Example sheet 1a

Probability revision

Network Performance—DJW—2010/11

You should be able to draw Venn diagrams to represent events. You should be able to calculate with probabilities, including conditional probability and independence. You should know the factorial function and the binomial coefficient.

Notation. Write $\mathbb{P}(A)$ for the probability of an event A. Write $x \wedge y$ for $\min(x,y)$ and $x \vee y$ for $\max(x,y)$.

Question 1. I throw a coin 5 times. The throws are independent. In each throw, the probability of heads is p. What is the probability that exactly two throws yielded heads? What is the probability of two consecutive heads?

Question 2*. (i) When $\mathbb{P}(A) = 1/3$, $\mathbb{P}(B) = 1/2$ and $\mathbb{P}(A \cup B) = 3/4$, calculate $\mathbb{P}(A \cap B)$. What is the probability that A occurs or B occurs but not both?

(ii) When $\mathbb{P}(A) = 3/4$ and $\mathbb{P}(B) = 1/3$, show that $1/12 \leq \mathbb{P}(A \cap B) \leq 1/3$, and give examples to show that both extremes are possible. Find corresponding bounds for $\mathbb{P}(A \cup B)$.

Question 3. You are one of a group of five mafiosi. Each of you has a vendetta against one of the others (chosen at random), and has hired a hitman to kill him. What is the probability than no one has hired a hitman to kill you?

Question 4. In a group of 10 people, what is the probability that no two of them have the same birthday?

Question 5*. After being tested, you are diagnosed with a disease. You know that 1% of the population has the disease. Write D for the event that "you have the disease" and T for "the test says you have the disease". It is known that the test is imperfect: $\mathbb{P}(T|D) = 0.98$ and $\mathbb{P}(\text{not } T|\text{not } D) = 0.95$.

(i) Given that you test positive what is the probability that you truly have the disease?

(ii) You obtain a second opinion, i.e. an independent repetition of the test. You test positive again. Given this, what is the probability that you truly have the disease?

Question 6. I draw four cards at random from a standard deck of playing cards. What is the probability that I have one of each suit?

Question 7. I throw three fair dice, and sum up my throws. Compute the probability of scoring n, for all $3 \leq n \leq 18$.

Question 8. Consider the r-round coordination game, with n players. Calculate the probability that I have to pay out assuming that each students raises a hand with probability p, independently from student to student and from round to round.

* These questions are taken from *A Modern Introduction to Probability and Statistics* by Dekking, Kraaikamp, Lopuhaä and Meester, and from *One Thousand Exercises in Probability* by Grimmet and Stirzaker.

Example sheet 1b

Calculus revision

Network Performance—DJW—2009/10

You should be able to differentiate polynomials, log, the exponential function, sin etc. You should know the chain rule, and you should be able to differentiate products and divisions. You should be able to sketch curves, by looking for turning points and using the second derivative to calculate the type of the turning point, and by finding asymptotics. You should understand the derivative as the slope of a curve. You should be understand the $O(\cdot)$, $o(\cdot)$, $\Theta(\cdot)$ and $\cdot \sim \cdot$ notations.

Notation. Write $\mathbb{N}$ for $\{1,2,\dots\}$, $\mathbb{R}$ for the set of real numbers, and $\mathbb{R}_+$ for the set $\{x \in \mathbb{R} : x > 0\}$.

Question 9. Differentiate
(i) $x^{1/3}$
(ii) $\sqrt{1+x^2}$
(iii) $\sin(1/x)$
(iv) 4^x
(v) $2^x/(2^x-3^x)$
(vi) x^x (Hint: write it as $e^{x\log x}$)

Question 10. Given $x_1,\dots,x_n$, find $\lambda \in \mathbb{R}_+$ to maximize

$$\prod_{i=1}^{n} \lambda e^{-\lambda x_i}.$$

(Hint: take logs first.) Sketch this curve.

Question 11. Given $n \in \mathbb{N}$ and $r \in \mathbb{N}$ with $1 \le r \le n$, find p to maximize

$$\binom{n}{r} p^r (1-p)^{n-r}.$$

Sketch this curve. Here, $\binom{n}{r}$ is the binomial coefficient, $n!/(n-r)!r!$

Question 12. Given $x_1,\dots,x_n$, find $\mu \in \mathbb{R}$ and $\sigma \in \mathbb{R}_+$ to maximize

$$\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x_i-\mu)^2/2\sigma^2}.$$

Sketch this surface.

Question 13. Which of the following are true?
(i) $2x$ is $O(x^2)$ as $x \to \infty$
(ii) $2x$ is $o(x^2)$ as $x \to \infty$
(iii) $2x$ is $\Theta(x^2)$ as $x \to \infty$
(iv) $2x$ is $o(x^2)$ as $x \to 0$
(v) $(n+\log n)^2$ is $O(n\log n)$ as $n \to \infty$
(vi) $\sin x$ is $O(1)$ as $x \to \infty$
(vii) $\sin x$ is $O(1)$ as $x \to 0$
(viii) If $f(x)$ is $O(g(x))$ as $x \to \infty$ then $g(x)$ is $o(f(x))$ as $x \to \infty$
(ix) $\sum_{i=1}^{n} i^2$ is $O(n^3)$ as $n \to \infty$