Example sheet 6

Advanced random processes

Network Performance—DJW—2010/11

Question 1. M. Ahmed, a former PhD student at UCL, studied epidemic-based models of trust and assurance. He proposed the following system. There are n agents in total. Each agent may be either sound or mistaken. Agents may switch: when a sound and a mistaken agent talk they both become mistaken; and a mistaken agent will after a time become sound again.

We will model this as a Markov process. Suppose that each agent initiates conversations as a Poisson process with rate 1/2, and whenever it initiates a conversation it picks one of the other $n-1$ agents at random to talk to. Suppose that once an agent becomes mistaken, it remains mistaken for a length of time which is exponentially distributed with mean d.

(i) For an arbitrary agent, show that the mean time until it is involved in a conversation is 1.

(ii) Let I_t be the number of mistaken agents at time t. Find the state space diagram for the Markov process I_t, and calculate the transition rates.

(iii) Show that when i agents are mistaken, the drift in I_t is $i(1-i/n)-i/d$.

(iv) Sketch the drift diagram. Find any fixed points. Are they stable or unstable?

Question 2. This question describes a simple model for a wireless MAC protocol. New packets arise as a Poisson process of rate λ, and immediately attempt transmission. A transmission may fail, if another packet is already being transmitted; in this case the packet becomes backlogged. Each backlogged packet attempts retransmission at rate μ, and these retransmissions may also fail, in which case the packet remains backlogged. Let the time to transmit a packet be Δ.

(i) The probability that there is no new packet being transmitted is $e^{-\lambda\Delta}$, and the probability that a given backlogged packet is not being transmitted is $e^{-\mu\Delta}$. Let B_t be the number of backlogged packets. Explain why, when $B_t = b$, the drift in B_t is

$$\lambda(1-e^{-\lambda\Delta}e^{-b\mu\Delta}) - b\mu\, e^{-\lambda\Delta}e^{-(b-1)\mu\Delta}.$$

(ii) Sketch the drift diagram. Find any fixed points. Are they stable or unstable? How do you expect the system to behave?

The drift formula here makes an invalid assumption, namely that the outcomes of successive transmission attempts are independent. The next question describes a slotted-time model in which the independence assumption is valid.

Question 3. This question describes a slotted-time model for a wireless MAC protocol with a finite number of stations. Let there be a fixed number of stations n, each wanting to transmit packets. Each station may be either backlogged or free; let B_t be the number that are backlogged at timeslot t, and $n-B_t$ the number that are free.

In each timeslot, stations may choose to transmit. Each backlogged station attempts to transmit with probability p, and each free station decides to transmit a new packet with probability q. Let $R_t \sim \text{Bin}(B_t, p)$ be the number of backlogged stations that attempt to transmit, let $N_t \sim \text{Bin}(n-B_t, q)$ be the number of free stations that attempt to transmit a new packet, and let $X_t = R_t + N_t$ be the total number of stations that attempt to transmit.

If $X_t = 1$ and $R_t = 1$ then the backlogged station that transmitted becomes free, i.e. $B_{t+1} = B_t - 1$. If $X_t = 1$ and $N_t = 1$ then the free station that transmitted remains free, i.e. $B_{t+1} = B_t$. Otherwise all the free stations that attempted to transmit become backlogged, i.e. $B_{t+1} = B_t + N_t$.

(i) Find the drift in B_t. Sketch the drift diagram. Find the fixed points. Are they stable or unstable? How do you expect the system to behave?

(ii) Pick interesting values for n, p and q, and compute the equilibrium distribution.
(iii) Comment on the relationship between the drift diagram and the equilibrium distribution.

Question 4. Consider a FIFO queue with Poisson arrivals of rate λ, and suppose that the service time for a job is the sum of two exponential random variables, say $\mathrm{Exp}(\mu_1) + \mathrm{Exp}(\mu_2)$. The first might represent the time it takes the server to do a 'context switch' and start work on a new job, and the second might represent how long the job takes to complete.

Let the state be (X_t, W_t) where X_t is the number of jobs in the queue (including the job being served) and $W_t \in \{0,1\}$, $W_t = 0$ if the server is currently switching context or idle and $W_t = 1$ if the server is working on a job.
(i) Draw the state space diagram, and find the transition rates. Justify your answer.
(ii) Assuming that the queue does not empty, draw a state space diagram for W_t, and find its equilibrium distribution π.
(iii) Suppose that $X_t = x$. Find the drift in x, assuming that $\mathbb{P}(W_t = 0) = \pi_0$ and $\mathbb{P}(W_t = 1) = \pi_1$.

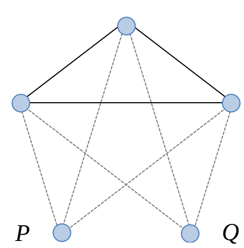
We are treating the variation in W_t as fast-timescale jitter, and the variation in X_t as slow-timescale drift. There are many natural systems in which there is a natural timescale decomposition, e.g. what you eat on a given day is random in quantity and timing, and it affects your overall fitness but over a longer timescale. More generally, the fast-timescale jitter might depend on the slow-timescale quantity.

Question 5. Consider the two-link three-route system described in Section 3.3 of lecture notes. Let $\mu = 1$, $\lambda_0 = 3$, $\lambda_1 = 5$, $\lambda_2 = 7$, $C_1 = 10$, $C_2 = 12$.
(i) Used the fixed point method to compute B_1 and B_2, the probabilities that links 1 and 2 respectively are full. Calculate the blocking probability for each of the three types of call.
(ii) Plot a two-dimensional drift diagram, in which B_1 is on the x-axis, B_2 is on the y-axis, and arrows show (drift in B_1,drift in B_2). On the same plot, show the successive iterations of the fixed point method.

Question 6. Consider a network of n nodes running the Dynamic Alternative Routing algorithm. When the network is fully connected, the probability B that a given link has all its circuits busy can be found by solving the fixed-point equation

$$B = E\left[\frac{1}{\mu}\left(\lambda + 2\lambda B(1-B)\right), C\right].$$

In this question we will investigate what happens when one of the link fails.



Suppose that P and Q are two of the n nodes in the network, and the link between P and Q fails. We would expect that the links which terminate at P or Q should become slightly busier, as traffic between P and Q shifts to alternative paths. Label a link *red* if it terminates at P or Q; there are $2(n-2)$ of these. Label the other links *black*. Let B_{red} be the probability that a red link has all its circuits busy, and let B_{black} be the probability that a black link has all its circuits busy. Derive fixed-point equations for B_{red} and B_{item}. Explain your reasoning. *[Hint. First consider a red link, say between P and some other node M, and list all the different routes that might use this link. Then consider a black link, and do the same.]*

Question 7. This question concerns an operational law known as the *interactive response time law*. There are n users sharing a processor. Each user alternates between waiting and thinking: he/she submits a job, waits for the processor to process it, receives the response, spends some time thinking, then submits another job and so on. Let W be the average wait time, let T be the average think time, and let λ be the total rate at which jobs are submitted. By counting up the total time spent thinking and waiting over an interval, and the total number of jobs submitted in that interval, show that

$$W = \frac{n}{\lambda} - T.$$

Question 8. A timesharing system is being shared by 10 users. They submit jobs to a CPU. The CPU may complete the job directly (with probability 0.04), or it may route it to a device. If the job is routed to a device, it goes to A, B or C, with probabilities 0.32, 0.44 and 0.24 respectively. Device A sends the job back to the CPU; device B sends the job on to device D; device C sends it to D (with probability 0.27) or back to the CPU. The average service times for a job are 5ms at the CPU, 14ms at device A, 20ms at device B, 10ms at device C, 15ms at device D. Average user think time is 5s.

(i) Let λ be the total rate at which jobs are submitted by users to the CPU. Calculate the rate at which jobs arrive to each device.

(ii) Calculate the utilization of each device. Which device is the bottleneck?

(iii) Suppose the utilization of device B is 8%. Calculate the average time that a user has to wait for a job to be processed.