Sample path LDP for traffic  
(in the long-timescale limit)

Large Deviations and Queues—Damon Wischik

1 Model

Let $\mathcal{X}$ be the set of discrete-time traffic processes

$$\mathcal{X} = \{ x : \mathbb{Z}_+ \to \mathbb{R}, x(0) = 0 \}.$$  

Let $\mathcal{C}$ be the set of continuous-time traffic processes

$$\mathcal{C} = \{ x : \mathbb{R}_+ \to \mathbb{R}, x(0) = 0 \}.$$  

Let $\mathcal{A}$ be the subset of $\mathcal{C}$ consisting of absolutely continuous traffic processes.  

Let $\mathcal{X}^T$ be and $\mathcal{C}^T$ be

$$\mathcal{X}^T = \{ x : \{0, \ldots, T\} \to \mathbb{R}, x(0) = 0 \},$$

$$\mathcal{C}^T = \{ x : [0, T] \to \mathbb{R}, x(0) = 0 \},$$

and define $\mathcal{A}^T$ similarly. Interpret $x(t)$ as the amount of work arriving in the interval $(-t, 0]$. Say that a traffic process $x$ has mean rate $\mu$ if $\lim_{t \to \infty} x(t)/t = \mu$. Write $\mathcal{X}_\mu$, $\mathcal{C}_\mu$ and $\mathcal{A}_\mu$ for the restrictions of $\mathcal{X}$, $\mathcal{C}$ and $\mathcal{A}$ to traffic processes with mean rate $\mu$.

Define the scaled uniform norm $\| \cdot \|$ on these spaces by

$$\| x \| = \sup_{t \geq 0} \left| \frac{x(t)}{t+1} \right|$$

Also define $\pi$, the topology of uniform convergence on compact intervals.

Given $x \in \mathcal{X}$, define the polygonalized version $\tilde{x} \in \mathcal{A}$ to be

$$\tilde{x}(t) = ([t+1] - t)x([t]) + (t - [t])x([t+1]).$$

Given $x \in \mathcal{C}$, define the speeded-up version $x^{\mu L} \in \mathcal{C}$ to be

$$x^{\mu L}(t) = x(Lt).$$

Use the following extended notation: write

- $x(-t, 0]$ for $x(t)$
- $x|_{(-t,0]}$ for the restriction of $x$ to $[0,t]$  
- $\tilde{x}_{-t}$ for $dx(t)/dt$, where it is defined, for $x \in \mathcal{C}$
- $\tilde{x}_{-t}$ for $x(t+1) - x(t)$, for $x \in \mathcal{X}$

2 Probabilistic setup

Let $A$ be a random discrete-time traffic process, taking values in $\mathcal{X}$. Suppose that the $A_{-t}$ are independent and identically distributed. Let

$$\Lambda(\theta) = \log E \exp(\theta A_0),$$

and assume that $\Lambda(\cdot)$ is finite in a neighbourhood of the origin.
3 Cramér’s theorem

By Cramér’s theorem, $L^{-1}A^{\infty L}(1)$ satisfies an LDP in $\mathbb{R}$ with good convex rate function

$$\Lambda^*(x) = \sup_{\theta \in \mathbb{R}} \theta x - \Lambda(\theta).$$

4 Finite horizon SP-LDP

By Mogulskii’s theorem, $L^{-1}\tilde{A}^{\infty L}|_{[-1,0]}$ satisfies a sample path LDP in $(C^1, \pi)$ with rate function

$$I_1(x) = \begin{cases} \int_{-1}^{0} \Lambda^*(\dot{x}_s) \, ds & \text{if } x \in A^1 \\ \infty & \text{otherwise.} \end{cases}$$

This can easily be extended to a sample path LDP for $L^{-1}\tilde{A}^{\infty L}|_{[-T,0]}$ with rate function

$$I_T(x) = \begin{cases} \int_{-T}^{0} \Lambda^*(\dot{x}_s) \, ds & \text{if } x \in A^T \\ \infty & \text{otherwise.} \end{cases}$$

5 Infinite horizon SP-LDP

Write $\Pi_L : C \to C^T$ for the projection $x \mapsto x|_{[-T,0]}$. With these projections, $(C, \pi)$ is the projective limit of the collection of spaces $(C^T, \pi)$. By the Dawson-Gärtner theorem, $L^{-1}\tilde{A}^{\infty L}$ satisfies a sample path LDP in $(C, \pi)$ with good rate function

$$I(x) = \sup_{T \geq 0} I_T(\Pi_L x).$$

By the non-negativity of $\Lambda^*$, the supremum is

$$I(x) = \begin{cases} \int_{-\infty}^{0} \Lambda^*(\dot{x}_s) \, ds & \text{if } x \in A \\ \infty & \text{otherwise.} \end{cases}$$

6 Strengthening the topology

It can be shown that $L^{-1}\tilde{A}^{\infty L}$ is exponentially tight in $(C, \|\cdot\|)$. Therefore, using the inverse contraction principle, the sample path LDP for $L^{-1}\tilde{A}^{\infty L}$ holds in $(C, \|\cdot\|)$.

7 Restricting the space

Since $\Lambda(\cdot)$ is finite in a neighbourhood of the origin, it is differentiable at the origin. Let $\mu = \Lambda'(0)$. It can be shown that $\mathbb{P}(L^{-1}\tilde{A}^{\infty L} \in C_\mu) = 1$, and that this space is closed. Therefore the sample path LDP for $L^{-1}\tilde{A}^{\infty L}$ holds in $(C_\mu, \|\cdot\|)$.