Useful abstract LDPs

Large Deviations and Queues—Damon Wischik

1 Contraction principle

Let $\mathcal{X}$ and $\mathcal{Y}$ be Hausdorff spaces, and let $f : \mathcal{X} \to \mathcal{Y}$ be continuous. If $(X^L,\ L \in \mathbb{N})$ satisfies an LDP in $\mathcal{X}$ with good rate function I then the sequence $f(X^L)$ satisfies an LDP in $\mathcal{Y}$ with good rate function

$$J(y) = \inf_{x: f(x)=y} I(x).$$

2 Inverse contraction principle

Let $\mathcal{X}$ be a Hausdorff space. The sequence of $\mathcal{X}$-valued random variables $(X^L,\ L \in \mathbb{N})$ is *exponentially tight* if for every $\alpha \in \mathbb{R}_+$ there exists a compact set K_α such that

$$\limsup_{L\to\infty} \frac{1}{L} \log \mathbb{P}(X^L \notin K_\alpha) < -\alpha.$$

Let $\mathcal{X}$ and $\mathcal{Y}$ be Hausdorff spaces, and $f : \mathcal{X} \to \mathcal{Y}$ a continuous bijection. If $(X^L,\ L \in \mathbb{N})$ is exponentially tight in $\mathcal{X}$, and the sequence $f(X^L)$ satisfies an LDP in $\mathcal{Y}$ with rate function $J(\cdot)$, then the sequence X^L satisfies an LDP in $\mathcal{X}$ with good rate function

$$I(x) = J(f(x)).$$

3 Product spaces

Let $\mathcal{X}$ and $\mathcal{Y}$ be regular Hausdorff spaces, e.g. metric spaces. If $(X^L,\ L \in \mathbb{N})$ and $(Y^L,\ L \in \mathbb{N})$ satisfy LDPs in $\mathcal{X}$ and $\mathcal{Y}$ with good rate functions I and J, and X^L is independent of Y^L, then the sequence (X^L, Y^L) satisfies an LDP in $(\mathcal{X}, \mathcal{Y})$ with good rate function

$$K(x,y) = I(x) + J(y).$$

4 Dawson-Gärtner theorem

Let $(J, \leq)$ be a partially-ordered set with the property that for any $i, j \in J$ there exists $k \in J$ with $i \leq k$ and $j \leq k$.

A *projective system* is a collection of Hausdorff spaces $\mathcal{X}_i$ and functions $p_{ij} : \mathcal{X}_i \to \mathcal{X}_j$ $(i \geq j)$ such that $p_{jk} \circ p_{ij} = p_{ik}$ for $i \geq j \geq k$, and $p_{ii} = \mathrm{id}$.

Let $\mathcal{Y} = \prod_{i \in J} \mathcal{X}_i$ and let $p_i : \mathcal{Y} \to \mathcal{X}_i$ be the canonical projections. Let

$$\mathcal{X} = \big\{ x \in \mathcal{Y} : p_j(x) = p_{ij}(p_i(x)) \text{ whenever } i \geq j \big\}.$$

The product topology on $\mathcal{Y}$ induces a topology on $\mathcal{X}$, called the *projective limit topology*, which makes every p_i continuous. In this topology, every open set is the union of sets of the form $\{x \in \mathcal{X} : p_i(x) \in U_i\}$ where $i \in J$ and U_i is open in $\mathcal{X}_i$.

We call $\mathcal{X}$ equipped with this topology the *projective limit* of the projective system.

Let $(X^L,\ L \in \mathbb{N})$ be a sequence of $\mathcal{X}$-valued random variables. If for every $i \in J$, $p_i(X^L)$ satisfies an LDP in $\mathcal{X}_i$ with good rate function I_i, then the sequence X^L satisfies an LDP in $\mathcal{X}$ with good rate function

$$I(x) = \sup_{i \in J} I_i\big(p_i(x)\big).$$

5 Restriction

Let $(X_n,\ n \in \mathbb{N})$ be a sequence of random variabes taking values in some Hausdorff space $\mathcal{X}$. Let $\mathcal{E}$ be a measurable subset of $\mathcal{X}$ such that $\mathbb{P}(X_n \in \mathcal{E}) = 1$ for all $n \in \mathbb{N}$. Equip $\mathcal{E}$ with the topology induced by $\mathcal{X}$, and suppose $\mathcal{E}$ is closed.

If $(X_n,\ n \in \mathbb{N})$ satisfies an LDP in $\mathcal{X}$ with rate function I then it satisfies an LDP in $\mathcal{E}$ with the same rate function I.

6 Exponential equivalence

Let $\mathcal{X}$ be a metric space, with metric d. Let $(X_n,\ n \in \mathbb{N})$ and $(Y_n,\ n \in \mathbb{N})$ be sequences of random variables on $\mathcal{X}$. They are *exponentially equivalent* if

$$\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}\big(d(X_n, Y_n) > \delta\big) = -\infty \quad \text{for all } \delta > 0.$$

Then, if $(X_n,\ n \in \mathbb{N})$ satisfies a large devations principle with good rate function I, so does $(Y_n,\ n \in \mathbb{N})$.