskell. The same goes for the `Functor` type class: `fmap :: (a -> b) -> (f a -> f b)` is enriched functoriality, not ordinary functoriality.

Proposition 5.8. *Assume the setting of Proposition 5.4, with an action of* $\mathbf{V}$ *on* $\mathbf{C}$ *that has a corresponding enrichment. There is a bijection between strong monads on* $\mathbf{C}$ *and enriched monads on* $\mathbf{C}$*, and this bijection preserves the underlying ordinary monads. If* T, S *are strong and* $\tau : \underline{\mathsf{T}} \to \underline{\mathsf{S}}$ *is a monad morphism, then* τ *is strong if and only if it is enriched.*

Note that the definition of enriched monad involves only 3 equations whereas the definition of strong monad has 4 and the definition of monad with a strength has as many as 12 (7 equations of a monad and 5 equations of a strength for a monad).

6 Powering

We now turn to the final perspective on strength that we consider. Enrichment fits into the picture by considering right adjoints $X \multimap -$ to $- \triangleright X$. If instead the functors $\Gamma \triangleright - : \mathbf{C} \to \mathbf{C}$ have right adjoints $\Gamma \pitchfork - : \mathbf{C} \to \mathbf{C}$, then they form a *powering* of $\mathbf{C}$ over $\mathbf{V}$ in the following sense; we call $\Gamma \pitchfork X$ the *power* of Γ and X.[5]

Definition 6.1. A *powering* of a category $\mathbf{C}$ over a monoidal category $(\mathbf{V}, I, \otimes)$ is a functor $\pitchfork : \mathbf{V}^{\mathrm{op}} \times \mathbf{C} \to \mathbf{C}$ equipped with natural isomorphisms

$$i_X : X \to I \pitchfork X \qquad p_{\Gamma,\Gamma',X} : \Gamma \pitchfork (\Gamma' \pitchfork X) \to (\Gamma' \otimes \Gamma) \pitchfork X$$

satisfying the following coherence conditions:

$$\begin{array}{ccc} \Gamma \pitchfork (I \pitchfork X) & \xrightarrow{p_{\Gamma,I,X}} & (I \otimes \Gamma) \pitchfork X \\ \uparrow{\scriptstyle \Gamma \pitchfork i_X} & & \uparrow{\scriptstyle \lambda_\Gamma \pitchfork X} \\ \Gamma \pitchfork X & = & \Gamma \pitchfork X \\ \downarrow{\scriptstyle i_{\Gamma \pitchfork X}} & & \uparrow{\scriptstyle \rho_\Gamma \pitchfork X} \\ I \pitchfork (\Gamma \pitchfork X) & \xrightarrow[p_{I,\Gamma,X}]{} & (\Gamma \otimes I) \pitchfork X \end{array}$$

$$\begin{array}{ccccc} & \xrightarrow{\Gamma_3 \pitchfork p_{\Gamma_2,\Gamma_1,X}} & \Gamma_3 \pitchfork ((\Gamma_1 \otimes \Gamma_2) \pitchfork X) & \xrightarrow{p_{\Gamma_3,\Gamma_1 \otimes \Gamma_2,X}} & \\ \Gamma_3 \pitchfork (\Gamma_2 \pitchfork (\Gamma_1 \pitchfork X)) & & & & ((\Gamma_1 \otimes \Gamma_2) \otimes \Gamma_3) \pitchfork X \\ & \xrightarrow{p_{\Gamma_3,\Gamma_2,\Gamma_1 \pitchfork X}} & (\Gamma_2 \otimes \Gamma_3) \pitchfork (\Gamma_1 \pitchfork X) \xrightarrow[p_{\Gamma_2 \otimes \Gamma_3,\Gamma_1,X}]{} (\Gamma_1 \otimes (\Gamma_2 \otimes \Gamma_3)) \pitchfork X & \xrightarrow{\alpha_{\Gamma_1,\Gamma_2,\Gamma_3} \pitchfork X} & \end{array}$$

[5] The terminology here comes from the fact that, just as $\Gamma \triangleright X$ is a copower (tensor) in the enriched sense when $\multimap$ exists, $\Gamma \pitchfork X$ is a power (cotensor) in the enriched sense.

Example 6.2. If $\mathbf{V}$ is right closed in the sense that each $\Gamma \otimes - : \mathbf{V} \to \mathbf{V}$ has a right adjoint $\Gamma \multimap^{\mathrm{R}} - : \mathbf{V} \to \mathbf{V}$, then $\multimap^{\mathrm{R}}$ gives a powering of $\mathbf{V}$ over itself (by Proposition 6.4 below). This right adjoint is naturally isomorphic to $\Gamma \multimap -$ exactly when $\mathbf{V}$ is symmetric. Even when $\mathbf{V}$ is symmetric, the definitions of powered functor and powered monad are different from the enriched versions (but they are in bijection).

If a category $\mathbf{C}$ has small products, then it is powered over $\mathbf{Set}$ by defining $\Gamma \mathrel{-\!\!\in} X = \Gamma \pitchfork X = \prod_{\gamma \in \Gamma} X$. If $\mathbf{C}$ also has small coproducts, then we have adjunctions $\Gamma \bullet - \dashv \Gamma \pitchfork -$.

We define powered notions of functor and natural transformation analogous to the strong and enriched notions.

Definition 6.3. If $\mathbf{C}$ and $\mathbf{D}$ are powered categories, then a *powered functor* $F : \mathbf{C} \to \mathbf{D}$ consists of an object $FX \in \mathbf{D}$ for each $X \in \mathbf{C}$, and a $\mathbf{D}$-morphism $F^{(\Gamma)} f : FX \to \Gamma \mathrel{-\!\!\in}_{\mathbf{D}} FY$ for each $\mathbf{C}$-morphism $f : X \to \Gamma \mathrel{-\!\!\in}_{\mathbf{C}} Y$ such that $F^{(\Gamma)}$ is natural in $\Gamma \in \mathbf{V}$ and

$$\begin{array}{rcll} F^{(I)} i_X &=& i_{FX} & \text{for } X \in \mathbf{C} \\ F^{(\Gamma' \otimes \Gamma)}(p_{\Gamma,\Gamma',Z} \circ (\Gamma \mathrel{-\!\!\in}_{\mathbf{C}} g) \circ f) &=& p_{\Gamma,\Gamma',FZ} \circ (\Gamma \mathrel{-\!\!\in}_{\mathbf{D}} F^{(\Gamma')} g) \circ F^{(\Gamma)} f & \text{for } f : X \to \Gamma \mathrel{-\!\!\in}_{\mathbf{C}} Y,\ g : Y \to \Gamma' \mathrel{-\!\!\in}_{\mathbf{C}} Z \end{array}$$

If $F, G : \mathbf{C} \to \mathbf{D}$ are powered functors, then a *powered natural transformation* $\tau : F \Rightarrow G$ consists of a $\mathbf{D}$-morphism $\tau_X : FX \to GX$ for each $X \in \mathbf{C}$ such that $(\Gamma \mathrel{-\!\!\in}_{\mathbf{D}} \tau_Y) \circ F^{(\Gamma)} f = G^{(\Gamma)} f \circ \tau_X$ for $f : X \to \Gamma \mathrel{-\!\!\in}_{\mathbf{C}} Y$.

If $F : \mathbf{C} \to \mathbf{D}$ is a powered functor, then we obtain an ordinary functor $\underline{F} : \mathbf{C} \to \mathbf{D}$ by defining $\underline{F}X = FX$ and $\underline{F}f = i_{FY}^{-1} \circ F^{(I)}(i_Y \circ f)$ for $f : X \to Y$.

Equivalently, a powered functor is an ordinary functor F with a powering, i.e. family of morphisms $\mathrm{pow}_{\Gamma,Y} : F(\Gamma \mathrel{-\!\!\in}_{\mathbf{C}} Y) \to \Gamma \mathrel{-\!\!\in}_{\mathbf{D}} FY$ natural in $\Gamma \in \mathbf{V}$ and $Y \in \mathbf{C}$, subject to two equations. We will not discuss this definition further.

The relationship between strength and powering is as follows. Similar to the relationship between strength and enrichment (Proposition 5.4), this proposition enables us to look at strength from the perspective of powering.

Proposition 6.4. *Suppose for each* $\Gamma \in \mathbf{V}$ *an adjunction* $\Gamma \triangleright - \dashv \Gamma \mathrel{-\!\!\in} - : \mathbf{C} \to \mathbf{C}$. *There is a bijection between the additional data required for* $\triangleright$ *to form an action of* $\mathbf{V}$ *on* $\mathbf{C}$ *and the additional data required for* $\mathrel{-\!\!\in}$ *to form a powering of* $\mathbf{C}$ *over* $\mathbf{V}$. *If both* $\mathbf{C}$ *and* $\mathbf{D}$ *are equipped with an action and a powering related by this bijection, then there is a bijection between strong functors and powered functors* $\mathbf{C} \to \mathbf{D}$*; this preserves the underlying ordinary functors. Under this bijection, natural transformations are strong if and only if they are powered.*

We can connect enrichment and powering by combining this proposition with Proposition 5.4, but also directly by a natural isomorphism $\mathbf{V}(\Gamma, X \mathrel{-\!\!\triangleright} Y) \cong \mathbf{C}(X, \Gamma \mathrel{-\!\!\in} Y)$; we omit the precise statement.

Definition 6.5. If $\mathbf{C}$ is a powered category, then a *powered monad* $\mathsf{T} = (T, \eta, (-)^{\dagger})$ consists of an object $TX \in \mathbf{C}$ and morphism $\eta_X : X \to TX$ for each $X \in \mathbf{C}$, and a morphism $f^{\dagger} : TX \to \Gamma \mathrel{-\!\!\in} TY$ for each $f : X \to \Gamma \mathrel{-\!\!\in} TY$, such that $(-)^{\dagger}$ is natural in Γ and

$$\begin{array}{rcll} (i_{TX} \circ \eta_X)^{\dagger} &=& i_{TX} & \text{for } X \in \mathbf{C} \\ f^{\dagger} \circ \eta_X &=& f & \text{for } f : X \to \Gamma \mathrel{-\!\!\in} TY \\ p_{\Gamma,\Gamma',TZ} \circ (\Gamma \mathrel{-\!\!\in} g^{\dagger}) \circ f^{\dagger} &=& (p_{\Gamma,\Gamma',TZ} \circ (\Gamma \mathrel{-\!\!\in} g^{\dagger}) \circ f)^{\dagger} & \text{for } f : X \to \Gamma \mathrel{-\!\!\in} TY,\ g : Y \to \Gamma' \mathrel{-\!\!\in} TZ \end{array}$$

If S and T are powered monads, then a *powered monad morphism* $\tau : \mathsf{S} \to \mathsf{T}$ consists of a morphism $\tau_X : SX \to TX$ for each $X \in \mathbf{C}$ such that $\tau_X \circ \eta_X = \eta_X$ for each $X \in \mathbf{C}$ and such that $(\Gamma \mathrel{-\!\!\in} \tau_Y) \circ f^{\dagger} = ((\Gamma \mathrel{-\!\!\in} \tau_Y) \circ f)^{\dagger} \circ \tau_X$ for each $f : X \to \Gamma \mathrel{-\!\!\in} SY$.

If $\mathsf{T} = (T, \eta, (-)^\dagger)$ is a powered monad, then T forms a powered functor $\mathbf{C} \to \mathbf{C}$ by defining $T^{(\Gamma)} f = ((\Gamma \mathrel{-\!\!\in} \eta_Y) \circ f)^\dagger$ for each $f : X \to \Gamma \mathrel{-\!\!\in} Y$. There is also a monad $\underline{\mathsf{T}} = (\underline{T}, \eta, \mu)$, with multiplication $\mu_X = i_{TX}^{-1} \circ i_{TX}^\dagger$.

As for the action perspective, the powering perspective gives rise to several equivalent notions of monad, given in the following proposition. We emphasize the characterization (3) below in particular. This characterization is useful when the monad T is constructed so that the Eilenberg-Moore category matches some particular category (for example, the models of an algebraic theory); in which case one way of making T into a strong monad is to first obtain a powered monad using (3), and then obtain a strong monad using Proposition 6.4.

Proposition 6.6. *For each monad $\mathsf{T} = (T, \eta, \mu)$ on a powered category $\mathbf{C}$, there is a bijection between:*

1. *powered monads $\hat{\mathsf{T}}$ such that $\underline{\hat{\mathsf{T}}} = \mathsf{T}$;*
2. *powered functors $\hat{T}$ such that $\underline{\hat{T}} = T$ and such that η and μ are powered natural transformations $\mathrm{Id} \Rightarrow \hat{T}$ and $\hat{T} \cdot \hat{T} \Rightarrow \hat{T}$;*
3. *liftings of $\mathrel{-\!\!\in}$ to the Eilenberg-Moore category of T, i.e. powerings $\mathrel{-\!\!\in}_{\mathsf{T}}$ of $\mathbf{Alg}\,\mathsf{T}$ over $\mathbf{V}$, such that the following diagram commutes (up to equality, where U_{T} is the forgetful functor).*

$$\begin{array}{ccc}
\mathbf{V}^{\mathrm{op}} \times \mathbf{Alg}\,\mathsf{T} & \xrightarrow{\mathrel{-\!\!\in}_{\mathsf{T}}} & \mathbf{Alg}\,\mathsf{T} \\
{\scriptstyle \mathbf{V}^{\mathrm{op}} \times U_{\mathsf{T}}}\downarrow & & \downarrow{\scriptstyle U_{\mathsf{T}}} \\
\mathbf{V}^{\mathrm{op}} \times \mathbf{C} & \xrightarrow[\mathrel{-\!\!\in}]{} & \mathbf{C}
\end{array}$$

If S, T are monads equipped with the equivalent data from this bijection, then the following conditions on monad morphisms $\tau : \mathsf{S} \to \mathsf{T}$ are equivalent: (1) τ is a powered monad morphism $\hat{\mathsf{S}} \to \hat{\mathsf{T}}$; (2) τ is a powered natural transformation $\hat{S} \Rightarrow \hat{T}$; (3) τ makes the diagram below commute.

$$\begin{array}{ccc}
\mathbf{V}^{\mathrm{op}} \times \mathbf{Alg}\,\mathsf{T} & \xrightarrow{\mathrel{-\!\!\in}_{\mathsf{T}}} & \mathbf{Alg}\,\mathsf{T} \\
{\scriptstyle \mathbf{V}^{\mathrm{op}} \times \mathbf{Alg}\,\tau}\downarrow & & \downarrow{\scriptstyle \mathbf{Alg}\,\tau} \\
\mathbf{V}^{\mathrm{op}} \times \mathbf{Alg}\,\mathsf{S} & \xrightarrow[\mathrel{-\!\!\in}_{\mathsf{S}}]{} & \mathbf{Alg}\,\mathsf{S}
\end{array}$$

6.1 Free monads on powered endofunctors

As an application of Proposition 6.6, we show that, unlike in the case of strength with respect to an action, if T is an algebraically free monad on a powered functor, then T is powered in a canonical way. In light of Proposition 6.4, this explains why algebraic freeness suffices to construct a (left) strength with respect to a monoidal right-closed structure.

Proposition 6.7. *If F is a powered endofunctor on a powered category $\mathbf{C}$ and T is the algebraically free monad on $\underline{F}$, then T forms a powered monad.*

If $\triangleright$ is an action of $\mathbf{V}$ on $\mathbf{C}$, related to $\mathrel{-\!\!\in}$ as in Proposition 6.4, and F is a strong endofunctor, then every free $\underline{F}$-algebra is strongly free.

Proof. If T is algebraically free there is an isomorphism $\mathbf{alg}\,\underline{F} \cong \mathbf{Alg}\,\mathsf{T}$ that commutes with the forgetful functors. By Proposition 6.6, to make T into a powered monad it therefore suffices to show that the powering $\mathrel{-\!\!\in}$ lifts to $\mathbf{alg}\,\underline{F}$. To do this, define $\mathrel{-\!\!\in}_{\underline{F}} : \mathbf{V}^{\mathrm{op}} \times \mathbf{alg}\,\underline{F} \to \mathbf{alg}\,\underline{F}$ on objects by $\Gamma \mathrel{-\!\!\in}_{\underline{F}} (A, a) = (\Gamma \mathrel{-\!\!\in} A, (\Gamma \mathrel{-\!\!\in} a) \circ F^{(\Gamma)} \mathrm{id}_{\Gamma \mathrel{-\!\!\in} A})$ and on morphisms by $\sigma \mathrel{-\!\!\in}_{\underline{F}} f = \sigma \mathrel{-\!\!\in} f$.

Given an action $\triangleright$ as in Proposition 6.4, strong functors and monads are in bijection with powered functors and monads, so the algebraically free monad T then forms a strong monad. To construct the unique maps $\Gamma \triangleright TX \to A$ of Definition 4.5, we can therefore use the strength $\Gamma \triangleright TX \to T(\Gamma \triangleright X)$ and the fact that $T(\Gamma \triangleright X)$ is free on $\Gamma \triangleright X$. □

7 Conclusion

We have shown and commented on a number of different equivalent definitions of strong functor and of strong monad, and explained how and why they arise. These definitions differ significantly in the amount of data and the equations they involve, and they serve different applications. We presented some sufficient conditions for uniqueness and existence of strengths for all functors, in particular the condition of weak functional completeness, which is new as far as we know, and some examples of absence and multiplicity of strengths, which we crafted to demonstrate that these conditions are not necessary.

There are some questions we could not settle; for example, we could neither find a Cartesian category with multiple weak functional completeness structures nor show that there is none. We would like to identify interesting examples of unique existence, absence and multiplicity of strengths for non-symmetric monoidal categories and non-self-actions.

A finer analysis of strength could proceed from a generally non-symmetric non-monoidal closed category $\mathbf{V}$, this being the minimal structure needed for self-enrichment of $\mathbf{V}$. A further possible direction of refinement would be to work with skew monoidal/closed categories and actions, cf. [30]. There are no immediate indications of obstacles, but we would also like to find interesting applications of this level of generality.

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A Biactions, bistrong functors, commutative monads

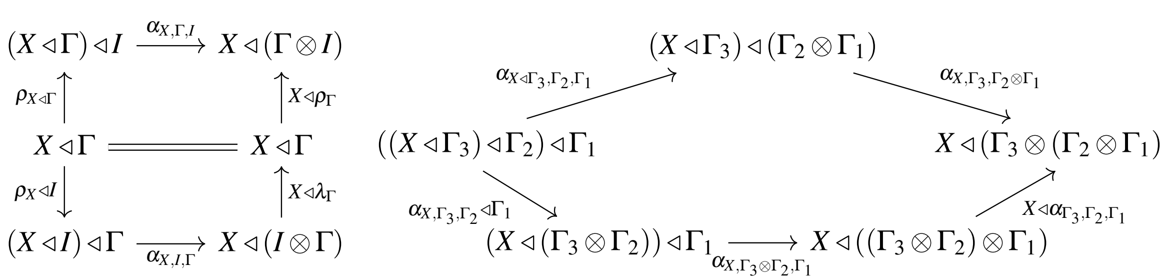
A.1 Biactions, bistrong functors

If a monoidal category **V** acts on a category **C** from both the left and the right, the two actions can be required to agree with each other.

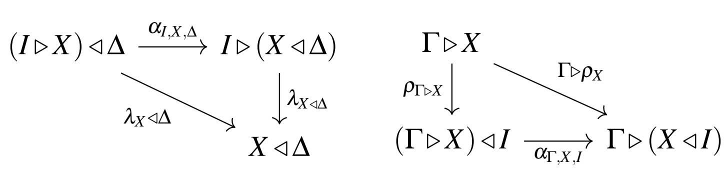
Definition A.1. A *right action* $\triangleleft$ of a monoidal category $(\mathbf{V}, I, \otimes)$ on a category **C** consists of a functor $\triangleleft : \mathbf{C} \times \mathbf{V} \to \mathbf{C}$ and natural isomorphisms

$$\rho_X : X \to X \triangleleft I \qquad \alpha_{X,\Gamma,\Gamma'} : (X \triangleleft \Gamma) \triangleleft \Gamma' \to X \triangleleft (\Gamma \otimes \Gamma')$$

satisfying the following coherence conditions:



Definition A.2. A *biaction* of a monoidal category $(\mathbf{V}, I, \otimes)$ on a category **C** consists of a left action $\triangleright : \mathbf{V} \times \mathbf{C} \to \mathbf{C}$, a right action $\triangleleft : \mathbf{C} \times \mathbf{V} \to \mathbf{C}$, and a natural isomorphism $\alpha_{\Gamma,X,\Delta} : (\Gamma \triangleright X) \triangleleft \Delta \to \Gamma \triangleright (X \triangleleft \Delta)$ such that



$$
\begin{array}{ccccc}
((\Gamma\otimes\Gamma')\rhd X)\lhd\Delta & & \xrightarrow{\alpha_{\Gamma\otimes\Gamma',X,\Delta}} & & (\Gamma\otimes\Gamma')\rhd(X\lhd\Delta) \\
{\scriptstyle \alpha_{\Gamma,\Gamma',X}\lhd\Delta}\downarrow & & & & \downarrow{\scriptstyle \alpha_{\Gamma,\Gamma',X\lhd\Delta}} \\
(\Gamma\rhd(\Gamma'\rhd X))\lhd\Delta & \xrightarrow[\alpha_{\Gamma,\Gamma'\rhd X,\Delta}]{} & \Gamma\rhd((\Gamma'\rhd X)\lhd\Delta) & \xrightarrow[\Gamma\rhd\alpha_{\Gamma',X,\Delta}]{} & \Gamma\rhd(\Gamma'\rhd(X\lhd\Delta))
\end{array}
$$

$$
\begin{array}{ccccc}
((\Gamma\rhd X)\lhd\Delta)\lhd\Delta' & \xrightarrow{\alpha_{\Gamma,X,\Delta}\lhd\Delta'} & (\Gamma\rhd(X\lhd\Delta))\lhd\Delta' & \xrightarrow{\alpha_{\Gamma,X\lhd\Delta,\Delta'}} & \Gamma\rhd((X\lhd\Delta)\lhd\Delta') \\
{\scriptstyle \alpha_{\Gamma\rhd X,\Delta,\Delta'}}\downarrow & & & & \downarrow{\scriptstyle \Gamma\rhd\alpha_{X,\Delta,\Delta'}} \\
(\Gamma\rhd X)\lhd(\Delta\otimes\Delta') & & \xrightarrow[\alpha_{\Gamma,X,\Delta\otimes\Delta'}]{} & & \Gamma\rhd(X\lhd(\Delta\otimes\Delta'))
\end{array}
$$

An example is $\mathbf{C}=\mathbf{V}$ and $\rhd=\lhd=\otimes$.

In a biaction situation, if a functor has both a left strength and a right strength, these can be required to cohere as follows.

Definition A.3. Suppose a biaction of a monoidal category $\mathbf{V}$ on a category $\mathbf{D}$. A *bistrength* for a functor $F:\mathbf{C}\to\mathbf{D}$ is a pair of a left strength $\mathrm{str}_{\Gamma,X}:\Gamma\rhd_{\mathbf{D}}FX\to F(\Gamma\rhd_{\mathbf{C}}X)$ and a right strength $\mathrm{str}^{\mathrm{R}}_{X,\Delta}:FX\lhd_{\mathbf{D}}\Delta\to F(X\lhd_{\mathbf{C}}\Delta)$ such that

$$
\begin{array}{ccccc}
(\Gamma\rhd_{\mathbf{D}}FX)\lhd_{\mathbf{D}}\Delta & \xrightarrow{\mathrm{str}_{\Gamma,X}\lhd_{\mathbf{D}}\Delta} & F(\Gamma\rhd_{\mathbf{C}}X)\lhd_{\mathbf{D}}\Delta & \xrightarrow{\mathrm{str}^{\mathrm{R}}_{\Gamma\rhd_{\mathbf{C}}X,\Delta}} & F((\Gamma\rhd_{\mathbf{C}}X)\lhd_{\mathbf{C}}\Delta) \\
{\scriptstyle \alpha_{\Gamma,FX,\Delta}}\downarrow & & & & \downarrow{\scriptstyle F\alpha_{\Gamma,X,\Delta}} \\
\Gamma\rhd_{\mathbf{D}}(FX\lhd_{\mathbf{D}}\Delta) & \xrightarrow[\Gamma\rhd_{\mathbf{D}}\mathrm{str}^{\mathrm{R}}_{X,\Delta}]{} & \Gamma\rhd_{\mathbf{D}}F(X\lhd_{\mathbf{C}}\Delta) & \xrightarrow[\mathrm{str}_{\Gamma,X\lhd_{\mathbf{C}}\Delta}]{} & F(\Gamma\rhd_{\mathbf{C}}(X\lhd_{\mathbf{C}}\Delta))
\end{array}
$$

A natural transformation between two bistrong functors is bistrong if it is both left strong and right strong.

Consider the case $\mathbf{C}=\mathbf{D}=\mathbf{V}$ and $\rhd=\lhd=\otimes$. If $\mathbf{V}$ is symmetric, with braiding $c_{X,Y}:X\otimes Y\to Y\otimes X$, then any left strength str of a functor F induces a right strength $\mathrm{str}^{\mathrm{R}}$ via $\mathrm{str}^{\mathrm{R}}_{X,\Delta}=Fc_{\Delta,X}\circ\mathrm{str}_{\Delta,X}\circ c_{FX,\Delta}$. The two strengths together form a bistrength. But the right strength does not have to be related to the left strength like this, not even when $\mathbf{V}$ is a Cartesian category. For example, take $\mathbf{V}$ to be the category of pointed sets with its Cartesian structure. The identity functor is bistrong with $\mathrm{str}_{\Gamma,X}(\gamma,x)=(\gamma,x)$ and $\mathrm{str}^{\mathrm{R}}_{X,\Delta}(x,\delta)=(x,\star)$.

A.2 Commutative monads

Kock [12] studied what he named commutative monads for the case of a symmetric monoidal category. His commutative monads were left-strong monads subject to an additional equational condition.

Symmetry is in fact not needed. The concept of commutative monad makes sense for a general monoidal category $\mathbf{V}$; Kock's condition can be formulated for any bistrength for the tensor as a biaction (where the right strength need not in general be defined in terms of the left strength like we did above).

Definition A.4. Suppose a monoidal category $(\mathbf{V},I,\otimes)$. A *commutative monad* is a monad $\mathsf{T}=(T,\eta,\mu)$ with a bistrength $(\mathrm{str},\mathrm{str}^{\mathrm{R}})$ of T (wrt. $\otimes$ as a biaction of $\mathbf{V}$ on itself) such that η, μ are bistrong and moreover the following diagram commutes:

$$
\begin{array}{ccccc}
TX\otimes TY & \xrightarrow{\mathrm{str}_{TX,Y}} & T(TX\otimes Y) & \xrightarrow{T\mathrm{str}^{\mathrm{R}}_{X,Y}} & T(T(X\otimes Y)) \\
{\scriptstyle \mathrm{str}^{\mathrm{R}}_{X,TY}}\downarrow & & & & \downarrow{\scriptstyle \mu_{X\otimes Y}} \\
T(X\otimes TY) & \xrightarrow[T\mathrm{str}_{X,Y}]{} & T(T(X\otimes Y)) & \xrightarrow[\mu_{X\otimes Y}]{} & T(X\otimes Y)
\end{array}
$$

Commutative monads in this sense are exactly the same as lax monoidal monads. Even when **V** is symmetric, the bistrength of a commutative monad does not need to be defined by symmetry, so this notion of commutative monad (i.e. lax monoidal monad) is strictly more general than Kock's. For example, consider the writer monad Wr_{M} on $\mathbf{Act}\,\mathsf{M}$ from Example 4.4, where M is any commutative monoid. From the two strengths given there, we can make a bistrength

$$\mathrm{str}_{\Gamma,X}(\gamma,(x,m)) = ((\gamma,x),m) \qquad \mathrm{str}^{\mathrm{R}}_{X,\Delta}((x,m),\delta) = ((x,\delta * m),m)$$

and Wr_{M} equipped with this bistrength is a commutative monad. Kock's commutative monads are the same as symmetric lax monoidal monads.